ON THE FRACTIONAL DIFFERENTIATION OF A FUNCTION OF SEVERAL VARIABLES

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1. In [5] a characterization for fractional differentiation of a function of a real variable is given. Here, the results are extended to the case of a function of several variables.

Before we state these results we must review some definitions. By $x, y, t, \dots$ we denote points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n), \dots$ of n -dimensional Euclidean space. We will consider integrable functions, $f: E_n \rightarrow R$, where R is the set of real numbers. We denote the measure of a measurable set E by $|E|$. The symbols $x+y$ and λx , where λ is a scalar, have the usual meaning. By $|x|$ we mean $(\sum_{i=1}^n x_i^2)^{1/2}$, by $\langle x, y \rangle$ we mean $\sum_{i=1}^n x_i y_i$, and if $j=(j_1, j_2, \dots, j_n)$ where the j_i are integers we use the notation $|j|$ to mean $j_1 + j_2 + \dots + j_n$. By x^j we mean $x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$. Σ denotes the unit sphere of E_n , σ, μ denote elements of Σ , and $d\sigma$ the usual area element of Σ .

We will use the symbol C , sometimes with subscripts and sometimes without, for an absolute constant or a constant dependent only on the dimension and the parameters of the problem.

By C_0^k we denote the class of functions with compact support and k continuous derivatives, and we write $(\partial/\partial x)^j$ for $(\partial^{j_1}/\partial x_1^{j_1})(\partial^{j_2}/\partial x_2^{j_2}) \dots (\partial^{j_n}/\partial x_n^{j_n})$ and $(\partial/\partial x)^j f = f_j$.

For a function f , we denote its β th integral by f_β and write

$$f_\beta(x) = \int_{E_n} \frac{f(t) dt}{|x-t|^{n-\beta}} \quad (0 < \beta < 1).$$

Whenever the region of integration for an integral is E_n we may simply write $\int$ for $\int_{E_n}$.

A function defined in a neighborhood of a point x_0 is said to have a k derivative if

$$(1) \quad f(x_0 + t) = P_{x_0}(t) + R_{x_0}(t)$$

where $P_{x_0}(t)$ is a polynomial in the variable t of degree $\leq k$ and $R(t) = R_{x_0}(t) = O(|t|^k)$ as $|t| \rightarrow 0$. For $1 \leq p < \infty$, f is said to have a derivative of order k in the L^p sense if $\{\rho^{-n} \int_{|t| \leq \rho} |R(t)|^p dt\}^{1/p} = o(\rho^k)$ as $\rho \rightarrow 0$.

For an integer $k \geq 0$, f is said to have a derivative of order α at x_0 , $k < \alpha < k+1$, if f_β has a $k+1$ derivative at x_0 ($\alpha + \beta = k+1$). f is said to have an α derivative at x_0 in the L^p sense if f_β has a $k+1$ derivative at x_0 in the L^p sense.

Furthermore, with $R(t)$ defined as in (1), f is said to satisfy Λ_α at x_0 if $R(t) = O(|t|^\alpha)$ as $|t| \rightarrow 0$. In this case we write $f \in \Lambda_\alpha$. The corresponding L^p definition, $\Lambda_{\alpha,p}^p$, is satisfied if

$$(2) \quad \left\{ \frac{1}{\rho^n} \int_{|t| < \rho} |R(t)|^p dt \right\}^{1/p} = O(\rho^\alpha).$$

Finally f is said to satisfy the condition N_α^p if

$$(3) \quad \int_{|t| \leq \rho} \frac{|R(t)|^p}{|t|^{n+p\alpha}} dt < \infty \quad \text{for some } \rho > 0.$$

The main theorems are as follows.

THEOREM 1. *Suppose f satisfies the condition Λ_α at every point of a measurable set E , E has positive measure. Then the necessary and sufficient condition for f to have a derivative of order α almost everywhere in E is that f satisfy the condition N_α^2 almost everywhere in E .*

The condition N_α^2 is the characterizing notion for the fractional differentiability of functions, and if Λ_α is relaxed to Λ_α^2 we have:

THEOREM 2. *f satisfies the condition N_α^2 almost everywhere in the set E if and only if f satisfies the condition Λ_α^2 and f has an α derivative in the L^2 sense almost everywhere in E .*

That N_α^2 is the characterizing notion for differentiability is perhaps underlined by:

THEOREM 2'. *f has an α derivative in the sense of L^p ($2 \leq p < \infty$) and satisfies the condition Λ_α^p almost everywhere in a set E if and only if f satisfies both N_α^2 and N_α^p almost everywhere in the set E .*

The proofs of these results rely heavily on some theorems first proved in [4] for the one-dimensional case and then recently extended to the n -dimensional case in [3]. To state these results we make some further remarks.

Suppose that f has a $k-1$ derivative at x_0 in either the ordinary sense or the L^p sense and the polynomial for this case as corresponds to (1) is $P_{x_0}(t)$. Write $\Delta_{x_0}^k(t) = R_{x_0}(t) + (-1)^{k-1} R_{x_0}(-t)$. f is said to satisfy the condition Λ_k^* at x_0 if $\Delta_{x_0}^k(t) = O\{|t|^k\}$ as $|t| \rightarrow 0$ and it is said to satisfy the condition N_k^p if there is a $\rho > 0$ such that

$$\int_{|t| \leq \rho} \frac{|\Delta_{x_0}^p(t)|^p}{|t|^{n+p k}} dt < \infty \quad (1 \leq p < \infty).$$

The results we speak of are given below.

THEOREM A. *Suppose f satisfies the condition Λ_k^* at every point of a set E . Then f has a k derivative at almost every x in E if and only if f satisfies N_k^2 almost everywhere in E .*

THEOREM B. *The necessary and sufficient condition for f to have a k derivative in the L^p sense ($2 \leq p < \infty$) almost everywhere in a set E is that f satisfies the conditions N_k^p, N_k^2 almost everywhere in E .*

Theorem B is not proved in [3] but this result can be obtained in much the same way as it is done in the one-dimensional case in [4].

The proof of Theorem 2' does not differ from the proof of Theorem 2 and we will only fully prove Theorem 1 and Theorem 2. The existence of the α derivative at x_0 of f depends on its local properties and hence altering f outside a neighborhood of x_0 does not change the existence of the α derivative.

