THE PLANCHEREL TRANSFORM ON THE NILPOTENT PART OF G_2 AND SOME APPLICATIONS TO THE REPRESENTATION THEORY OF G_2

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INTRODUCTION

This paper is concerned with the development of the Plancherel transform on the nilpotent part of the complex simple group of type G_2 and its application to some problems in the representation theory of G_2. The impetus for this investigation was supplied by the results of Kunze and Stein [13] concerning the construction of intertwining operators for the equivalent representations in the (nondegenerate) principal series of unitary representations on a complex semisimple Lie group G. These representations $T(\cdot, \chi)$ can be described as *induced* from unitary characters χ of a maximal solvable subgroup B of G, and can be conveniently realized as *multiplier* representations acting in the space $L^2(V)$ where V is a maximal nilpotent subgroup of G which we shall call the nilpotent part of G. It is a result of Bruhat that the representations corresponding to χ_1 and χ_2 are unitarily equivalent if and only if they are conjugate under the Weyl group; i.e., $p\chi_1=\chi_2$ for some element p of the Weyl group. In the case when p is a simple Weyl reflection, say p_j, Kunze and Stein were able to construct *explicit* intertwining operators $A(p_j, \chi)$; i.e., for the simple Weyl reflections $p_1, \dots, p_r$, $A(p_j, \chi)T(\cdot, \chi)=T(\cdot, p_j\chi)A(p_j, \chi)$. They then showed that the problem of constructing intertwining operators for general elements of the Weyl group could be reduced to the specific case of the rank two simple groups; namely, the classical groups SL (3, C) and Sp (2, C) and the exceptional group G_2. In particular they had to establish for each of these three groups the validity of an operator equation—the so-called *intertwining equation*—involving the operators $A(p_1, \chi)$ and $A(p_2, \chi)$. For the case of the classical groups the result was obtained by finding *commutative families* of operators involving certain products of $A(p_1, \chi)$ and $A(p_2, \chi)$; but for G_2 no commutative families could be found and the intertwining equation was established by "normalizing" the intertwining operators $A(p_1, \chi)$ and $A(p_2, \chi)$ and using the result of Bruhat on the irreducibility of those representations $T(\cdot, \chi)$ in general position. The existence of commutative families is of significant independent interest because they are used to *analytically continue* the principal series. Since for the classical rank two groups the Plancherel transform of V was the basis of the initial stage in the proof of existence of commutative families, it seemed likely that the Plancherel transform on the nilpotent part of G_2 should be investigated.

Received by the editors March 10, 1967.

Let us now describe the content of this paper. In Chapter I we follow the well-known procedure for realizing the Lie algebra (Lie group) of type G_2 as a subalgebra $\mathfrak{G}$ of the linear Lie algebra gl (7, $\boldsymbol{C}$) (closed subgroup G of the general linear group GL (7, $\boldsymbol{C}$)). In §1 we recall the formulation of the Cayley algebra over the complex numbers, and in §2 realize $\mathfrak{G}$ by means of a faithful representation of the derivation algebra. In §3 we obtain a convenient root space decomposition for $\mathfrak{G}$ and an explicit formulation of the action of the Weyl group of $\mathfrak{G}$. In §4 we set down criteria for determining when a 7×7 matrix is in G, and in §5 we compute various subgroups of G; namely, the nilpotent part V, the maximal solvable subgroup B, the Cartan subgroup C, the maximal compact subgroup K. We also write down representatives in K for the Weyl group of G and point out that the subgroups C and V together with the simple Weyl reflections p and q generate G.

In Chapter II we construct the Plancherel transform on the group V, which is of complex dimension six. The organization is as follows: §1 reviews some prerequisite notions; in §2 we find all the irreducible unitary representations of V; in §3 we determine the Plancherel transform and state the Plancherel-theorem; and in §4 we compute the resulting decomposition of the left and right regular representations, the Plancherel transform of an important representation of C, and we outline a proof of the ontoness of the Plancherel transform.

It is appropriate here to point out that the harmonic analysis of V is significantly more complicated than that of the nilpotent parts of the rank two classical groups. For one thing the representations of V involved in the Plancherel transform are induced in *two* stages rather than one. This unfortunate structure is manifested in a Plancherel transform which is quite far removed from the ordinary Fourier transform of the complex plane. In the end, one can see that the basic complication is that the representations involved when restricted to the two generating complex one-parameter subgroups of V do not separate into multiplication operators and translation operators respectively.

Chapter III is devoted to representation theory of G_2, in particular to applications of the Placherel transform. In §1 we describe the principal series and determine the explicit operators corresponding to a set of generators for G. We are able to compute the commuting algebra of the restriction of the principal series to the maximal solvable subgroup $H=CV$. However, the Plancherel transforms of the operators $T(p, \chi)$ and $T(q, \chi)$ must be evaluated before the analogous problem can be attacked on any properly larger subgroup of G. In §2 we consider the intertwining operators mentioned at the outset and show that there is a natural candidate for a commutative family. Although we can evaluate the Plancherel transforms of the operators, we do not see at this time a proof of commutativity. In §3 we exhibit two *complementary* series of representations of G.

Finally, I would like to acknowledge my indebtedness to Professor Ray A. Kunze for his constant advice and assistance.

CHAPTER I. CONSTRUCTION OF G_2

1. **The Cayley algebra $\mathscr{A}$.** In this section we recall the pertinent facts about the Cayley algebra over the complex numbers $\boldsymbol{C}$.

By a *quaternion algebra* over $\boldsymbol{C}$ we mean a four-dimensional algebra Q over $\boldsymbol{C}$ having an ordered basis $\{1, u_1, u_2, u_3\}$ relative to which the multiplication table is

$$\begin{aligned} u_ju_k &= u_l = -u_ku_j \quad \text{for a cyclic permutation } (jkl) \text{ of } (123); \\ u_j^2 &= -1 \qquad\qquad \text{for } j = 1, 2, 3. \end{aligned} \tag{1.1.1}$$

We shall write $Q=\{1, u_1, u_2, u_3\}$ to mean that Q is a quaternion algebra with the indicated ordered basis satisfying (1.1.1).

