THE C^k -CLASSIFICATION OF CERTAIN OPERATORS IN $L_p^{(1)}$

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Introduction. We investigate in this paper the one-parameter family of operators

$$T_\alpha = M + \alpha J$$

acting in $L_p(0, 1)$ ($1 \leq p < \infty$), where $\alpha \in \mathbb{C}$ (the complex field), $M: f(x) \rightarrow xf(x)$ and $J: f(x) \rightarrow \int_0^x f(t) dt$.

Extending a result of Sakhnovič [8] from the case $p=2$ to the case $1 < p < \infty$, Kalisch [4] established recently that T_α is similar to M if $\operatorname{Re} \alpha = 0$.

The situation becomes quite different if $\operatorname{Re} \alpha \neq 0$; thus, T_α is *not* similar to M if $|\operatorname{Re} \alpha| \geq 1$, and more generally, T_α is not similar to T_β if $[\operatorname{Re} \alpha] \neq [\operatorname{Re} \beta]$ (Proposition 13). This result will follow from our discussion of the C^k -operational calculus (both “global” and “local”, in the sense of [5], [6], [7]) for the operators T_α . We recall briefly the terminology, restricting ourselves to bounded operators $T: X \rightarrow X$ (X a Banach space) with *real* spectrum $\sigma(T)$.

Fix a compact interval $\Delta \subset \mathbb{R}$ (the real line) which contains $\sigma(T)$. For $n=0, 1, 2, \dots$, let $C^n(\Delta)$ denote the Banach algebra of all complex valued functions of class C^n on Δ , with the norm

$$\|\varphi\|_{n,\Delta} = \sum_{j=0}^n \sup_{\Delta} |\varphi^{(j)}|/j!$$

(We shall write $\|\varphi\|_n$ when $\Delta=[0, 1]$.) We say that T is of class C^n (and we write $T \in (C^n)$) if there exists a continuous representation $\varphi \rightarrow T(\varphi)$ of $C^n(\Delta)$ on X such that $T(\varphi)=I$ (the identity operator) for $\varphi(t) \equiv 1$ and $T(\varphi)=T$ for $\varphi(t) \equiv t$. Such a representation is unique when it exists, and is called the C^n -operational calculus for T . For example, $T_0=M$ is of class C ($=C^0$), and its C -operational calculus is $\varphi \rightarrow M(\varphi)$, where

$$M(\varphi): f(x) \rightarrow \varphi(x)f(x), f \in L_p(0, 1).$$

If $W \subset X$ is a linear manifold, we denote by $T(W)$ the algebra of all linear transformations of X with domain W and range contained in W . If W is invariant for T , a C^n -operational calculus for T on W is an algebra homomorphism $\varphi \rightarrow T(\varphi)$ of $C^n(\Delta)$ into $T(W)$ with the following properties:

- (i) $T(\varphi)=I/W$ for $\varphi(t) \equiv 1$;
- (ii) $T(\varphi)=T/W$ for $\varphi(t) \equiv t$;
- (iii) for each $x \in W$, the mapping $\varphi \rightarrow T(\varphi)x$ of $C^n(\Delta)$ into X is continuous.

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For each $n=0, 1, 2, \dots$, there exists a maximal invariant linear manifold $W_n(T)$ on which T admits a C^n -operational calculus, and the latter is uniquely determined on $W_n(T)$. In fact, $W_n(T)$ is the set of all $x \in X$ for which

$$|x|_n = \sup \{ \|\varphi(T)x\|; \varphi \text{ a polynomial with } \|\varphi\|_{n,\Delta} \leq 1 \} \text{ is finite.}$$

The "semisimplicity manifold" $W_0(T)$ is particularly important (cf. Theorem 2.1 in [6]); it contains trivially the eigenvectors of T . We shall write $W_n(T_\alpha; p)$ instead of $W_n(T_\alpha)$ when it will be necessary to specify the L_p space under consideration.

We may now describe the main results of this paper (for $1 < p < \infty$):

1°. T_α is of class C^n if $|\operatorname{Re} \alpha| \leq n$ and only if $|\operatorname{Re} \alpha| < n+1$ (Theorem 6); the C^n -operational calculus for T_α ($|\operatorname{Re} \alpha| \leq n$) is given in Theorems 8 and 9.

2°. The W_k manifolds of T_α are dense for $\operatorname{Re} \alpha < 0$ and trivial for $\operatorname{Re} \alpha \geq 1$ and $k < [\operatorname{Re} \alpha]$ (Theorems 10 and 11).

3°. T_α is not spectral (in Dunford's sense) for $|\operatorname{Re} \alpha| \geq 1$ (it is of course spectral for $\operatorname{Re} \alpha = 0$, by the Kalisch-Sakhnovič result).