2. In addition to the above results several lemmas are used in the proofs.

LEMMA 1. (See [2, p. 148] for a similar case.) *Let $f(x, y)$ be a measurable function on the product of two measure spaces $M_1 \times M_2$ with the measures μ_1 and μ_2 respectively. Then using the usual notation for L^p norms, $1 \leq p < \infty$, $\| \int f(\cdot, y) d\mu_2 \|_p \leq \int \| f(\cdot, y) \|_p d\mu_2$ where the L^p norm is taken in the variable x .*

LEMMA 2. *Suppose that k is an integer, $0 \leq k < \alpha < k+1$, and $\alpha + \beta = k+1$. Let g be defined by*

$$g(x) = \int_{E_n} \frac{G_\sigma(x+t)}{|t|^{n+\beta-1}} dt$$

where $\sigma = t/|t|$, G_σ is the direction derivative of G in the direction of σ , and $G \in C_0^{k+1}$. Then g satisfies the condition Λ_α uniformly and g satisfies N_α^2 and N_α^p for almost all x in E_n , $2 \leq p < \infty$.

Proof. Let μ be a unit vector in E_n and by $g_{\mu^k}(x)$ denote the direction derivative of g of order k in the direction of μ . Consider

$$\begin{aligned} g_{\mu^k}(x+\mu\tau) - g_{\mu^k}(x) &= \Delta_g^k(x, \mu, \tau) = \int_{E_n} G_{\sigma, \mu^k}(x+t+\mu\tau) - G_{\sigma, \mu^k}(x+t) \frac{dt}{|t|^{n+\beta-1}} \\ &= \int_{\Sigma} d\sigma \int_0^\infty (G_{\sigma, \mu^k}(x+r\sigma+\tau\mu) - G_{\sigma, \mu^k}(x+r\sigma)) \frac{dr}{r^\beta} \\ &= \int_{\Sigma} d\sigma \int_0^\infty (G_{\sigma, \mu^k}(x+r\sigma+\tau\mu) - G_{\sigma, \mu^k}(x+\tau\sigma+r\sigma)) \frac{dr}{r^\beta} \\ &\quad + \int_{\Sigma} d\sigma \int_0^\infty (G_{\sigma, \mu^k}(x+\tau\sigma+r\sigma) - G_{\sigma, \mu^k}(x+r\sigma)) \frac{dr}{r^\beta} \\ &= I_1 + I_2. \\ |I_2| &= \left| \int_{\Sigma} d\sigma \left[\int_0^\tau G_{\sigma, \mu^k}(x+s\sigma) \frac{ds}{s^\beta} + \int_\tau^\infty G_{\sigma, \mu^k}(x+s\sigma) \left\{ \frac{1}{|s-\tau|^\beta} - \frac{1}{s^\beta} \right\} ds \right] \right| \\ &\leq |\Sigma| \|G_{\sigma, \mu^k}\|_\infty \left| \int_0^\infty \frac{ds}{|s-\tau|^\beta - s^\beta} \right| \leq C |\tau|^{1-\beta}; \\ |I_1| &= \int_{\Sigma} d\sigma \left\{ \int_\tau^\infty + \int_0^\tau \Delta^k G_\sigma(x, \mu, \tau, r) \frac{dr}{r^\beta} \right\} \end{aligned}$$

where $\Delta^k G_\sigma = \Delta^k G_\sigma(x, \mu, \tau, r)$ is the integrand of I_1 . The first of these integrals can be estimated by first integrating by parts and noting that G_{μ^k} is Lipschitz. The second integral can be estimated by replacing $\Delta^k G_\sigma$ by its maximum and integrating. Hence $|I_2| \leq C|\tau|^{1-\beta}$. It only remains to apply Taylor's theorem to g to see that g is in Λ_α , with the polynomial in (1) for g being its Taylor's development of order k .

Next, to show g satisfies the condition N_α^2 , define the function $\omega_{x,\mu}(t)$ by

$$\omega_{x,\mu}(t)|t|^{1-\beta} = |g_{\mu^k}(x+t) - g_{\mu^k}(x)|$$

with $\mu = t/|t|$. We first show

$$(4) \quad \int_{E_n} \frac{|\omega_{x,\mu}(t)|^2}{|t|^n} dt < \infty.$$

To do this integrate (4) with respect to x . After an application of Parseval's equality we have

$$(5) \quad \frac{1}{(2\pi)^n} \int_{E_n} \left\{ \int_{E_n} \frac{|\omega_{x,\mu}(t)|^2}{|t|^n} dx \right\} = \int_{E_n} |t|^{2(\beta-1)-n} dt \int_{E_n} 2 \sin^2 \frac{\langle y, t \rangle}{2} |g_{\mu^k}(y)|^2 dy$$

where $g_{\mu^k}(y) = \int_{E_n} g_{\mu^k}(x) \cdot e^{-i\langle x, y \rangle} dx$. The integral

$$\int_{E_n} |t|^{2(\beta-1)-n} |y|^{2(1-\beta)} 2 \sin^2 \frac{\langle y, t \rangle}{2} dt \leq C$$

where C is a constant depending on only β and n . The remaining integral is equal to the square of

$$\begin{aligned} & \left\{ \int_{E_n} \left[\int_{\Sigma} d\sigma \int_0^\infty e^{-i\langle y, r\sigma \rangle} G_{\sigma, \mu^k}(y) \cdot |y|^{\beta-1} \frac{dr}{r^\beta} \right]^2 dy \right\}^{1/2} \\ &= \left\{ \int_{E_n} \left[\int_{\Sigma} G_{\sigma, \mu^k}(y) \cdot |y|^{\beta-1} d\sigma \int_0^\infty e^{-i\langle y, r\sigma \rangle} \frac{dr}{r^\beta} \right]^2 dy \right\}^{1/2}. \end{aligned}$$

But $\left| \int_0^\infty e^{-i\langle y, r\sigma \rangle} r^{-\beta} dr \right| = \text{constant} \cdot |\langle y, \sigma \rangle|^{1-\beta} < C|y|^{1-\beta}$. Using this and applying Lemma 1 we have (5) is $\leq C \int_{\Sigma} \left\{ \int_{E_n} |G_{\sigma, \mu^k}(y)|^2 dy \right\}^{1/2} d\sigma \leq C|\Sigma| \|G_{\sigma, \mu^k}\|_2 < \infty$.