As in [9] we define a Cayley algebra over $\boldsymbol{C}$ to be an algebra $\mathscr{A}$ generated over a quaternion algebra $Q=\{1, u_1, u_2, u_3\}$ by two elements u_4 and 1, where multiplication is given as follows: With $z=q_1+q_2u_4$ and $w=r_1+r_2u_4$ in $\mathscr{A}$ (where $q_1, q_2, r_1, r_2 \in Q$), we have

$$zw = (q_1r_1-\bar{r}_2q_2)+(r_2q_1+q_2\bar{r}_1)u_4; \tag{1.1.2}$$

where $q \to \bar{q}$ denotes the involutive anti-automorphism of Q defined by

$$(\alpha_0 1+\alpha_1u_1+\alpha_2u_2+a_3u_3)^- = \alpha_0 1-(\alpha_1u_1+\alpha_2u_2+\alpha_3u_3).$$

$\mathscr{A}$ is an eight-dimensional nonassociative algebra over $\boldsymbol{C}$ which is *alternative* in the sense that $z^2w=z(zw)$ and $wz^2=(wz)z$ for all $z, w \in \mathscr{A}$. We shall call the ordered basis

$$u_0 = 1, u_1, u_2, u_3, u_4, u_5 = u_1u_4, u_6 = u_2u_4, u_7 = u_3u_4 \tag{1.1.3}$$

of $\mathscr{A}$ a *standard basis* (relative to $Q=\{1, u_1, u_2, u_3\}$).

The most important subspace of $\mathscr{A}$ for our purposes is the subspace $\mathscr{B}$ generated by the elements $u_1, \ldots, u_7$ of the standard basis (1.1.3). $\mathscr{B}$ may also be described as the subspace of $\mathscr{A}$ generated by all commutators $[z, w]=zw-wz$ in $\mathscr{A}$. In $\mathscr{B}$ we consider the symmetric two-form $(\cdot, \cdot)$ defined by

$$(z, w) = \sum_{j=1}^{7} z_jw_j \quad \text{where} \quad z = \sum_{j=1}^{7} z_ju_j,\ w = \sum_{j=1}^{7} w_ju_j \qquad (z_j, w_j \in \boldsymbol{C}). \tag{1.1.4}$$

There is also a critical skew-symmetric three-form T on $\mathscr{B}$ defined by

$$T(u, z, w) = \tfrac{1}{2}(u, [z, w]) \quad \text{for } u, z, w \in \mathscr{B}. \tag{1.1.5}$$

An appropriate reference for the above material is [9].

2. **The Lie algebra $\mathfrak{G}$ of type G_2.** As is well known, the simple complex Lie algebra of type G_2 in the Cartan classification can be realized as the derivation algebra of the Cayley algebra $\mathscr{A}$. In this section we capitalize on this to obtain the simple Lie algebra of type G_2 as a Lie algebra $\mathfrak{G}$ of 7×7 complex matrices for which a root-space decomposition is readily exhibited.

Let $\mathscr{D}$ denote the Lie algebra of all C-derivations of $\mathscr{A}$. The mapping $D \to D|_{\mathscr{B}}$ (the restriction of D to $\mathscr{B}$) is a faithful representation of $\mathscr{D}$ in linear transformations on $\mathscr{B}$. Denote by $\mathscr{D}'$ the range of this restriction map. We have the following characterization, proved in [9], of $\mathscr{D}'$:

PROPOSITION 1.2.1. *A linear transformation $D: \mathscr{B} \to \mathscr{B}$ is in $\mathscr{D}'$ if and only if the following two conditions are satisfied*:

$$(1.2.1) \qquad \begin{aligned} &\text{(i)} \quad (Dz, w) = -(z, Dw), \\ &\text{(ii)} \quad T(Dz, w, u) = -T(z, Dw, u) - T(z, w, Du), \end{aligned}$$

for all $z, w, u \in \mathscr{B}$, where $(\cdot, \cdot)$ is the two-form (1.1.4) *and T is the three-form* (1.1.5).

One observes from the form of the elements of the standard basis that the linear transformations in $\mathscr{D}'$ are uniquely determined by their action on u_1, u_2, and u_4. In [9] it is shown that a linear transformation $D: \mathscr{B} \to \mathscr{B}$ is in $\mathscr{D}'$ if and only if it acts on u_1, u_2, and u_4 as follows:

$$\begin{aligned} Du_1 &= \lambda_1 u_1 + \cdots + \lambda_7 u_7, \\ Du_2 &= \mu_1 u_1 + \cdots + \mu_7 u_7, \\ Du_4 &= \nu_1 u_1 + \cdots + \nu_7 u_7, \end{aligned}$$

where $\lambda_1, \ldots, \lambda_7, \mu_1, \ldots, \mu_7, \nu_1, \ldots, \nu_7$ are complex numbers subject only to the relations

$$\lambda_1 = \mu_2 = \nu_4 = 0, \quad \lambda_2 + \mu_1 = 0, \quad \lambda_4 + \nu_1 = 0,$$
$$\mu_4 + \nu_2 = 0, \quad \lambda_6 + \nu_3 - \mu_5 = 0.$$

From this, it is easy to compute the matrix of $D \in \mathscr{D}'$ in the standard basis $u_1, \ldots, u_7$ of $\mathscr{B}$. Indeed, let $\mathfrak{G}'$ denote the Lie algebra of complex 7×7 matrices X' which represent linear transformations in $\mathscr{D}'$ relative to the standard basis of $\mathscr{B}$. A matrix X' is in $\mathfrak{G}'$ if and only if it has the form

$$(1.2.2) \quad X' = \begin{bmatrix} 0 & -\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 \\ \lambda_2 & 0 & -\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 \\ \lambda_3 & \mu_3 & 0 & \mu_5 - \lambda_6 & -\lambda_7 - \mu_4 & \lambda_4 - \mu_7 & \lambda_5 + \mu_6 \\ \lambda_4 & \mu_4 & \lambda_6 - \mu_5 & 0 & -\nu_5 & -\nu_6 & -\nu_7 \\ \lambda_5 & \mu_5 & \lambda_7 + \mu_4 & \nu_5 & 0 & -\lambda_2 - \nu_7 & \nu_6 - \lambda_3 \\ \lambda_6 & \mu_6 & \mu_7 - \lambda_4 & \nu_6 & \lambda_2 + \nu_7 & 0 & -\mu_3 - \nu_5 \\ \lambda_7 & \mu_7 & -\lambda_5 - \mu_6 & \nu_7 & \lambda_3 - \nu_6 & \mu_3 + \nu_5 & 0 \end{bmatrix}$$

with $\lambda_1, \ldots, \lambda_7, \mu_3, \ldots, \mu_7, \nu_5, \nu_6, \nu_7 \in C$. In particular, $\mathfrak{G}'$ is 14-dimensional over C.