1. Five lemmas. Let $\{J^\zeta; \operatorname{Re} \zeta > 0\}$ be the Riemann-Liouville holomorphic semigroup, acting in $L_p(0, 1)$ ($1 \leq p < \infty$):

$$(J^\zeta f)(x) = (1/\Gamma(\zeta)) \int_0^x (x-t)^{\zeta-1} f(t) dt$$

($f \in L_p(0, 1)$, $x \in [0, 1]$, $\operatorname{Re} \zeta > 0$). It is known (cf. [3] for $p=2$, and [4] for $1 < p < \infty$) that if $1 < p < \infty$, the semigroup $\{J^\zeta\}$ admits a strongly continuous boundary group $\{J^{i\gamma}; \gamma \in \mathbf{R}\}$ of bounded operators, and

$$\|J^{i\gamma}\| \leq e^{\pi|\gamma|/2} \quad (\gamma \in \mathbf{R})$$

(see also the estimates at the end of Kalisch's paper [4]). The operator J^ζ ($\operatorname{Re} \zeta > 0$) is one-to-one in $L_p(0, 1)$ ($p \geq 1$); its inverse, with domain $\mathscr{D}_{-\zeta} = \mathscr{R}(J^\zeta)$ (the range of J^ζ), is a closed operator, which we denote by $J^{-\zeta}$.

For $1 < p < \infty$, note that $\mathscr{R}(J^{\beta+i\gamma}) = \mathscr{R}(J^\beta)$ ($\beta > 0$, $\gamma \in \mathbf{R}$), since $J^{\beta+i\gamma} = J^\beta J^{i\gamma}$ and $J^{i\gamma}$ is nonsingular.

For $p=1$ and $\gamma \in \mathbf{R}$, we define $J^{i\gamma}$ as follows. Its domain is $\mathscr{D}_{i\gamma} = U\{\mathscr{R}(J^\zeta); \operatorname{Re} \zeta > 0\}$; if $f \in \mathscr{D}_{i\gamma}$, say $f = J^\zeta h$ for some $h \in L_1(0, 1)$ and $\zeta \in \mathbf{C}$ with $\operatorname{Re} \zeta > 0$, then $J^{i\gamma} f = J^{\zeta+i\gamma} h$. One verifies easily that $J^{i\gamma}$ is well defined.

We shall use the notation $\mathscr{D}_\zeta$ also for $\operatorname{Re} \zeta > 0$, in which case $\mathscr{D}_\zeta = L_p(0, 1)$ (the domain of J^ζ !); similarly $\mathscr{D}_{i\gamma} = L_p(0, 1)$ for $1 < p < \infty$.

LEMMA 1. *For any $\zeta \in \mathbf{C}$, $\mathscr{D}_\zeta$ is invariant under M , and the following (trivially equivalent) identities are valid on $\mathscr{D}_\zeta$:*

$$(1) \quad MJ^\zeta - J^\zeta M = \zeta J^{\zeta+1},$$

$$(2) \quad J^\zeta M = T_{-\zeta} J^\zeta,$$

$$(3) \quad MJ^\zeta = J^\zeta T_\zeta.$$

Proof. If $\operatorname{Re} \zeta > 0$ ($\operatorname{Re} \zeta \geq 0$ if $1 < p < \infty$), the first statement is trivial, since $\mathcal{D}_\zeta = L_p(0, 1)$. For $g \in L_p(0, 1)$ and $\operatorname{Re} \zeta > 0$,

$$\begin{aligned}(MJ^\zeta - J^\zeta M)g(x) &= \Gamma(\zeta)^{-1} \int_0^x (x-t)^{\zeta-1} (x-t)g(t) dt \\ &= \Gamma(\zeta)^{-1} \Gamma(\zeta+1) J^{\zeta+1} g(x) = \zeta J^{\zeta+1} g(x).\end{aligned}$$

This proves the first, and hence all three identities of the lemma (for $1 \leq p < \infty$ and $\operatorname{Re} \zeta > 0$). By (3), $M\mathcal{R}(J^\zeta) \subset \mathcal{R}(J^\zeta)$, i.e. $\mathcal{D}_{-\zeta}$ is invariant under M . Apply $J^{-\zeta}$ to both sides of (3):

$$J^{-\zeta} M J^\zeta = T_\zeta,$$

i.e., $J^{-\zeta} M = T_\zeta J^{-\zeta}$ on $\mathcal{D}_{-\zeta}$. This proves (2) (and hence the lemma) for $\operatorname{Re} \zeta < 0$.

Let $\mathcal{D} = U\{\mathcal{D}_\zeta; \operatorname{Re} \zeta < 0\}$. $\mathcal{D}$ is M -invariant, and dense in $L_p(0, 1)$ ($1 \leq p < \infty$). Let $g \in \mathcal{D}$; then $g = J^\zeta h$ for some $\zeta \in \mathbb{C}$ with $\operatorname{Re} \zeta > 0$ and some $h \in L_p(0, 1)$. Using (3) twice (for $\operatorname{Re} \zeta > 0$), we obtain (for $\gamma \in \mathbb{R}$):

$$\begin{aligned}M J^{i\gamma} g &= M J^{\zeta+i\gamma} h = J^{\zeta+i\gamma} T_{\zeta+i\gamma} h \\ &= J^{i\gamma} J^\zeta (T_\zeta + i\gamma J) h = J^{i\gamma} (M J^\zeta + i\gamma J^{\zeta+1}) h \\ &= J^{i\gamma} (M + i\gamma J) J^\zeta h = J^{i\gamma} T_{i\gamma} g.\end{aligned}$$

If $p=1$, this finishes the proof of the lemma for $\operatorname{Re} \zeta=0$. If $1 < p < \infty$, this shows that (3) is valid on the dense subset $\mathcal{D}$ of $L_p(0, 1)$ (for $\zeta=i\gamma$); since $J^{i\gamma}$ is a bounded operator, (3) is valid everywhere on $L_p(0, 1)$.