Now let x_0 be a point where (4) is finite. Writing $g(x_0+t) = P_{x_0}(t) + R_{x_0}(t)$ where $P_{x_0}(t)$ is the usual polynomial in the development of g we see that

$$|R_{x_0}(t)| = \frac{1}{(k-1)!} \left| \int_0^{|t|} (|t|-u)^{k-1} \omega_{x_0, \sigma}(u\sigma) |u|^{1-\beta} du \right|$$

where $\sigma = t/|t|$. Hence

$$\begin{aligned} \int_{E_n} \frac{|R_{x_0}(t)|^2}{|t|^{n+2\alpha}} dt &= \int_{\Sigma} d\sigma \int_0^\infty \frac{|R_{x_0}(r\sigma)|^2}{r^{1+2\alpha}} dr \\ &\leq C \int_{\Sigma} d\sigma \int_0^\infty \left\{ \int_0^r r^{k-\beta} |\omega_{x_0, \sigma}(u\sigma)| du \right\}^2 \frac{dr}{r^{1+2\alpha}} \\ &= C \int_{\Sigma} d\sigma \int_0^\infty \left\{ \frac{1}{r} \int_0^r |\omega_{x_0, \sigma}(u\sigma)| du \right\}^2 \frac{dr}{r}. \end{aligned}$$

After an application of Hölder's inequality and a change of order of integration of the inner two integrals this is less than or equal to a constant times

$$\int_{\Sigma} d\sigma \int_0^{\infty} |\omega_{x_0, \sigma}(u\sigma)|^2 \left\{ \int_u^{\infty} \frac{dr}{r^2} \right\} du = \int_{E_n} \frac{|\omega_{x_0, \sigma}(t)|^2}{t^n} dt < \infty.$$

For the case $2 \leq p < \infty$ let A be an upper bound for $\omega_{x_0}(t)$. Then one has

$$\int_{E_n} \frac{|\omega_{x_0}(t)|^p}{|t|^n} dt \leq \frac{A^p}{A^2} \int_{E_n} \frac{|\omega_{x_0}(t)|^2}{|t|^n} dt < \infty.$$

LEMMA 3. Suppose that $f \in L^2(E_n)$ and has finite support, and that $F(x) = f_{\beta}(x)$, $0 < \beta < 1$. Then

$$\frac{1}{A_{\beta}} \int_{|t| \geq \varepsilon} \frac{F(x+t) - F(x)}{|t|^{n+\beta}} dt$$

converges to $f(x)$ in the L^2 norm as $\varepsilon \rightarrow 0$ for a suitable choice of A_{β} . A_{β} is a nonzero constant depending only on β and n .

Proof. The Fourier transform of $F(x+t) - F(x)$ is $B_{\beta} \hat{f}(x) \{e^{it \cdot x} - 1\} |x|^{-\beta}$ where B_{β} is the constant such that $\{1/|x|^{n-\beta}\}^{\wedge} = B_{\beta} |x|^{-\beta}$. Therefore the Fourier transform of $\int_{|t| \geq \varepsilon} ((F(x+t) - F(x))/|t|^{n+\beta}) dt = \hat{f}(x) M_{\varepsilon}(x)$ where

$$M_{\varepsilon}(x) = B_{\beta} |x|^{-\beta} \int_{|t| \geq \varepsilon} \frac{e^{it \cdot x} - 1}{|t|^{n+\beta}} dt.$$

It is easy to see that $M_{\varepsilon}(x)$ is bounded in x and ε uniformly for $x \neq 0$ and that the limit of $M_{\varepsilon}(x)$ as $\varepsilon \rightarrow 0$ is nonzero. An application of Plancherel's theorem gives the required result.

LEMMA 4. (See [1, p. 184].) Let P be a closed set and $U = \{x : d(x, P) < 1\}$. There exists a covering of $U - P$ by nonoverlapping cubes K with the property that $\text{diam } K \leq d(P, K) \leq 3 \text{ diam } K$. $d(x, P)$ and $d(P, K)$ represent the distance from x to P and from P to K respectively and $\text{diam } K$ represents the diameter of the cube K .

Here the conclusion has been slightly altered from that of [1] but it is not essentially different in the proof.

LEMMA 5. (See [6, p. 130].) Let P be a closed set and U be as above. Set $\Delta(x) = d(x, P)$ for x in U and zero otherwise. Then for $\lambda > 0$ we have

$$\int \frac{\Delta^{\lambda}(x_0 + t)}{|t|^{n+\lambda}} dt < \infty$$

for almost all $x_0 \in P$.

LEMMA 5'. (See [1, p. 189].) Suppose that $\lambda > 0$ and

$$\frac{1}{h^n} \int_{|t| < h} H(x_0 + t) dt \leq Ah^{\lambda}, \quad 0 < h < \infty,$$

for every x_0 in P . Then $\int (H(x_0 + t)/|t|^{n+\lambda}) dt < \infty$ for almost all x_0 in P .

LEMMA 5". (See [6, p. 131].) Let K_μ be the sets of the cover in Lemma 4. Then

$$\sum_{\mu} \frac{(\text{diam } K_{\mu})^{n+\lambda}}{|x_0 - x_{\mu}|^{n+\lambda}} < \infty \quad \text{for almost all } x_0 \text{ in } P.$$

Here x_{μ} is a point of P such that $d(x_{\mu}, K) = d(P, K)$ and $\lambda > 0$.

LEMMA 6. (See [1, p. 183].) Suppose $\alpha > 0$, with $k < \alpha < k+1$, and that h has the development $h(x_0 + t) = \sum_{|j|=0}^k h_j(x_0)t^j + R_{x_0}(t)$ for each x_0 in P , with $(1/\rho^n) \int_{|t| \leq \rho} |R_{x_0}(t)|^p dt \leq A\rho^\alpha$ for $0 < \rho \leq \delta$. Then with x_0 in P and $x_0 + t$ in P ,

$$h_e(x_0 + t) = \sum_{|j|=0}^{k-|e|} \frac{t^j}{j!} h_{j+e}(x_0) + O(|t|^{\alpha+|e|})$$

for $|e| = 0, 1, \dots, k$; O is uniform for $x_0, x_0 + t$ in P , and $|t| \leq \delta$.