$\mathfrak{G}'$ is, in point of fact, a simple Lie algebra of type G_2, but to obtain an explicit root space decomposition it is convenient to consider a conjugate Lie algebra. To this end we define a new basis $v_1, \dots, v_7$ of $\mathscr{B}$ by

$$
\begin{aligned}
v_1 &= -iu_4, \\
v_2 &= 2^{-1/2}(u_1+iu_5), \quad v_5 = 2^{-1/2}(iu_5-u_1), \\
v_3 &= 2^{-1/2}(u_2+iu_6), \quad v_6 = 2^{-1/2}(iu_6-u_2), \\
v_4 &= 2^{-1/2}(u_3+iu_7), \quad v_7 = 2^{-1/2}(iu_7-u_3),
\end{aligned}
\tag{1.2.3}
$$

where $i=\sqrt{-1}$, and $u_1, \dots, u_7$ is the standard basis of $\mathscr{B}$. The matrix of this coordinate change is

$$
P = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} 0 & I_3 & -I_3 \\ \hline -i\sqrt{2} & 0 & 0 \\ \hline 0 & iI_3 & iI_3 \end{array} \right]
\tag{1.2.4}
$$

(where I_3 is the 3×3 unit matrix), which is seen to be unitary. We consider the Lie algebra

$$\mathfrak{G} = P^{-1}\mathfrak{G}'P$$

conjugate to $\mathfrak{G}'$ in GL (7, $\boldsymbol{C}$). By multiplication of the matrices involved and a change of coordinates, one obtains the following form for the elements $X=P^{-1}X'P$ of $\mathfrak{G}$ (with X' given by (1.2.2)):

$$
X = \left[\begin{array}{c|ccc|ccc}
0 & -\sqrt{2}\,w_1 & -\sqrt{2}\,w_3 & -\sqrt{2}\,z_4 & -\sqrt{2}\,z_1 & -\sqrt{2}\,z_3 & -\sqrt{2}\,w_4 \\ \hline
\sqrt{2}\,z_1 & h_1 & w_2 & z_5 & 0 & z_4 & -w_3 \\
\sqrt{2}\,z_3 & z_2 & h_2 & z_6 & -z_4 & 0 & w_1 \\
\sqrt{2}\,w_4 & w_5 & w_6 & -h_1-h_2 & w_3 & -w_1 & 0 \\ \hline
\sqrt{2}\,w_1 & 0 & w_4 & -z_3 & -h_1 & -z_2 & -w_5 \\
\sqrt{2}\,w_3 & -w_4 & 0 & z_1 & -w_2 & -h_2 & -w_6 \\
\sqrt{2}\,z_4 & z_3 & -z_1 & 0 & -z_5 & -z_6 & h_1+h_2
\end{array} \right]
\tag{1.2.5}
$$

where $z_1, \dots, z_6, w_1, \dots, w_6, h_1, h_2 \in \boldsymbol{C}$.

We state three categorizations of $\mathfrak{G}$ in the following theorem:

THEOREM 1.2.1. (1) *$\mathfrak{G}$ is the complex Lie algebra of all 7×7 matrices which represent the restrictions of derivations of $\mathscr{A}$ to the subspace $\mathscr{B}$, relative to the basis $v_1, \dots, v_7$ of $\mathscr{B}$ given by* (1.2.3).

(2) *$\mathfrak{G}$ consists of all matrices* (1.2.5) *with $z_1, \dots, z_6, w_1, \dots, w_6, h_1, h_2 \in \boldsymbol{C}$.*

(3) *Let*

$$
S = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_3 \\ \hline 0 & I_3 & 0 \end{array} \right]
\tag{1.2.6}
$$

where I_3 is the 3×3 unit matrix. Also, let T be the skew-symmetric three-form on C^7 given by

$$T = v_1^* \wedge v_6^* \wedge v_3^* + v_1^* \wedge v_5^* \wedge v_2^* + v_1^* \wedge v_7^* \wedge v_4^* + \sqrt{2}\,(v_2^* \wedge v_3^* \wedge v_4^* - v_5^* \wedge v_6^* \wedge v_7^*) \tag{1.2.7}$$

where $v_1,\dots,v_7$ is the usual basis for C^7 (i.e., v_j is a column vector with 1 in the jth entry and 0 elsewhere), $v_1^*,\dots,v_7^*$ is the corresponding dual basis, and the wedge-product is that of the exterior algebra of C^7. Then $\mathfrak{G}$ consists of those complex 7×7 matrices X for which the following two conditions hold:

(a) *$XS=-S\,{}^tX$ (where ${}^tX$ denotes the matrix transpose to X).*

(b) *$T(Xu,v,w)+T(u,Xv,w)+T(u,v,Xw)=0$ for all vectors $u,v,w\in C^7$.*

Proof. The first and second statements have already been established. The third follows from Proposition 1.2.1 by use of the basis (1.2.3) of $\mathscr{B}$.