LEMMA 2. For $\beta, \gamma \in \mathbb{R}$ arbitrary, the operators $T_{\beta+i\gamma}$ and T_β acting in $L_p(0, 1)$ ($1 < p < \infty$) are similar, with $J^{i\gamma}$ implementing the similarity:

$$J^{i\gamma} T_{\beta+i\gamma} J^{-i\gamma} = T_\beta,$$

(for $\beta=0$ and $p=2$, this is due to Sakhnovič [8]).

Proof. By (3),

$$J^{i\gamma} T_{\beta+i\gamma} J^{-i\gamma} = J^{i\gamma} (T_{i\gamma} + \beta J) J^{-i\gamma} = M + \beta J = T_\beta.$$

REMARK. For any $1 \leq p < \infty$ and any complex numbers α and ζ , T_α is unboundedly similar to T_ζ (in particular, T_α is unboundedly similar to M). Indeed,

$$J^{-(\zeta-\alpha)} T_\alpha J^{\zeta-\alpha} = T_\zeta \quad \text{on } \mathcal{D}_{\zeta-\alpha},$$

where everything makes sense by Lemma 1. This suggests considering the map $\varphi \rightarrow J^{-\alpha} M(\varphi) J^\alpha$ as a "possible" operational calculus for T_α .

LEMMA 3. For any integer $n \geq 0$, and $1 \leq p < \infty$, the operator T_n acting in $L_p(0, 1)$ belongs to $(C^n) - (C^{n-1})$, and the C^n -operational calculus for T_n is given by

$$(4) \quad T_n(\varphi) = J^{-n} M(\varphi) J^n, \quad \varphi \in C^n[0, 1].$$

REMARK. (1) $(C^{-1}) = \Phi$ by convention. (2) $T_n(\varphi)$ is well defined, since $\mathcal{R}(J^n)$ is a $C^n[0, 1]$ -module.

Proof. By Leibnitz' formula,

$$(5) \quad T_n(\varphi) = \sum_{j=0}^n \binom{n}{j} M(\varphi^{(j)}) J^j, \quad \varphi \in C^n[0, 1].$$

Thus $T_n(\varphi)$ is a bounded operator on $L_p(0, 1)$. In fact, since $\|J^j\| \leq 1/j!$ (cf. [3, p. 664]), we have

$$(6) \quad \|T_n(\varphi)\| \leq \binom{n}{[n/2]} \|\varphi\|_n, \quad \varphi \in C^n[0, 1].$$

The map $\varphi \rightarrow T_n(\varphi)$ is clearly an algebra homomorphism of $C^n[0, 1]$ into the bounded operators on $L_p(0, 1)$, which is continuous by (6). If $\varphi(x) \equiv 1$, $T_n(\varphi)$ is trivially the identity operator. If $\varphi(x) \equiv x$, $T_n(\varphi) = T_n$ by (3). Thus, $T_n \in (C^n)$ and its C^n -operational calculus is given by (4). Finally, apply (5) to the functions

$$\varphi_t(x) = e^{itx} \quad (x, t \in R).$$

Thus

$$(e^{itT_n} g)(x) = \sum_{j=0}^n \binom{n}{j} (it)^j e^{itx} (J^j g)(x),$$

and consequently $\|e^{itT_n}\| \neq O(|t|^{n-1})$ ($n = 1, 2, \dots$). Therefore $T_n \notin (C^{n-1})$ ($n \geq 1$) by Lemma 2.11 in [5].

LEMMA 4. Let $n \geq 0$ be an integer. Then, for φ of class C^n and $g \in \mathcal{R}(J^n)$ (or vice versa, or for both φ and g in $\mathcal{R}(J^n)$), the following identity is valid:

$$J^n(\varphi J^{-n} g) = \sum_{j=0}^n \binom{n}{j} (-1)^j J^j (g J^{-j} \varphi).$$

REMARK. Here, as usual, $\mathcal{R}(J^n)$ denotes the range of J^n in $L_p(0, 1)$, with p arbitrary ($1 \leq p < \infty$). Note the analogy with Leibnitz' formula (in fact, the latter may be used to prove the lemma).