LEMMA 7. (See [1, p. 189].) Suppose that F has a $k+1$ derivative in the L^2 sense on a set E , $|E| > 0$, uniformly. Also assume F has finite support. Then there exists a function $G \in C_0^{k+1}$ such that $F = G + H$ where $H(x_0) = 0$ for x_0 in E and

$$\frac{1}{\rho^n} \int_{|t| < \rho} |H(x_0 + t)|^2 dt = o(|t|^{2k+2}) \quad \text{for } x_0 \text{ in } E \text{ uniformly.}$$

3. In this section we show the conditions of Theorem 1 are sufficient for f to have an α derivative almost everywhere in E . As mentioned earlier we will not do the L^p case since it differs from the L^2 case in an unessential way. In view of Theorem A it is enough to show that $F = f_\beta$ satisfies condition Λ_{k+1}^* and N_{k+1}^2 almost everywhere in E .

Suppose that x_0 in E is the origin and that f has support contained in the sphere $S_a = \{x : |x| \leq a\}$. Let $\lambda(t)$ be a function which is infinitely differentiable and has support in the sphere $S_b = \{x : |x| \leq b\}$ with $b > a$ and $\lambda(t) = 1$ for $|t| \leq a$. Hence $f(t) = \lambda(t)f(t)$. Let $P(t)$ be the polynomial in (1) and $R(t)$ be the remainder. Then $f(t) = \lambda(t)P(t) + \lambda(t)R(t)$. The integral $\int_{E_n} \lambda(t)P(t)|x-t|^{\beta-n} dt$ represents an infinitely differentiable function. Hence we can make the simplifying assumptions that x_0 is the origin and that $f(t) = R(t)$ satisfies the condition Λ_α^2 at x_0 and has support in S_b . We also assume k even since the case k odd is similar. Assume $0 < |h| < b/2$. Then

$$\begin{aligned} \frac{1}{2}\{F(h) + F(-h)\} &= \int_{S_b} R(t) \frac{1}{2}\{|h-t|^{\beta-n} + |h+t|^{\beta-n}\} dt \\ &= \int_{S_b} \frac{R(t) + R(-t)}{2} \cdot |h-t|^{\beta-n} dt \\ &= \int_{S_b} \omega(t) |t|^\alpha \cdot |h-t|^{\beta-n} dt. \end{aligned}$$

We first show the last integral is the sum of a polynomial in h of degree $\leq k+1$ and a remainder which is $O(|h|^{k+1})$ as $|h| \rightarrow 0$.

Split this integral into the two integrals $\int_{|t| \leq 2|h|} + \int_{2|h| \leq |t| \leq b}$. The first is $\leq O(|h|^\alpha) \int_{|t| < 2|h|} |h-t|^{\beta-n} dt \leq O(|h|^{k+1})$. For the second, expand $|h-t|^{\beta-n}$ and $|h+t|^{\beta-n}$ in their respective Taylor's developments to obtain

$$\int_{2|h| \leq |t| \leq b} \frac{R(t)}{2} |t|^{\beta-n} \left\{ P\left(\frac{h}{|t|}\right) + O\left(\left|\frac{h}{|t|}\right|^{k+2}\right) \right\} dt$$

where P is a polynomial of degree $\leq k+1$ containing even terms only, i.e., a term of $P(x)$ is $a_j x^j$ where $|j|$ is even. Since $R(t) = O(|t|^\alpha)$

$$\int_{2|h| \leq |t| \leq b} O\left(\left|\frac{h}{|t|}\right|^{k+2}\right) \frac{R(t)}{2} dt = O(|h|^{k+1}).$$

For $|j|=0, 2, 4, \dots, \frac{1}{2}k$ we have

$$\begin{aligned} & \int_{2|h| \leq |t| \leq b} \frac{R(t)}{2} |t|^{\beta-n} a_j \frac{h^j}{|t|^{|j|}} dt \\ &= a_j h^j \int_{|t| \leq b} \frac{R(t)}{2} |t|^{\beta-n-|j|} dt + a_j h^j \int_{|t| < 2|h|} O(|t|^{\beta-n-|j|}) dt \\ &= a_j h^j \int_{|t| \leq b} \frac{R(t)}{2} |t|^{\beta-n-|j|} dt + O(|h|^{k+1}). \end{aligned}$$

Collecting these results we see we have F equal to a polynomial of degree $\leq k$ plus a term which is $O(|h|^{k+1})$.

Now it remains to show that if $\eta(t)|t|^{k+1} = F(t) - P(t)$, where $P(t)$ is the polynomial obtained in the above argument, then

$$\int_{|t| \leq \rho} \frac{\eta^2(t)}{|t|^n} dt < \infty$$

for some $\rho > 0$. This will be accomplished if we show $\int_{|t| \leq \rho} (\eta_i^2(t)/|t|^n) dt < \infty$, $i=1, 2, 3$, where

$$\begin{aligned} \eta_1(t) &= |t|^{-(k+1)} \int_{|h| \leq 2|t|} R(h) |h-t|^{\beta-n} dh \\ \eta_2(t) &= |t| \int_{2|t| \leq |h| \leq b} R(h) |h|^{-n-\alpha-1} dh \\ \eta_3(t) &= |t|^{-(k+1)+|j|} \int_{|h| \leq 2|t|} R(h) |h|^{\beta-n-|j|} dh \end{aligned}$$

where $|j|=0, 2, 4, \dots, k$.