3. **A root space decomposition of $\mathfrak{G}$.** We establish some notation: Let e_j ($j=1,2,3$) denote the usual basis for C^3 (i.e., e_j is a column vector with jth entry equal to 1 and all other entries 0); let E_{jk} ($j,k=1,2,3$) denote the usual elementary 3×3 matrices (i.e., E_{jk} has 1 as its jkth entry and 0 in all other entries); and let H_j $j=1,2,3$) denote the 7×7 matrix

$$H_j = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & E_{jj} & 0 \\ \hline 0 & 0 & -E_{jj} \end{array}\right].$$

Finally, δ_{jk} denotes the Kronecker symbol; i.e.,

$$\begin{aligned} \delta_{jk} &= 1 \quad \text{if } j = k, \\ &= 0 \quad \text{if } j \neq k; \end{aligned}$$

and the symbol $\varepsilon(jkl)$ is defined by

$$\begin{aligned} \varepsilon(jkl) &= 1 \quad \text{if } (jkl) \text{ is a cyclic permutation of } (123), \\ &= -1 \quad \text{if } (jkl) \text{ is a noncyclic permutation of } (123), \\ &= 0 \quad \text{if } (jkl) \text{ is not a permutation of } (123). \end{aligned}$$

THEOREM 1.3.1. *Let $\mathfrak{H}=\{H=d_1H_1+d_2H_2+d_3H_3 \mid d_1+d_2+d_3=0\}$. $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{G}$, relative to which the roots are $\pm d_j$ ($j=1,2,3$) and d_j-d_k ($j,k=1,2,3$ with $j\neq k$). Corresponding root vectors are*

$$E_{d_j} = \left[\begin{array}{c|c|c} 0 & 0 & (-\sqrt{2})\,{}^te_j \\ \hline \sqrt{2}\,e_j & 0 & 0 \\ \hline 0 & E_{kl}-E_{lk} & 0 \end{array}\right] \qquad \textit{for } j = 1,2,3 \textit{ with } \varepsilon(jkl) = 1,$$

$$E_{-d_j} = \left[\begin{array}{c|c|c} 0 & (-\sqrt{2})^t e_j & 0 \\ \hline 0 & 0 & E_{kl}-E_{lk} \\ \hline \sqrt{2}\, e_j & 0 & 0 \end{array}\right]$$

$$E_{d_j-d_k} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & E_{jk} & 0 \\ \hline 0 & 0 & -E_{kj} \end{array}\right].$$

The proof is straightforward. In fact one verifies that $\mathfrak{H}$ is a maximal abelian subalgebra of $\mathfrak{G}$, that $\mathfrak{H}$ together with the matrices $E_{\pm d_j}$ and $E_{d_j-d_k}$ span $\mathfrak{G}$, and that one indeed has

$$[H, E_{\pm d_j}] = \pm d_j E_{\pm d_j}$$

and

$$[H, E_{d_j-d_k}] = (d_j-d_k)E_{d_j-d_k}, \qquad \text{for all } H \in \mathfrak{H}.$$

As simple roots we choose $\alpha=-d_1$, $\beta=d_1-d_2$; hence, relative to the lexicographical ordering of the dual (see [11, p. 119]) the complete set of positive roots is

$$\text{(1.3.1)} \qquad \begin{gathered} \alpha = -d_1, \quad \beta = d_1-d_2, \quad \alpha+\beta = -d_2, \quad 2\alpha+\beta = d_3, \\ 3\alpha+\beta = d_3-d_1, \quad 3\alpha+2\beta = d_3-d_2. \end{gathered}$$

(It is now clear, by the way, that $\mathfrak{G}$ is a simple complex Lie algebra of type G_2.)

Let $\mathfrak{B}$ denote the nilpotent subalgebra of $\mathfrak{G}$ generated by those root vectors corresponding to negative roots. $\mathfrak{B}$ has a basis composed of the matrices

$$\text{(1.3.2)} \qquad \begin{gathered} X_1 = E_{d_1}, \quad X_2 = E_{d_2-d_1}, \quad X_3 = E_{d_2}, \quad X_4 = E_{-d_3}, \\ X_5 = E_{d_1-d_3}, \quad X_6 = E_{d_2-d_3}; \end{gathered}$$

relative to which the multiplication table for $\mathfrak{B}$ is

$$\text{(1.3.3)} \qquad \begin{gathered} [X_1, X_2] = -X_3, \quad [X_1, X_3] = -2X_4, \\ [X_1, X_4] = -3X_5, \quad [X_2, X_5] = X_6, \ [X_3, X_4] = -3X_6 \end{gathered}$$

(where all other brackets not determined by skew-symmetry are zero). The elements of $\mathfrak{B}$ are of the form

$$X = z_1X_1+\cdots+z_6X_6 = \left[\begin{array}{c|ccc|ccc} 0 & 0 & 0 & -\sqrt{2}\,z_4 & -\sqrt{2}\,z_1 & -\sqrt{2}\,z_3 & 0 \\ \hline \sqrt{2}\,z_1 & 0 & 0 & z_5 & 0 & z_4 & 0 \\ \sqrt{2}\,z_3 & z_2 & 0 & z_6 & -z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -z_3 & 0 & -z_2 & 0 \\ 0 & 0 & 0 & z_1 & 0 & 0 & 0 \\ \sqrt{2}\,z_4 & z_3 & -z_1 & 0 & -z_5 & -z_6 & 0 \end{array}\right]$$

(1.3.4)

Similarly, one defines the nilpotent subalgebra $\mathfrak{N}$ of $\mathfrak{G}$ generated by the root vectors corresponding to positive roots. One observes that $\mathfrak{N}$ consists precisely of the matrices transpose to the elements of $\mathfrak{V}$; i.e.,

$$\mathfrak{N} = {}^t\mathfrak{V}. \tag{1.3.5}$$

The subalgebra

$$\mathfrak{B} = \mathfrak{H} \oplus \mathfrak{N} \tag{1.3.6}$$

is a maximal solvable subalgebra of $\mathfrak{G}$, and we have the following important decomposition of $\mathfrak{G}$:

$$\mathfrak{G} = \mathfrak{B} \oplus \mathfrak{V}. \tag{1.3.7}$$

We conclude this section with a discussion of the Weyl group of $\mathfrak{G}$. The Killing form B is nondegenerate on $\mathfrak{H}$, so for each $\xi \in \mathfrak{H}^*$ (the dual of $\mathfrak{H}$), there exists a unique $H_\xi \in \mathfrak{H}$ such that $B(H, H_\xi) = \xi(H)$ for all $H \in \mathfrak{H}$. Thus, there is a symmetric bilinear form on $\mathfrak{H}^*$, defined by $(\xi|\xi') = B(H_\xi, H_{\xi'})$ for all $\xi, \xi' \in \mathfrak{H}^*$, which has the property that its restriction to the real subspace $\mathfrak{A}^*$ spanned by the roots is a real inner product on $\mathfrak{A}^*$. The *simple Weyl reflections* p_α and p_β, which are in the orthogonal group of this inner product, are the linear transformations of $\mathfrak{A}^*$ defined by

$$\begin{aligned} p_\alpha(\xi) &= \xi - 2(\xi|\alpha)\alpha/(\alpha|\alpha), \\ p_\beta(\xi) &= \xi - 2(\xi|\beta)\beta/(\beta|\beta) \end{aligned} \tag{1.3.8}$$

for all $\xi \in \mathfrak{A}^*$. p_α and p_β generate a group $\mathfrak{W}$ of order twelve called the *Weyl group* of $\mathfrak{G}$, which is isomorphic to the symmetry group of a hexagon. The relations on the generators of $\mathfrak{W}$ are

$$p_\alpha^2 = p_\beta^2 = e, \quad (p_\alpha p_\beta)^6 = e, \tag{1.3.9}$$

e being the identity of $\mathfrak{W}$.

We can choose convenient coordinates in $\mathfrak{A}^*$ to explicitly formulate the action of $\mathfrak{W}$. In fact, parametrize $\mathfrak{A}^*$ as follows: If $\xi \in \mathfrak{A}^*$, there exists a unique triple (s_1, s_2, s_3) of real numbers such that $s_1+s_2+s_3=0$ and $\xi = s_1 d_1 + s_2 d_2 + s_3 d_3$ (recall that d_1, d_2, $d_3 = -(d_1+d_2)$ are roots). Thus, we identify ξ with the triple (s_1, s_2, s_3); i.e., $\mathfrak{A}^* = \{\xi = (s_1, s_2, s_3) \in \boldsymbol{R}^3 \mid s_1+s_2+s_3=0\}$. Since $\xi = (s_1, s_2, s_3) = -3(s_1+s_2)\alpha - (s_1+2s_2)\beta$, we have

$$2(\xi|\beta)/(\beta|\beta) = s_1 - s_2, \quad 2(\xi|\alpha)/(\alpha|\alpha) = -3s_1 \tag{1.3.10}$$

(see [11, p. 116, equation (18)]). From (1.3.8) and (1.3.10) one computes that

$$p_\alpha(s_1, s_2, s_3) = (-s_1, -s_3, -s_2), \quad p_\beta(s_1, s_2, s_3) = (s_2, s_1, s_3). \tag{1.3.11}$$

Finally, since $\mathfrak{H}^* = \mathfrak{A}^* + i\mathfrak{A}^*$, we can extend the action of $\mathfrak{W}$ to $\mathfrak{H}^*$ by the definition $p(\xi + i\eta) = p(\xi) + ip(\eta)$ for all $p \in \mathfrak{W}$, and $\xi, \eta \in \mathfrak{A}^*$. We can then parametrize the elements of $\mathfrak{H}^*$ by triples $(s_1, s_2, s_3) \in \boldsymbol{C}^3$ such that $s_1+s_2+s_3=0$ in the same fashion as above, and the Weyl group acts on these triples according to the same formulae (1.3.11).

4. **The Lie group G of type G_2.** The automorphism group of the Cayley algebra $\mathscr{A}$ is a simple complex Lie group of type G_2, for its Lie algebra is the derivation algebra of $\mathscr{A}$. As for derivations, an automorphism A of $\mathscr{A}$ is uniquely determined by its restriction to the subspace $\mathscr{B}$; i.e., $A \to A|_{\mathscr{B}}$ is a faithful representation of the automorphism group. The range of this restriction map is determined by the following proposition.

PROPOSITION 1.4.1. *A linear transformation $A: \mathscr{B} \to \mathscr{B}$ is the restriction to $\mathscr{B}$ of an automorphism of $\mathscr{A}$ if and only if the following two conditions hold*:

(i) $(Az, Aw) = (z, w)$,

(ii) $T(Az, Aw, Au) = T(z, w, u)$,

for all $z, w, u \in \mathscr{B}$, where the notation is as in Proposition 1.2.1.

The proof of this result is quite analogous to that of Proposition 1.2.1 and can be found in [10].

Let $\mathfrak{G}$ be the Lie algebra categorized in Theorem 1.2.1. With the notation of that theorem preserved, we have its group-theoretic analogue:

THEOREM 1.4.1. *Let G denote the group of complex* 7×7 *matrices which represent automorphisms of $\mathscr{A}$ (restricted to $\mathscr{B}$) in the basis $v_1, \ldots, v_7$ of $\mathscr{B}$. Then*

(1) *G is a simple complex Lie group whose Lie algebra is $\mathfrak{G}$.*

(2) *Let S and T be as in Theorem* 1.2.1. *Then G consists of all complex* 7×7 *matrices A which satisfy the following two conditions*:

(a) *A leaves the symmetric bilinear form of S invariant; i.e., ${}^tASA = S$.*

(b) *A leaves the skew-symmetric form T invariant; i.e., $T(Au, Av, Aw) = T(u, v, w)$ for all $u, v, w \in \mathbf{C}^7$.*

(3) *G is a closed connected subgroup of the group* SL $(7, \mathbf{C})$ *of complex* 7×7 *matrices of determinant one.*

Proof. (1) follows, by standard arguments, from Propositions 1.2.1 and 1.4.1. (See, for example, [4, p. 137, Proposition 1].) (2) follows from Proposition 1.4.1 by use of the basis (1.2.3) of $\mathscr{B}$. That G is closed in GL $(7, \mathbf{C})$ is evident from (2). In [10] it is shown that the automorphism group of $\mathscr{A}$ is algebraically simple; i.e., G contains no nontrivial normal subgroups, and is therefore connected. By part (a) of (2), G is a subgroup of the orthogonal group of the form determined by S; thus, since G is connected all the matrices in G must have determinant one. This completes the proof of (3).