Proof. We use induction on n . The lemma is trivial for $n=0$. Suppose it is true for $n=k$. Let φ and g be as required for $n=k+1$. Write $g' = J^{-1}g$. Then φ and g' satisfy the hypothesis for $n=k$. Therefore

$$\begin{aligned} J^{k+1}(\varphi J^{-(k+1)} g) &= J J^k(\varphi J^{-k} g') \\ &= \sum_{j=0}^k (-1)^j \binom{k}{j} J^j J(g' J^{-j} \varphi). \end{aligned}$$

An integration by parts shows that

$$J(g' J^{-j} \varphi) = g J^{-j} \varphi - J(g J^{-(j+1)} \varphi).$$

Thus

$$\begin{aligned} J^{k+1}(\varphi J^{-(k+1)}g) &= \sum_{j=0}^k (-1)^j \binom{k}{j} \{J^j(gJ^{-j}\varphi) - J^{j+1}(gJ^{-(j+1)}\varphi)\} \\ &= g\varphi + \sum_{j=1}^k (-1)^j \left[\binom{k}{j} + \binom{k}{j-1} \right] J^j(gJ^{-j}\varphi) \\ &\quad + (-1)^{k+1} J^{k+1}(gJ^{-(k+1)}\varphi). \end{aligned}$$

Since

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j},$$

we obtain the correct identity for $n=k+1$, Q.E.D.

LEMMA 5. For any integer $n \geq 0$, and for $1 \leq p < \infty$, the operator T_{-n} acting in $L_p(0, 1)$ belongs to $(C^n) - (C^{n-1})$, and the C^n -operational calculus for T_{-n} is given by

$$(7) \quad T_{-n}(\varphi) = \sum_{j=0}^n \binom{n}{j} (-1)^j J^j M(\varphi^{(j)}), \quad \varphi \in C^n[0, 1].$$

Proof. The map $\varphi \rightarrow T_{-n}(\varphi)$ of $C^n[0, 1]$ into the bounded operators on $L_p(0, 1)$ is clearly linear and continuous, in fact,

$$\|T_{-n}(\varphi)\| \leq \binom{n}{[n/2]} \|\varphi\|_n.$$

For $g \in \mathcal{R}(J^n)$, we have by Lemma 4:

$$(8) \quad T_{-n}(\varphi)g = J^n(\varphi J^{-n}g).$$

Therefore $\mathcal{R}(J^n)$ is invariant under $T_{-n}(\varphi)$ (for all $\varphi \in C^n[0, 1]$) and $T_{-n}(\varphi\psi) = T_{-n}(\varphi)T_{-n}(\psi)$ on $\mathcal{R}(J^n)$ (for all $\varphi, \psi \in C^n[0, 1]$). Since $\mathcal{R}(J^n)$ is dense in $L_p(0, 1)$ and $T_{-n}(\varphi)$ is continuous, it follows that $T_{-n}(\cdot)$ is multiplicative on $C^n[0, 1]$. The relations $T_{-n}(\varphi) = I(T_{-n})$ for $\varphi(x) \equiv 1$ ($\equiv x$) are trivial on $\mathcal{R}(J^n)$ by (8) and Lemma 1; by density, they are true throughout $L_p(0, 1)$.

Finally, one verifies that $T_{-n} \notin (C^{n-1})$ just as in Lemma 3.

2. Global classification ($1 < p < \infty$).

THEOREM 6. The operator T_α acting in $L_p(0, 1)$ ($1 < p < \infty$) is of class C^n ($n = 0, 1, 2, \dots$) if $|\operatorname{Re} \alpha| \leq n$ and only if $|\operatorname{Re} \alpha| < n + 1$.

In other words, T_α is of class C^n in the strip $|\operatorname{Re} \alpha| \leq n$ and is not of class C^n outside the strip $|\operatorname{Re} \alpha| < n + 1$.

The theorem is an immediate corollary of Lemmas 3 and 5, together with the following

LEMMA 7. Suppose that, for some integer $n \geq 0$ and some $\alpha_0 \in \mathcal{C}$, the operator T_{α_0} is of class C^n (when acting in $L_p(0, 1)$, $1 < p < \infty$). Then T_α is of class C^n for all α in the strip $-n \leq \operatorname{Re} \alpha \leq \operatorname{Re} \alpha_0$ if $\operatorname{Re} \alpha_0 \geq 0$ ($\operatorname{Re} \alpha_0 \leq \operatorname{Re} \alpha \leq n$ if $\operatorname{Re} \alpha_0 \leq 0$).

Proof. To fix the ideas, suppose $\operatorname{Re} \alpha_0 = \beta_0 \geq 0$. Write $\alpha = \beta + i\gamma$ ($\beta, \gamma \in \mathbf{R}$). Fixing a polynomial φ and elements $f \in L_p(0, 1)$, $g \in L_q(0, 1)$ ($p^{-1} + q^{-1} = 1$), we define

$$\Phi(\alpha) = \langle e^{\pi\alpha^2} \varphi(T_\alpha) f, g \rangle, \quad \alpha \in \mathbf{C}.$$

Since $|e^{\pi\alpha^2}| \leq e^{\pi\beta^2}$, and since $\varphi(T_\alpha)$ is a polynomial in α (with operator coefficients), we have $|\Phi(\alpha)| = O(e^{\varepsilon|\gamma|})$ (for $|\gamma| \rightarrow \infty$) in the strip $-n \leq \beta \leq \beta_0$, for any $\varepsilon > 0$.

By Lemma 2 and the estimate $\|J^{i\gamma}\| \leq e^{\pi|\gamma|/2}$, we have:

$$\begin{aligned} |\Phi(\beta + i\gamma)| &\leq \exp \pi(\beta^2 - \gamma^2 + |\gamma|) \cdot \|f\|_p \|g\|_q \|\varphi(T_\beta)\| \\ &\leq \exp \pi(\beta^2 + 1/4) \cdot \|f\|_p \|g\|_q \|\varphi(T_\beta)\| \end{aligned}$$

for all $\beta, \gamma \in \mathbf{R}$.