In each of these cases a similar argument is used. We will do the argument in full for the case of η_1 . Let $g(t)$ be a function such that

$$\int_{|t| \leq \rho} g^2(t) \frac{dt}{|t|^{2(\alpha+\beta)+n}} = 1$$

and assume that $\int_{|h| \leq 2\rho} (R^2(h)/|h|^{2\alpha+n}) dh < \infty$. If we can show

$$\int_{|t| \leq \rho} g(t) \int_{|h| \leq 2|t|} R(h) |h-t|^{\beta-n} dh \frac{dt}{|t|^{2(\alpha+\beta)+n}}$$

is finite, this will prove

$$\int_{|t| \leq \rho} \left\{ \int_{|h| \leq 2|t|} R(h) |h-t|^{\beta-n} dh \right\}^2 \frac{dt}{|t|^{2(\alpha+\beta)+n}}$$

is finite as required. Rewriting the above using the notation that $t = |t|\sigma$ where σ is the unit vector $t|t|^{-1}$ and $\tau = h|t|^{-1}$ we obtain

$$\begin{aligned} \int_{0 \leq |t| \leq \rho} g(t) \frac{dt}{|t|^{2(\alpha+\beta)+n}} \int_{|\tau| \leq 2} \frac{|t|^\beta R(|t|\tau)}{|\tau-\sigma|^{n-\beta}} d\tau \\ = \int_{0 \leq |t| \leq \rho; |\tau| \leq 2} \frac{g(t)}{|t|^{\alpha+\beta}} \cdot \frac{R(|t|\tau)}{|t|^\alpha} \frac{dt}{|t|^n} \cdot \frac{d\tau}{|\tau-\sigma|^{n-\beta}}. \end{aligned}$$

Applying Hölder's inequality with the measure $(dt/|t|^n) \cdot (d\tau/|\tau-\sigma|^{n-\beta})$ this is

$$\begin{aligned} \leq \left\{ \int_{0 \leq |t| \leq \rho; |\tau| \leq 2} \frac{g^2(t)}{|t|^{2(\alpha+\beta)}} \frac{dt}{|t|^n} \frac{d\tau}{|\tau-\sigma|^{n-\beta}} \right\}^{1/2} \\ \times \left\{ \int_{0 \leq |t| \leq \rho; |\tau| \leq 2} \frac{R^2(|t|\tau)}{|t|^{2\alpha}} \frac{dt}{|t|^n} \frac{d\tau}{|\tau-\sigma|^{n-\beta}} \right\}^{1/2}. \end{aligned}$$

The square of the first integral in this product is

$$\leq \int_{0 \leq |t| \leq \rho} g^2(t) \frac{dt}{|t|^{2(\alpha+\beta)+n+\beta}} \int_{|h| \leq 2|t|} \frac{dh}{|h-t|^{n-\beta}}.$$

The inner integral is $\leq C|t|^\beta$ and hence the first term in the product is finite. The square of the second term is equal to

$$\begin{aligned} \int_{|t| \leq \rho} \int_{|h| \leq 2|t|} \frac{R^2(h)}{|h|^{2\alpha+n}} \frac{|\tau|^{2\alpha+n-\beta} |h|^\beta}{|h-t|^{n-\beta}} dh dt \\ \leq (2\rho)^\beta (2)^{2\alpha+n-\beta} \int_{|t| \leq \rho} \int_{|h| \leq 2\rho} \frac{R^2(h)}{|h|^{2\alpha+n}} dh \frac{dt}{|h-t|^{n-\beta}} \\ \leq (2\rho)^\beta (2)^{2\alpha+n-\beta} \int_{|h| \leq 2\rho} \int_{|t| \leq 3\rho} \frac{R^2(h)}{|h|^{2\alpha+n}} dh \frac{dt}{|t|^{n-\beta}} < \infty. \end{aligned}$$

4. We consider the necessity of the conditions of Theorem 1 and Theorem 2 from this point on.

The assumption $(1/\rho^n) \int_{|t| \leq \rho} |f(x_0+t) - P_{x_0}(t)|^p dt = O(\rho^{p\alpha})$, $2 \leq p < \infty$, clearly implies f is locally integrable to the p th power. Thus we may modify f to have finite support and to be in L^p . In addition, we may limit our consideration to a closed set $P \subset E$, where $|E-P| < \varepsilon$ and $\varepsilon > 0$ is arbitrary, on which $f_\beta = F$ has a $k+1$ derivative in the L^p sense uniformly and satisfies the condition Λ_α^p uniformly there.

By Lemma 7 we may write $F(x) = G(x) + H(x)$ where $G(x) \in C_0^{k+1}$ and $G_j(x_0) = F_j(x_0)$ represents the j th coefficient of the polynomial in (1). Also we know that $H = F - G$ is zero on P and because of the uniform differentiability of F and the fact that H has compact support we have

$$(6) \quad \int_{|t| \leq \rho} |H(x_0 + t)|^p dt \leq \text{constant } \rho^{p(k+1)+n}, \quad 0 < \rho < \infty,$$

for x_0 in P .

Apply the inversion formula of Lemma 3 to F to obtain

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{A_\beta} \int_{|t| \geq \varepsilon} \frac{F(x+t) - F(x)}{|t|^{n+\beta}} dt$$

where the limit is taken in the L^2 norm. Also consider the function $g(x) = \lim_{\varepsilon \rightarrow 0} (1/A_\beta) \int_{|t| \geq \varepsilon} (G(x+t) - G(x))/|t|^{n+\beta} dt$. Since $G \in C_0^{k+1}$ this limit exists in the L^2 norm uniformly in x . Calculating $g(x)$ we find

$$\begin{aligned} g(x) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{A_\beta} \int_{\Sigma} d\sigma \int_{\varepsilon}^{\infty} \frac{G(x+r\sigma) - G(x)}{r^{1+\beta}} dr \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{A_\beta} \int_{\Sigma} d\sigma \int_{\varepsilon}^{\infty} \left\{ \int_0^r G_\sigma(x+s\sigma) ds \right\} \frac{dr}{r^{1+\beta}}. \end{aligned}$$

After changing the order of integration of the inner integrals and letting $\varepsilon \rightarrow 0$ this becomes

$$\frac{1}{\beta A_\beta} \int_{\Sigma} d\sigma \int_0^{\infty} G_\sigma(x+s\sigma) \frac{ds}{s^\beta}.$$

Note that $g(x)$ is a function which satisfies the hypothesis of Lemma 2.