5. **Some subgroups of G.** Let $\mathfrak{V}$, $\mathfrak{N}$, $\mathfrak{H}$, and $\mathfrak{B}$ be the subalgebras of $\mathfrak{G}$ determined in §3, and let V, N, C, and B be the respective analytic subgroups of G.

We first compute V. Let $X = z_1X_1 + \cdots + z_6X_6$ be the typical element (1.3.4) of $\mathfrak{V}$, and let $v = \exp X = \sum_{n=0}^{6} X^n/n!$. Make the change of coordinates

$$
\begin{aligned}
&v_1 = z_1, \quad v_2 = z_2, \quad v_3 = z_3 + \tfrac{1}{2} z_1 z_2, \quad v_4 = z_4 - \tfrac{1}{6} z_1^2 z_2, \\
&v_5 = z_5 - \tfrac{1}{2} z_1 z_4 + \tfrac{1}{12} z_1^3 z_2, \quad v_6 = z_6 + \tfrac{1}{2} z_2 z_5 - \tfrac{1}{2} z_3 z_4 + \tfrac{1}{12} z_1^2 z_2 z_3 + \tfrac{1}{60} z_1^3 z_2^2,
\end{aligned} \tag{1.5.1}
$$

in terms of which one obtains

$$
v = v(v_1, \ldots, v_6) =
\left[\begin{array}{c:cccc:cc:c}
1 & 0 & 0 & -\sqrt{2}\, v_4 & -\sqrt{2}\, v_1 & & \sqrt{2}\,(v_1 v_2 - v_3) & 0 \\
\hdashline
\sqrt{2}\, v_1 & 1 & 0 & v_5 & -v_1^2 & & v_4 - v_1 v_3 + v_1^2 v_2 & 0 \\
\sqrt{2}\, v_3 & v_2 & 1 & v_6 & -v_4 - v_1 v_3 & & v_2 v_4 - v_3^2 + v_1 v_2 v_3 & 0 \\
0 & 0 & 0 & 1 & 0 & & 0 & 0 \\
\hdashline
0 & 0 & 0 & -v_3 & 1 & & -v_2 & 0 \\
0 & 0 & 0 & v_1 & 0 & & 1 & 0 \\
\sqrt{2}\, v_4 & v_3 - v_1 v_2 & -v_1 & v_3 v_5 - v_1 v_6 - v_4^2 & -v_5 - v_1 v_4 & & -v_6 + v_2 v_5 - v_3 v_4 + v_1 v_2 v_4 & 1
\end{array}\right]
$$

(1.5.2)

The collection of such matrices v form a closed subgroup V of G, and exp maps $\mathfrak{V}$ homeomorphically onto V. V is a maximal nilpotent subgroup of G which we call the *nilpotent part of* G. The complex dimension of V is six, and $v_1, \ldots, v_6$ are global coordinates relative to which the group operations in V are as follows:

$$v(v_1, \ldots, v_6) v(w_1, \ldots, w_6) = v(t_1, \ldots, t_6)$$

where

$$
\begin{aligned}
t_1 &= v_1 + w_1, \quad t_2 = v_2 + w_2, \quad t_3 = v_3 + w_3 + v_2 w_1 \\
t_4 &= v_4 + w_4 + v_3 w_1 - v_1 w_3 - v_1 v_2 w_1, \\
t_5 &= v_5 + w_5 + v_4 w_1 - 2 v_1 w_4 + v_1^2 w_3 - v_1 v_3 w_1 + v_1^2 v_2 w_1, \\
t_6 &= v_6 + w_6 + v_2 w_5 + v_4 w_3 - 2 v_3 w_4 + v_1 v_3 w_3 + v_2 v_4 w_1 - v_3^2 w_1 + v_1 v_2 v_3 w_1.
\end{aligned} \tag{1.5.3}
$$

$$v(v_1, \ldots, v_6)^{-1} = v(w_1, \ldots, w_6)$$

where

$$
\begin{aligned}
&w_1 = -v_1, \quad w_2 = -v_2, \quad w_3 = -v_3 + v_1 v_2, \quad w_4 = -v_4, \\
&w_5 = -v_5 - v_1 v_4, \quad w_6 = -v_6 + v_2 v_5 - v_3 v_4 + v_1 v_2 v_4.
\end{aligned} \tag{1.5.4}
$$

Having found V it is clear from (1.3.5) that $N = {}^t V$; i.e.,

$$N = \{ n = {}^t v \mid v \in V \}. \tag{1.5.5}$$

Since the Cartan subalgebra $\mathfrak{H}$ of Theorem 1.3.1 consists of the diagonal matrices $H = \operatorname{diag}(0, d_1, d_2, d_3, -d_1, -d_2, -d_3)$ with $d_1 + d_2 + d_3 = 0$, the analytic subgroup C of G whose Lie algebra is $\mathfrak{H}$ consists precisely of the diagonal matrices

$$c = c(c_1, c_2) = \operatorname{diag}(1, c_1, c_2, c_3; c_1^{-1}, c_2^{-1}, c_3^{-1}) \tag{1.5.6}$$

with $c_1 c_2 c_3 = 1$.

To find the analytic subgroup B whose Lie algebra is the subalgebra $\mathfrak{B}$ of (1.3.6), we notice that $C \cap N = \{e\}$ and $cNc^{-1} \subset N$ for all $c \in C$. Thus CN is a closed connected subgroup of G (namely, a semidirect product of C and N), and since its Lie algebra is spanned by $\mathfrak{H}$ and $\mathfrak{N}$ we have

$$B = CN. \tag{1.5.7}$$

Next, consider the following collection of elements of $\mathfrak{G}$:

(1.5.8)
(i) The real subspace $\mathfrak{A}$ of $\mathfrak{H}$ generated by the two matrices $H_\alpha = (H_2+H_3-2H_1)/24$ and $H_\beta = (H_1-H_2)/8$. (We refer to the discussion preceding (1.3.8).)
(ii) The elements $W_j = E_{d_j}+E_{-d_j}$ and $U_j = i(E_{d_j}-E_{-d_j})$ where $j=1, 2, 3$, $i=\sqrt{-1}$, and $E_{\pm d_j}$ are defined in Theorem 1.3.1.
(iii) The elements $Y_{jk} = E_{d_j-d_k}-E_{d_k-d_j}$ and $Z_{jk} = i(E_{d_j-d_k}+E_{d_k-d_j})$ where $j<k$ and $j, k=1, 2, 3$.