Since T_{-n} and T_{β_0} are of class C^n (by Lemma 5, the hypothesis and Lemma 2), there exists a constant K (depending only on n, β_0 and p) such that

$$\|\varphi(T_{-n})\| \leq K \|\varphi\|_n \quad \text{and} \quad \|\varphi(T_{\beta_0})\| \leq K \|\varphi\|_n.$$

Hence

$$|\Phi(-n + i\gamma)| \leq M \|f\|_p \|g\|_q \|\varphi\|_n$$

and

$$|\Phi(\beta_0 + i\gamma)| \leq M \|f\|_p \|g\|_q \|\varphi\|_n$$

for all real γ , where $M = K \exp \pi(\delta^2 + 1/4)$ and $\delta = \max(n, \beta_0)$. By the Phragmén-Lindelöf principle (cf. [9, p. 180]), it follows that $|\Phi(\alpha)| \leq M \|f\|_p \|g\|_q \|\varphi\|_n$ for $-n \leq \operatorname{Re} \alpha \leq \beta_0$. Hence, for such α ,

$$\|\varphi(T_\alpha)\| \leq M \exp \pi(\gamma^2 - \beta^2) \|\varphi\|_n,$$

and the lemma follows.

The next two theorems give explicitly the C^n -operational calculus for T_α ($|\operatorname{Re} \alpha| \leq n$) acting in $L_p(0, 1)$, with $1 < p < \infty$.

THEOREM 8. *Let n be a nonnegative integer. Then for $0 \leq \operatorname{Re} \alpha \leq n$ and $\varphi \in C^n[0, 1]$ the range of J^α (i.e., the domain of $J^{-\alpha}$) is invariant under $M(\varphi)$, and the C^n -operational calculus for T_α is given by*

$$T_\alpha(\varphi) = J^{-\alpha} M(\varphi) J^\alpha, \quad \varphi \in C^n[0, 1].$$

Proof. By Lemma 1 (3),

$$(9) \quad \varphi(M) J^\alpha = J^\alpha \varphi(T_\alpha)$$

for any polynomial φ . In particular,

$$(10) \quad \varphi(M) \mathcal{R}(J^\alpha) \subset \mathcal{R}(J^\alpha), \quad \varphi = \text{a polynomial.}$$

Let $\varphi \in C^n[0, 1]$, and choose polynomials φ_k which converge to φ in $C^n[0, 1]$. In particular, $\varphi_k \rightarrow \varphi$ uniformly in $[0, 1]$, and therefore

$$(11) \quad \mathcal{D}_{-\alpha} \ni \varphi_k J^\alpha g \rightarrow \varphi J^\alpha g$$

in $L_p(0, 1)$, for any $g \in L_p(0, 1)$. By (9), we have:

$$(12) \quad J^{-\alpha}(\varphi_k J^\alpha g) = \varphi_k(T_\alpha) = T_\alpha(\varphi_k) \rightarrow T_\alpha(\varphi)$$

in the uniform operator topology, since $T_\alpha \in (C^n)$ by Theorem 6 (for $|\operatorname{Re} \alpha| \leq n$ and $1 < p < \infty$) and $\varphi_k \rightarrow \varphi$ in $C^n[0, 1]$. Since $J^{-\alpha}$ is a closed operator, it follows from (11) and (12) that $\varphi J^\alpha g \in \mathcal{D}_{-\alpha}$ and

$$J^{-\alpha}(\varphi J^\alpha g) = T_\alpha(\varphi), \quad \text{Q.E.D.}$$

We consider next the range $-n \leq \operatorname{Re} \alpha < 0$ ($n = 1, 2, \dots$). Note that $\operatorname{Re}(\alpha + n) \geq 0$. The notation $T_{-n}(\varphi)$ is that of Lemma 5.

THEOREM 9. *Let n be a nonnegative integer. Then for $-n \leq \operatorname{Re} \alpha < 0$ and $\varphi \in C^n[0, 1]$, the range of $J^{\alpha+n}$ (i.e., $\mathcal{D}_{-(\alpha+n)}$) is invariant under $T_{-n}(\varphi)$, and the C^n -operational calculus for T_α is given by*

$$T_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}, \quad \varphi \in C^n[0, 1].$$

Proof. By (10), $\mathcal{R}(J^{\alpha+n})$ is invariant for $M(\varphi)$ for any polynomial φ ; it is therefore invariant for the operator

$$T_{-n}(\varphi) = \sum_{j=0}^n \binom{n}{j} (-1)^j J^j M(\varphi^{(j)}), \quad \varphi = \text{a polynomial.}$$

Thus, for any polynomial φ , the operator

$$S_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}$$

is everywhere defined. Being closed, it is *continuous* by the Closed Graph Theorem.