Set $h(x) = f(x) - g(x)$. Then we have

$$h(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{A_\beta} \int_{|t| \geq \varepsilon} \frac{H(x+t) - H(x)}{|t|^{n+\beta}} dt$$

in the L^2 norm. However $H(x) = 0$ if x in P and the integral $\int_{E_n} (H(x+t))/|t|^{n+\beta} dt$ converges for almost every x in P by Lemma 4' and (6). By shrinking P further we may assume that $h(x_0) = (1/A_\beta) \int_{E_n} (H(x_0+t))/|t|^{n+\beta} dt$ for all x_0 in P . Since g satisfies the hypothesis of Lemma 2, it satisfies the condition Λ_α uniformly and it satisfies the condition N_α^p , $2 \leq p < \infty$, almost everywhere. Also f satisfies the condition Λ_α uniformly in P ; hence the problem of showing that f satisfies the condition N_α^p almost everywhere in P reduces to showing that h satisfies the condition N_α^p almost everywhere in P . To do this we assume that x_0 is a point of density of P at which the Lemmas 5, 5', 5'' and 6 hold. We may assume that x_0 is the origin. Since the constant A_β plays no essential role from this point on, we drop it.

Hence we need to show we can write

$$(7) \quad h(x) = P(t) + \xi(t)|t|^\alpha$$

where $P(t) = \sum_{|j|=0}^k (a_j t^j / j!)$ and $\int_{|t| \leq 0} (|\xi(t)|^p / |t|^n) dt < \infty$. As before we will do the problem for the case $p=2$ since the case $2 \leq p < \infty$ is similar. If x is in P we can write

$$h(x) = \int_{E_n} \frac{H(t)}{|x-t|^{n+\beta}} dt = \int_{|t| \geq 2|x|} + \int_{|t| \leq 2|x|} = S + T.$$

If $|t| \geq 2|x|$ then $|x-t|^{-(n+\beta)} = |t|^{-(n+\beta)} \{P_t(x) + R\}$ where $P_t(x)$ is the Taylor's development of $|x/t| - t/|t|$ up to and including the k th terms and R is the remainder which is $O(|x/t|^{k+1})$. Thus

$$S = \sum_{|j|=0}^k A_j x^j \int_{E_n} \frac{H(t)}{|t|^{|j|} |t|^{n+\beta}} dt + R_1$$

where

$$R_1 = \sum_{|j|=0}^k A_j x^j \int_{|t| \leq 2|x|} \frac{H(t)}{|t|^{n+\beta+|j|}} dt + O(|x|^{k+1} \int_{|t| \geq 2|x|} \frac{|H(t)|}{|t|^{k+1+\beta+n}} dt).$$

One can easily see that

$$T = \int_{|t| \leq 2|x|} \frac{|H(t)|}{|t-x|^{n+\beta}} dt \leq \int_{|x/2| \leq |t| \leq 2|x|} \frac{|H(t)|}{|t-x|^{n+\beta}} dt + \text{constant} \int_{|t| \leq 2|x|} \frac{|H(t)|}{|x|^{n+\beta}} dt.$$

The first integral of S is a polynomial of degree less than or equal to k . To show that h satisfies the condition N_α^2 it is enough to show that

$$\int_{|x| \leq \delta} \frac{I_i^2(x) dx}{|x|^n} < \infty,$$

$i=1, 2, 3, 4$, for some $\delta > 0$, where

$$I_1(x) = |x|^\beta \int_{|t| \geq 2|x|} \frac{|H(t)|}{|t|^{k+1+\beta+n}} dt,$$

$$I_2(x) = |x|^{|j|-\alpha} \int_{|t| \leq 2|x|} \frac{|H(t)|}{|t|^{n+\beta+|j|}} dt, \quad |j| = 0, 1, \dots, k,$$

$$I_3(x) = |x|^{-\alpha} \int_{|x|/2 \leq |t| \leq 2|x|} \frac{|H(t)|}{|t-x|^{n+\beta}} dt,$$

$$I_4(x) = |x|^{-(\alpha+\beta+n)} \int_{|t| \leq 2|x|} |H(t)| dt.$$

To do this we do the integration over the set of points which are in the set P and later we consider the integration over the complement of P .

First consider the above integral with $i=1$. As in an earlier argument suppose that $g(x)$ is such that $\int_{|x| \leq \delta} (g^2(x)/|x|^n) dx = 1$. Then we must show $\int_{|x| \leq \delta} (g(x)I_1(x)/|x|^n) dx < \infty$. This integral is equal to

$$\begin{aligned} & \int_{|x| \leq \delta} \int_{|t| \geq 2|x|} g(x) |x|^{\beta/2} \frac{|x|^{\beta/2} |H(t)|}{|t|^{k+1}} \frac{dt}{|t|^{n+\beta}} \frac{dx}{|x|^n} \\ & \leq \left\{ \int_{|x| \leq \delta} \int_{|t| \geq 2|x|} g^2(x) |x|^\beta \frac{dt}{|t|^{n+\beta}} \frac{dx}{|x|^n} \right\}^{1/2} \left\{ \int_{|x| \leq \delta} \int_{|t| \geq 2|x|} \frac{|x|^\beta H^2(t)}{|t|^{2(k+1)}} \frac{dt}{|t|^{n+\beta}} \frac{dx}{|x|^n} \right\}^{1/2}. \end{aligned}$$

Changing the order of integration in the second integral of this product and integrating over all x gives

$$\int_{|t| \geq 0} \int_{|x| \leq |t|/2} |x|^\beta \frac{H^2(t)}{|t|^{2(k+1)}} \frac{dx}{|x|^n} \frac{dt}{|t|^{n+\beta}} \leq \text{constant} \int_{|t| \geq 0} \frac{H^2(t)}{|t|^{2(k+1)+n}} dt < \infty.$$

On the other hand, one can easily see the first term in the product is finite.

In a similar way one can obtain the required results in the cases for the integrals I_2 and I_4 . For the remainder of this section we concentrate on I_3 .

Let $U = \{x : d(x, P) < 1\}$. Let $Q = U - P$ and let K_μ be the elements of the cover of Q as given in Lemma 4. Let x be in P and consider the integral $\int_{t \in K_\mu} (|H(t)|/|x-t|^{n+\beta}) dt$. We have that, the diameter of $K = \text{diam}(K_\mu)$, $\text{diam}(K_\mu) \leq d(P, K_\mu) \leq 3 \text{diam}(K_\mu)$. For each K_μ we let a_μ be the center of K_μ , and x_μ be a point of P such that $d(x_\mu, K_\mu) = d(P, K_\mu)$. Let S_μ be the smallest sphere containing K_μ with center at x_μ .