Then we have:

THEOREM 1.5.1. (a) *The real vector space*

$$\mathfrak{G}_K = i\mathfrak{A}+\sum_{j=1}^{3}(\boldsymbol{R}W_j+\boldsymbol{R}U_j)+\sum_{j<k;\,j,k=1}^{3}(\boldsymbol{R}Y_{jk}+\boldsymbol{R}Z_{jk})$$

is a compact real form of $\mathfrak{G}$. (*See* [8, *Chapter* II, §§5 *and* 6, *and Chapter* III, §6].)

(b) *With* $\mathfrak{G}$ *considered as a real Lie algebra, we have* $\mathfrak{G}_K = \mathfrak{G} \cap \mathrm{su}(7, \boldsymbol{C})$, *where* $\mathrm{su}(7, \boldsymbol{C})$ *is the Lie algebra (over* $\boldsymbol{R}$*) of the special unitary group* $\mathrm{SU}(7, \boldsymbol{C})$; *i.e.,* $\mathrm{su}(7, \boldsymbol{C})$ *consists of all complex* 7×7 *matrices* X *such that* $X=-{}^t\bar{X}$ *and* $\mathrm{tr}\, X=0$.

(c) *The analytic subgroup* K *of* G *whose Lie algebra is* $\mathfrak{G}_K$ *is a maximal compact subgroup of* G *and, in fact,* $K=G\cap \mathrm{SU}(7, \boldsymbol{C})$.

Proof. (a) $\mathfrak{G}_K$ is a real subalgebra of $\mathfrak{G}$ on which the Killing form is negative definite. Hence by [8, II, Proposition 6.6] it follows that $\mathfrak{G}_K$ is compact. Since $\mathfrak{G}=\mathfrak{G}_K+i\mathfrak{G}_K$, $\mathfrak{G}_K$ is a compact real form. As for (b), one observes that $\mathfrak{G}_K \subset \mathrm{su}(7, \boldsymbol{C}) \cap \mathfrak{G}$, and since $\mathfrak{G}_K$ is a maximal compact subalgebra of $\mathfrak{G}$, $\mathfrak{G}_K = \mathrm{su}(7, \boldsymbol{C}) \cap \mathfrak{G}$. To see (c), we note that K is a maximal compact subgroup of G [8, VI, §2] which is contained in the compact group $G \cap \mathrm{SU}(7, \boldsymbol{C})$. Thus $K = G \cap \mathrm{SU}(7, \boldsymbol{C})$.

Let M denote the subgroup of K whose Lie algebra is $i\mathfrak{A}$, and M', its normalizer in K. Then the Weyl group $\mathfrak{W}$ defined in §3 can be identified with the quotient group M'/M. (See [8, VII, §2].) The elements of M' act on $\mathfrak{H}^*$ by the dual of the adjoint representation of M' acting on $\mathfrak{H}$; in fact, for $\xi \in \mathfrak{H}^*, p \in M'$, one defines $p\xi \in \mathfrak{H}^*$ by $\mathrm{Ad}(p)H_\xi = H_{p\xi}$. In this way, the elements of M' (viewed as coset representatives for M'/M) act as orthogonal linear transformations (relative to the symmetric form of §3) on $\mathfrak{H}^*$. By means of a straightforward computation, which is based on knowing H_α and H_β for the simple roots α and β, one can explicitly compute M'. Indeed, the linear transformations p_α and p_β given by (1.3.8) are of the form

$$p_\alpha = \mathrm{Ad}\,(p), \quad p_\beta = \mathrm{Ad}\,(q) \tag{1.5.9}$$

where p and q are the following elements of M':

$$p = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & R \\ 0 & R & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix}, \tag{1.5.10}$$

where R and T are the 3×3 matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(One checks the criteria of Theorems 1.4.1 and 1.5.1 to see that p and q are actually in K.)

Hence, pM and qM generate M'/M. As a complete set S of coset representatives for M'/M we take

$$S = \{e, p, q, pq, qpq, \ldots, q(pq)^4, (pq)^5\}. \tag{1.5.11}$$

One observes that the element $p_0 = (pq)^3$ is given by the matrix

$$p_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -I_3 \\ 0 & -I_3 & 0 \end{bmatrix},$$

and takes positive roots into negative roots. In other words,

$$p_0 N p_0^{-1} = V. \tag{1.5.12}$$

Bruhat's lemma (see, e.g., [5]), framed in terms of the set S above and the maximal solvable subgroup B of (1.5.7), says that $G = \bigcup_{r \in S} BrB$. From this fact together with (1.5.12), (1.5.7), and (1.5.11) we see that we have:

THEOREM 1.5.2. *G is generated by the subgroups C and V together with the elements p and q of S.*

We conclude with a structural theorem which follows from Bruhat's lemma.

THEOREM 1.5.3. *BV is an open subset of G whose complement is a set of Haar measure zero.*

For a proof, we refer to [6, Exposé 15].

CHAPTER II. HARMONIC ANALYSIS ON THE NILPOTENT PART V OF G

1. **Preliminaries.** We introduce here the terminology, notation, and basic concepts and theorems of use in the succeeding sections.