Let $g \in \mathcal{D}_\alpha = \mathcal{R}(J^{-\alpha})$, say $g = J^{-\alpha} h$ with $h \in L_p(0, 1)$. By Lemma 1,

$$\begin{aligned} S_\alpha(\varphi)g &= J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n} h = J^{-(\alpha+n)} J^n \varphi(M) h \\ &= J^{-\alpha} \varphi(M) h = \varphi(T_\alpha) J^{-\alpha} h \\ &= \varphi(T_\alpha) g \end{aligned}$$

for any polynomial φ .

This shows that the continuous operators $S_\alpha(\varphi)$ and $\varphi(T_\alpha) = T_\alpha(\varphi)$ coincide on the dense subset $\mathcal{D}_\alpha$ of $L_p(0, 1)$. Thus, for every polynomial φ ,

$$(13) \quad T_\alpha(\varphi) = J^{-(\alpha+n)} T_{-n}(\varphi) J^{\alpha+n}.$$

Let $\varphi \in C^n[0, 1]$, and let φ_k be polynomials converging to φ in $C^n[0, 1]$. Since T_α and T_{-n} are of class C^n (by Theorem 6), we have (in the uniform operator topology):

$$(14) \quad T_\alpha(\varphi_k) \rightarrow T_\alpha(\varphi); \quad T_{-n}(\varphi_k) \rightarrow T_{-n}(\varphi)$$

for any $k \rightarrow \infty$.

Fix $g \in L_p(0, 1)$. Then $T_{-n}(\varphi_k)J^{\alpha+n}g \in \mathcal{D}_{-(\alpha+n)}$ (cf. beginning of the proof) and

$$T_{-n}(\varphi_k)J^{\alpha+n}g \rightarrow T_{-n}(\varphi)J^{\alpha+n}g$$

for $k \rightarrow \infty$ (by (14)). Moreover

$$J^{-(\alpha+n)}[T_{-n}(\varphi_k)J^{\alpha+n}g] = T_\alpha(\varphi_k)g \rightarrow T_\alpha(\varphi)g$$

by (13) and (14). Since $J^{-(\alpha+n)}$ is closed, it follows that $T_{-n}(\varphi)J^{\alpha+n}g \in \mathcal{D}_{-(\alpha+n)} = \mathcal{R}(J^{\alpha+n})$ and $J^{-(\alpha+n)}T_{-n}(\varphi)J^{\alpha+n}g = T_\alpha(\varphi)g$, Q.E.D.

3. The local C^k -operational calculus. Note first that the results of §2 are also relevant to the case $p=1$, in the sense of the *local* C^k -operational calculus. Let $L = \bigcup_{1 < p < \infty} L_p(0, 1)$. This is a dense linear manifold in $L_1(0, 1)$, which is invariant under T_α for all $\alpha \in \mathbb{C}$. Let $n \geq 0$ be an integer, and let $|\operatorname{Re} \alpha| \leq n$. If $f \in L$, say $f \in L_p(0, 1)$ for some $1 < p < \infty$, then the mapping $\varphi \in C^n[0, 1] \rightarrow T_\alpha(\varphi)f \in L_1(0, 1)$ is continuous ($T_\alpha(\cdot)$ is given by Theorems 8 and 9) because $\|T_\alpha(\varphi)f\|_1 \leq \|T_\alpha(\varphi)f\|_p \leq \|T_\alpha(\cdot)\|_p \|f\|_p \|\varphi\|_n$, where $\|T_\alpha(\cdot)\|_p$ denotes the norm of the C^n -operational calculus for T_α acting in $L_p(0, 1)$. Thus $W_n(T_\alpha; 1) \supset L$ for $|\operatorname{Re} \alpha| \leq n$, and the C^n -operational calculus for T_α on L is provided by Theorems 8 and 9.

In the next two theorems, we study the manifolds $W_k(T_\alpha; p)$ for $k < |\operatorname{Re} \alpha|$ (they coincide with the whole space for $k \geq |\operatorname{Re} \alpha|$, at least for $1 < p < \infty$, by §2). It turns out that the situation is totally different in the right and left half-planes.

THEOREM 10. For $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha < 0$ and $1 < p < \infty$,

$$W_k(T_\alpha; p) \supset \mathcal{D}_{\alpha+k}, \quad 0 \leq k < |\operatorname{Re} \alpha|,$$

and the C^k -operational calculus for T_α on $\mathcal{D}_{\alpha+k}$ is given by

$$(15) \quad T_\alpha(\varphi) = J^{-(\alpha+k)}T_{-k}(\varphi)J^{\alpha+k}, \quad \varphi \in C^k[0, 1],$$

(where $T_{-k}(\varphi)$ is defined in Lemma 5).

Proof. Fix p , α and k as in the theorem, and define $T_\alpha(\cdot)$ by (15). One verifies easily that the mapping $\varphi \rightarrow T_\alpha(\varphi)$ is an algebra homomorphism of $C^k[0, 1]$ into $\mathcal{T}(\mathcal{D}_{\alpha+k})$ which sends the functions $\varphi(x) \equiv 1$ and $\varphi(x) \equiv x$ respectively to $I|_{\mathcal{D}_{\alpha+k}}$ and $T_\alpha|_{\mathcal{D}_{\alpha+k}}$ (cf. Lemma 1). Moreover, for each $g \in \mathcal{D}_{\alpha+k}$, the mapping $\varphi \rightarrow T_\alpha(\varphi)g$ of $C^k[0, 1]$ into $L_p(0, 1)$ is continuous, since

$$T_\alpha(\varphi)g = J^{-\alpha-k}T_{-k}(\varphi)h$$

for $g = J^{-\alpha-k}h$ with $h \in L_p(0, 1)$. Q.E.D.