Suppose that $d(x, x_\mu) \geq 4 \text{diam}(K_\mu)$. Then

$$\int_{K_\mu} \frac{|H(t)|}{|x-t|^{n+\beta}} dt \leq C|x-x_\mu|^{-(n+\beta)} \left\{ \int_{S_\mu} |H(t)| dt \right\},$$

since $|x-t| \geq \frac{1}{4}|x-x_\mu|$. By (6) this is $\leq C|x-x_\mu|^{-(n+\beta)} \text{diam}(K_\mu)^{k+1+n} = C|x-x_\mu|^{-(n+\beta)} \int_{K_\mu} \Delta^{k+1}(t) dt$ where $\Delta(t)$ is the distance of t from P . Since $|x-t| \leq 3|x-x_\mu|$ the last line is a constant times $\int_{K_\mu} (\Delta^{k+1}(t)/|x-t|^{n+\beta}) dt$. Suppose that $|x-x_\mu| < 4 \text{diam}(K_\mu)$. Then

$$\int_{K_\mu} \frac{|H(t)|}{|x-t|^{n+\beta}} dt \leq C \int_{K_\mu} \frac{|H(t)|}{|x_\mu-t|^{n+\beta}} dt$$

since $|x-t| \geq \text{diam}(K_\mu)$ and $|x_\mu-t| \leq 4 \text{diam}(K_\mu)$. We have $|x_\mu-a_\mu| \leq 4 \text{diam}(K_\mu) \leq 4|x_\mu-t|$ and the above integral is

$$\begin{aligned} &\leq C \int_{S_\mu} \frac{|H(t)|}{|x_\mu-a_\mu|^{n+\beta}} dt \leq C|x_\mu-a_\mu|^\alpha \\ &\leq C|x_\mu-a_\mu|^{-(n+\beta)} \int_{K_\mu} \Delta^{k+1}(t) dt. \end{aligned}$$

One can show $|x_\mu-a_\mu| \geq |x-t|/8$ for t in K_μ so we find that

$$(8) \quad \int_{K_\mu} \frac{|H(t)|}{|x-t|^{n+\beta}} dt \leq C \int_{K_\mu} \frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}} dt \quad \text{for } x \text{ in } P.$$

Since the origin is a point of density of P and $K_\mu \subset Q$, $\text{diam}(K_\mu) = o(\text{diam } S_\mu(o))$ as $\text{diam}(S_\mu(o)) \rightarrow 0$ where $S_\mu(o)$ is the smallest sphere containing K_μ with center at the origin. Hence there is a $\delta > 0$ such that for $\text{diam}(S_\mu(o)) < 2\delta$, $\text{diam}(K_\mu)$

$< \text{diam}(S_\mu(o))/6$. If we assume that $|x| < \delta$ then it is easy to see that if $E_x = \{t : |x|/2 \leq |t| \leq 2|x|\}$

$$\begin{aligned} \int_{|x|/2 \leq |t| \leq 2|x|} \frac{|H(t)|}{|x-t|^{n+\beta}} dt &\leq \sum_{K_\mu \cap E_x \neq \emptyset} \int_{K_\mu} \frac{H(t)}{|x-t|^{n+\beta}} dt \leq C \sum_{K_\mu \cap E_x \neq \emptyset} \int_{K_\mu} \frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}} dt \\ &\leq C \int_{|x|/6 \leq |t| \leq 4|x|} \frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}} dt. \end{aligned}$$

If we choose $\delta < 1/4$ the integration is over a region contained in U . Set $\Delta(t) = 0$ if $t \notin U$. By applying the estimation technique used on the other integrals one can now show that $\int_{|x| \leq \delta} (I_3^2(x)/|x|^n) dx < \infty$ after noting that x in P and, $\int (\Delta^\lambda(x)/|x|^{\lambda+n}) dx < \infty$ for $\lambda > 0$ by Lemma 5, where one will need to use $\lambda = 2(k+1)$.

5. To complete the demonstration that h satisfies the condition N_α^2 one must show that $\int_{Q; |x| \leq \delta} (\xi^2(x)/|x|^n) dx < \infty$. Since $\{K_\mu\}$ is a cover for Q the finiteness of this will be proved if we show the two sums

$$(9) \quad \sum_{\mu} \int_{K_\mu} [\rho(x) - \rho(x_\mu)]^2 \frac{dx}{|x|^{2\alpha+n}}$$

and

$$(10) \quad \sum_{\mu} \int_{K_\mu} [\rho(x_\mu)]^2 \frac{dx}{|x|^{2\alpha+n}}$$

are finite, where $\xi(x)|x|^\alpha = \rho(x)$.

We consider (10) first. Since x_μ is a point of P , $\xi(x_\mu)$ is majorized by the sum of the integrals I_1, I_2, I_3 and I_4 with $x = x_\mu$. Since the origin is a point of density there is a $\delta > 0$ such that for $K_\mu \subset \{x : |x| < \delta\}$ and x in K_μ , $(|x_\mu|/2) \leq |x| \leq 2|x_\mu|$ and hence in this case (10) is majorized by a constant times

$$(11) \quad \sum_{\mu} \int_{K_\mu} \frac{\xi^2(x_\mu)}{|x|^n} dx.$$

One sees that, for x in K_μ , $I_1(x_\mu)$ is increased by a constant times

$$|x|^\beta \int_{|t| \geq |x|} \frac{|H(t)|}{|t|^{k+1+\beta+n}} dt.$$

Likewise $I_2(x_\mu)$ and $I_4(x_\mu)$ majorized by constant multiples of

$$|x|^{|\beta|-\alpha} \int_{|t| \leq 4|x|} \frac{|H(t)|}{|t|^{n+\beta+|\beta|}} dt, \quad |j| = 0, 1, \dots, k,$$

and $|x|^{-(\alpha+\beta+n)} \int_{|t| \leq 4|x|} |H(t)| dt$ respectively. The contribution of these integrals to (11) can be estimated in the same way as in the previous case.