In particular, $W_k(T_\alpha; p)$ is dense in $L_p(0, 1)$ for $\operatorname{Re} \alpha < 0$ and $k \geq 0$ arbitrary. For $\operatorname{Re} \alpha \geq 1$, we get the "other" extreme.

THEOREM 11. *For $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 1$ and $1 < p < \infty$, $W_k(T_\alpha; p) = (0)$ if $k < [\operatorname{Re} \alpha]$. The same is true for $p = 1$ if α is an integer.*

Proof. If α is an integer, this is a trivial consequence of Lemma 3 and Leibnitz' formula.

Suppose then that $1 < p < \infty$, and that $f \in W_k(T_\alpha; p)$ for some fixed $k < m = [\operatorname{Re} \alpha]$. As in the proof of Lemma 7, we apply the Phragmén-Lindelöf principle in the strip $0 \leq \operatorname{Re} \zeta \leq \operatorname{Re} \alpha$ to the function $\Phi(\zeta) = \langle e^{\pi \tau^2} \varphi(T_\zeta) f, g \rangle$ where φ is a polynomial and $g \in L_q(0, 1)$ (both fixed). We then obtain that $f \in W_k(T_\zeta; p)$ for all ζ in the strip, hence in particular for $\zeta = m$. Since $k < m$, we conclude that f is the null function.

4. Similarity and spectrality.

LEMMA 12. *Let $\alpha \in \mathbb{C}$ and $1 < p < \infty$. Then every $s \in [0, 1) = \sigma(T_\alpha) \setminus \{1\}$ is an eigenvalue of $T_{-\alpha}$ for $\operatorname{Re} \alpha \geq 1$ ($\operatorname{Re} \alpha > 1$ or $\alpha = 1$ if $p = 1$), while T_α has no eigenvalue for $\operatorname{Re} \alpha \geq 0$ ($\operatorname{Re} \alpha > 0$ or $\alpha = 0$ if $p = 1$).*

Proof. Let C_s denote the characteristic function of the interval $[s, 1]$, $0 \leq s < 1$. One verifies easily that C_s is an eigenvector of T_{-1} corresponding to the eigenvalue s (for $1 \leq p < \infty$).

By Lemma 1, (3),

$$T_{-\alpha} J^{\alpha-1} C_s = J^{\alpha-1} T_{-1} C_s = s J^{\alpha-1} C_s,$$

i.e. $J^{\alpha-1} C_s$ (which is in $L_p(0, 1)$ for α as in the first statement of the lemma) is an eigenvector of $T_{-\alpha}$ corresponding to the eigenvalue s .

Next, suppose $T_\alpha g = \lambda g$ for $g \in L_p(0, 1)$ and $\lambda \in \mathbb{C}$. If $\operatorname{Re} \alpha \geq 0$ ($\operatorname{Re} \alpha > 0$ or $\alpha = 0$ if $p = 1$), we may apply J^α on both sides of this equation; by Lemma 1, (3), we obtain

$$M J^\alpha g = \lambda J^\alpha g.$$

Since M has no eigenvector $\neq 0$ and J^α is one-one, it follows that g is the zero element.

Let $\alpha, \beta \in \mathbb{C}$. By Lemma 2, T_α and T_β are similar if $\operatorname{Re} \alpha = \operatorname{Re} \beta$ (and $1 < p < \infty$). On the other hand, since the C^k -classification and the point spectrum are similarity invariants, it follows from Lemmas 3, 5 and 12 that T_α and T_β are *not* similar if α and β are distinct integers (for $1 \leq p < \infty$).

Conjecture. For $1 < p < \infty$ and $\alpha, \beta \in \mathbb{C}$, T_α and T_β are similar if and only if $\operatorname{Re} \alpha = \operatorname{Re} \beta$. (By Lemma 2, it would suffice to verify that T_α and T_β are not similar if α and β are distinct *real* numbers.)

PROPOSITION 13. *Let $\alpha, \beta \in \mathbb{C}$ and $1 < p < \infty$. Then T_α and T_β (acting in $L_p(0, 1)$) are not similar if $[\operatorname{Re} \alpha] \neq [\operatorname{Re} \beta]$.*

Proof. Assume, without loss of generality, that $\operatorname{Re} \alpha < \operatorname{Re} \beta$. If either $0 \leq \operatorname{Re} \alpha$ or $\operatorname{Re} \beta \leq 0$, this follows from Theorem 6 and the similarity invariance of the C^k -classification. If $\operatorname{Re} \alpha < 0 < 1 \leq \operatorname{Re} \beta$, $W_0(T_\alpha)$ is dense in $L_p(0, 1)$ (Theorem 10) while $W_0(T_\beta) = (0)$ (Theorem 11). Thus T_α and T_β are not similar.

If $\operatorname{Re} \alpha \leq -1 < 0 < \operatorname{Re} \beta$, every $s \in [0, 1)$ is an eigenvalue of T_α , while T_β has no eigenvalue (Lemma 12), and the conclusion follows from the similarity invariance of the point spectrum. Q.E.D.

We next discuss the spectrality of T_α in Dunford's sense [1].

LEMMA 14. *Let T be a bounded spectral operator with real spectrum, acting in the Banach space X . Let $T = S + N$ be its canonical decomposition (cf. [1]). Then:*

(a) *If $W_k(T)$ is dense in X for some integer $k \geq 0$, then T is of finite type $\leq k$ (i.e., $N^{k+1} = 0$).*

(b) *If T is of finite type k , then $W_j(T) \neq (0)$ for all $j \geq 0$; in fact, $W_j(T) \supset \mathcal{R}(N^{k-j})$ for $j = 0, \dots, k-1$, and trivially $W_j(T) = X$ for $j \geq k$.*

Proof. Fix a compact interval $\Delta \supset \sigma(T)$. Let $S(\cdot)$ be the C -operational calculus for S (defined on $C(\Delta)$), and let $\|S(\cdot)\|$ be its norm.

(a) Let $x \in W_k(T)$. The function $e^{izN}x$ ($z \in C$) is entire of order one and minimal type (since N is a quasi-nilpotent operator). For $z = t \in \mathbf{R}$, we have:

$$\|e^{itN}x\| = \|e^{-itS}e^{itT}x\| \leq \|S(\cdot)\| \|e^{itT}x\| \leq \|S(\cdot)\| |x|_k \|\varphi_t\|_{k,\Delta},$$

where $\varphi_t(s) = e^{its}$, $t, s \in \mathbf{R}$.

Thus $\|e^{itN}x\| = O(|t|^k)$, and therefore $e^{izN}x$ is a polynomial of order $\leq k$ by Theorem 3.13.8 in [3]. Hence $N^{k+1}x = 0$ for each $x \in W_k(T)$, and it follows that $N^{k+1} = 0$ since $W_k(T)$ is dense in X .

(b) We have $N^{k+1} = 0$ and $N^k \neq 0$. The analytic operational calculus for T takes the form (cf. [1]):

$$T(\varphi) = \sum_{m=0}^k S(\varphi^{(m)}) N^m / m!$$

If $x \in \mathcal{R}(N^{k-j})$, say $x = N^{k-j}y$ with $y \in X$ ($0 \leq j < k$), then

$$T(\varphi)x = \sum_{m=0}^j S(\varphi^{(m)}) N^m y / m!$$

In particular, $\|p(T)x\| \leq \|S(\cdot)\| \cdot \max_{0 \leq m \leq j} \|N^m y\| \|p\|_{j,\Delta}$ for any polynomial p , i.e. $x \in W_j(T)$, Q.E.D.

For simplicity, we state the following result for $1 < p < \infty$, although part of the conclusion remains valid for $p = 1$.

PROPOSITION 15. *Let $1 < p < \infty$. Then T_α is spectral for $\operatorname{Re} \alpha = 0$, and is not spectral for $|\operatorname{Re} \alpha| \geq 1$.*

Proof. The first statement is a trivial corollary of Lemma 2.

By Theorem 6, T_α is of class C^n if $n \geq |\operatorname{Re} \alpha|$. Thus, if T_α were spectral, it should be of finite type by Lemma 14(a). In particular, its point spectrum should be at most countable by [2, Theorem 1, p. 56]. But this contradicts Lemma 12 if $\operatorname{Re} \alpha \leq -1$. Also all $W_j(T_\alpha)$ ($j \geq 0$) should be nontrivial by Lemma 14(b), contradicting Theorem 11 if $\operatorname{Re} \alpha \geq 1$. Thus T_α is not spectral for $|\operatorname{Re} \alpha| \geq 1$.

5. Remarks. It is interesting to regard the results of this paper as statements about the operators $\alpha^{-1}T_\alpha = J + \alpha^{-1}M$ ($0 \neq \alpha \in \mathbb{C}$), which are perturbations of J by a scalar operator of arbitrarily small norm. Thus, if α and β are nonzero complex numbers, the following assertions can be made (for $1 < p < \infty$):

(a) If $[|\operatorname{Re} \alpha|] \neq [|\operatorname{Re} \beta|]$, $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ belong to distinct (C^k) -classes, although they differ only by the scalar operator $(\alpha^{-1} - \beta^{-1})M$, which is of arbitrarily small norm. This shows that the commutativity hypothesis in [5, Corollary 5.6] cannot be replaced by a restriction on the norm of the perturbing scalar operator.

(b) The perturbations $J - \alpha^{-1}M$ and $J + \alpha^{-1}M$ have respectively a dense and a trivial semisimplicity manifold, a "pure" point spectrum (up to the right end point of the spectrum $[0, \alpha^{-1}]$) and no point spectrum.

(c) The perturbations $J + \alpha^{-1}M$ and $J + \beta^{-1}M$ are not similar if $[\operatorname{Re} \alpha] \neq [\operatorname{Re} \beta]$.

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