It remains to estimate the contribution of $I_3(x_\mu)$ to the convergence of (11). In view of (8) and the fact that x_μ is in P we have (11) is finite if

$$\begin{aligned} & \sum_{\mu} \int_{K_{\mu}} \frac{dx}{|x|^n} \left\{ |x_{\mu}|^{-\alpha} \int_{|x_{\mu}|/4 \leq |t| \leq 4|x_{\mu}|} \frac{|H(t)|}{|t-x_{\mu}|^{n+\beta}} dt \right\}^2 \\ & \leq C \sum_{\mu} \int_{K_{\mu}} \frac{dx}{|x|^{n+\alpha}} \int_{|x|/16 \leq |t| \leq 16|x|} \frac{\Delta^{k+1}(t)}{|t-x_{\mu}|^{n+\beta}} dt \end{aligned}$$

is finite. The last inequality follows by a use of the inequalities $\Delta(t) \leq |t-x_{\mu}|$ and $|x_{\mu}|/2 \leq |x| \leq 2|x_{\mu}|$. The limits of integration can be refined.

Let S'_{μ} be the sphere with center at x_{μ} and radius twice that of S_{μ} . It is easy to see that $\int_{S'_{\mu}} (\Delta^{k+1}(t)/|x_{\mu}-t|^{n+\beta}) dt \leq C[\text{diam}(K_{\mu})]^{\alpha}$ by noting that $\Delta^{k+1}(t) \leq |x_{\mu}-t|$ and that $\text{diam}(S_{\mu}) \leq C \text{diam } K_{\mu}$. Also, since $|x| \geq |x_{\mu}|/16$, one can see that

$$\int_{K_{\mu}} \frac{dx}{|x|^{n+\alpha}} \int_{S'_{\mu}} \frac{\Delta^{k+1}(t)}{|x_{\mu}-t|^{n+\beta}} dt$$

is majorized by a constant times $[\text{diam } K_{\mu}]^{n+\alpha} \cdot |x_{\mu}|^{-(n+\alpha)}$. Hence by the Lemma 5" the contribution of (11) on the S'_{μ} is finite and we only need to consider the contribution over the t with $|x|/16 \leq |t| \leq 16|x|$ outside the S'_{μ} . For such t , $|t-x_{\mu}| \geq \frac{1}{2}|t-x|$ since $x \in K_{\mu} \subset S_{\mu}$. Let ψ_Q be the characteristic function of Q and let $\lambda_x(t)$ be equal to zero if $t \in S'_{\mu}$ and $x \in K_{\mu}$, and let $\lambda_x(t)$ be one otherwise. Then the part of (11) that remains to be considered is a constant multiple of

$$\begin{aligned} (12) \quad & \int_Q \frac{dx}{|x|^{n+\alpha}} \left\{ \int_{|x|/16 \leq |t| \leq 16|x|} \frac{\Delta^{k+1}(t)\lambda_x(t)}{|t-x|^{n+\beta}} dt \right\} \\ & \leq C \int_{E_n} \frac{\Delta^{k+1}(t)}{|t|^{n+\alpha}} dt \int_{|t|/16 \leq |x| \leq 16|t|} \frac{\psi_Q(x)\lambda_x(t)}{|t-x|^{n+\beta}} dx. \end{aligned}$$

Now the inner integral is taken over the exterior of Q to the sphere S'_{μ} which contains x in K_{μ} , i.e., it is

$$\leq C \int_{2|x-t| \geq \Delta(t)} \frac{dx}{|t-x|^{n+\beta}} \leq C\Delta^{-\beta}(t).$$

Combining this with the above we have that (12) is dominated by

$$C \int_{|t| \leq \infty} \frac{\Delta^{k+1-\beta}(t)}{|t|^{n+\beta}} dt = C \int \frac{\Delta^{\alpha}(t)}{|t|^{n+\alpha}} dt.$$

This is finite by Lemma 5.

It now remains to show that (9) is finite. To do this recall that h satisfies the condition Λ_{α}^2 uniformly in the set P , i.e., we can write $f(x_0+t) = P_{x_0}(t) + \rho_{x_0}(t)$ where $P_{x_0}(t)$ is a polynomial of degree $\leq k$ and $\int_{|t| \leq \rho} |\rho_{x_0}(t)|^2 dt = O(\rho^{2\alpha+n})$, as $\rho \rightarrow 0$, uniformly for x_0 in P . We can apply Lemma 6 to h to see that for x_0 the origin we have $h(x) = P(x) + \rho(x)$ as in (7) and if we write $x_{\mu} + t = x$ we have $h(x) = \sum_{|j|=0}^k (h_j(x_{\mu})t^j/j!) + \rho_{x_{\mu}}(x-x_{\mu})$. For x in P $(\partial/\partial x)^\alpha P(x) = h_j(x)$ and hence we

have $h_j(x_\mu) - (\partial/\partial x)^j P(x_\mu) = O(|x_\mu|^{\alpha-|j|})$. Also we have that $P(x) = \sum_{|j|=0}^k (\partial/\partial x)^j \times P(x_\mu)(x-x_\mu)^j/j!$. Combining all these we can write

$$\begin{aligned} \rho(x) - \rho(x_\mu) &= h(x) - h(x_\mu) - \{P(x) - P(x_\mu)\} \\ &= \sum_{|j|=1}^k \frac{h_j(x_\mu)}{j!} t^j + \rho_{x_\mu}(x - x_\mu) - \sum_{|j|=1}^k \frac{P_j(x_\mu)(x - x_\mu)^j}{j!}. \end{aligned}$$

Hence

$$|\rho(x) - \rho(x_\mu)| \leq \rho_{x_\mu}(x - x_\mu) + O\left(\sum_{|j|=1}^k |x_\mu|^{\alpha-|j|} |x - x_\mu|^{|j|}\right).$$

Using this to estimate (9) we see that with the definition of h in Λ_α^2

$$\sum_\mu \frac{[\rho(x) - \rho(x_\mu)]^2}{|x|^{2\alpha+n}} dx \leq C \sum_\mu \frac{[\text{diam}(K_\mu)]^{n+2\alpha}}{|x_\mu|^{n+2\alpha}} + C \sum_\mu \frac{[\text{diam}(K_\mu)]^{n+2}}{|x_\mu|^{n+2}}.$$

The last inequalities are demonstrated by recalling the inequalities involving x and x_μ where x is in K_μ . (9) is finite by Lemma 5".

As a final remark we point out that the condition N_α^2 easily implies the condition Λ_α^2 which is enough to complete the proof of Theorem 2 and Theorem 2'.

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