If X is a second countable, locally compact Hausdorff space, m a regular Borel measure on X, and H a separable Hilbert space, we define $L^2(X, H, m)$ to be the Hilbert space of all (weakly) measurable functions $f\colon X \to H$ (modulo null functions) such that $\int_X \|f(x)\|^2\, dm(x) < \infty$. If H is one-dimensional we delete H from the notation, and if X is a topological group and m is a Haar measure we drop m from the notation. We shall call an operator A on $L^2(X, H, m)$ a (generalized) *multiplication operator* if for each $f \in L^2(X, H, m)$ $(Af)(x) = A(x)(f(x))$ a.e., where $x \to A(x)$ is a (weakly) measurable operator-valued function on X. In some cases we shall write

$$A = \int_X^{\oplus} A(x)\, dm(x)$$

and call A the *direct integral* of the operator-valued function. A lemma of major importance in the sequel is the following:

LEMMA 2.1.1. *Let $\boldsymbol{R}^n$ denote n-dimensional Euclidean space with the usual inner product, let m denote Lebesgue measure on $\boldsymbol{R}^n$, K a separable Hilbert space, and $H = L^2(\boldsymbol{R}^n, K)$. Denote by M_a and τ_a the unitary operators on H defined for each $a \in \boldsymbol{R}^n$ as follows: For $f \in H$, $(M_a f)(x) = [\exp i(a, x)]f(x)$ a.e., and $(\tau_a f)(x) = f(x+a)$ a.e. We then have the following properties, which are classical theorems when K is one-dimensional*:

(a) *If A is a bounded operator on H which commutes with all the operators M_a, then A is the direct integral of some operator-valued function $x \to A(x)$ on $\boldsymbol{R}^n$.*

(b) *If, in addition, A also commutes with all the operators τ_a, then $x \to A(x)$ is essentially constant; i.e., there is a bounded operator A_0 on K such that $A(x) = A_0$ a.e.*

One need only utilize the proof in [15, p. 352] together with the uniqueness of the Fourier transform on $\boldsymbol{R}^n$ to prove (a). As for (b), one reduces the problem to the scalar case by considering the matrix entries of $A(x)$.

As a final remark on vector-valued L^2 spaces, we shall make use of the identification of the Hilbert spaces $L^2(X_1 \times X_2, m_1 \times m_2)$ and $L^2(X_1, L^2(X_2, m_2), m_1)$ given by the unitary map $f \to \tilde{f}$ where $(\tilde{f}(x_1))(x_2) = f(x_1, x_2)$.

Next, we consider some ideas in representation theory. Throughout this chapter, the term *representation* shall mean a strongly continuous unitary representation acting in a Hilbert space.

If G is a locally compact group, N a closed subgroup, and U a representation of N, there is a way of constructing a representation of G, called the *representation induced by U*, from the representation U of N. For the general definition of induced representation we refer to [1]. In the case of principal interest for the study at hand

G is separable and all its subgroups are unimodular, N is normal, and $G=NB$ is the semidirect product of N and a subgroup B. In this case, the representation of G induced by the representation U of N is equivalent to a multiplier representation T acting in $L^2(B, H)$, where H is the space of U, by the formula

$$(2.1.1) \qquad (T(n_0b_0)f)(b) = U(bn_0b^{-1})(f(bb_0)) \quad \text{a.e.}$$

Conversely, as a consequence of Mackey's imprimitivity theorem (see, for example, [2]) every *multiplier representation* of G is induced. That is to say, if T is a representation of $G=NB$ acting in $L^2(B, H)$ by the formula $(T(x_0)f)(b)=\alpha(b, x_0)(f(bb_0))$ a.e., where $x_0=n_0b_0$ and $b \to \alpha(b, x_0)$ is a measurable operator-valued function on B, then there is a representation U of N in H such that T is unitarily equivalent to a representation of the form (2.1.1).

The following lemmas are standard and are easily proven from (2.1.1).

LEMMA 2.1.2. *If the representation T induced by U is irreducible, then U is necessarily irreducible.*

LEMMA 2.1.3. *Suppose $G=NB$ as above. Let T be the representation of G induced by a representation U of N; let $\tilde{b} \in B$ be fixed, ϕ the conjugation of N defined by $\phi(n)=\tilde{b}n\tilde{b}^{-1}$; and let $\tilde{U}$ be the representation of N defined by $\tilde{U}=U \circ \phi$. Then T is unitarily equivalent to the representation $\tilde{T}$ of G induced by $\tilde{U}$; in fact, $LT=\tilde{T}L$ where L is the unitary operator on $L^2(B, H)$, H being the space of U, defined by $(Lf)(b)=f(\tilde{b}b)$, a.e.*

We conclude this section by reviewing some representation theory of the simplest nonabelian nilpotent complex Lie group. This is the group Γ of all upper triangular unipotent complex 3×3 matrices

$$(2.1.2) \qquad \gamma = \gamma(z_1, z_2, z_3) = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We shall call Γ the *standard 3-dimensional nilpotent group*. Γ has two generating complex one-parameter subgroups consisting respectively of the matrices $\alpha(z)=\gamma(z, 0, 0)$ and $\beta(z)=\gamma(0, z, 0)$; the center has complex dimension one and is composed of the matrices $\zeta(z)=\gamma(0, 0, z)$; and the relation which holds among these one parameter subgroups is

$$(2.1.3) \qquad \alpha(s)\beta(t) = \zeta(st)\beta(t)\alpha(s) \qquad \text{for all } s, t \in \mathbf{C}.$$

LEMMA 2.1.4. (a) *Let T be a representation of Γ in the space H; let A and B denote the complex one-parameter groups of unitary operators on H given by*

$$(2.1.4) \qquad A(z) = T(\alpha(z)), \qquad B(z) = T(\beta(z)) \qquad \textit{for } z \in \mathbf{C};$$

and suppose T acts nontrivially by scalars on the center of Γ; *i.e., there exists nonzero* $\lambda \in \boldsymbol{C}$ *such that*

$$T(\zeta(z)) = \exp i \operatorname{Re}(\lambda z) I, \qquad \text{for all } z \in \boldsymbol{C}, \tag{2.1.5}$$

where I is the identity operator on H. Then one has

$$A(s)B(t) = \{\exp i \operatorname{Re}(\bar{\lambda} st)\} B(t)A(s), \qquad \text{for all } s, t \in \boldsymbol{C}. \tag{2.1.6}$$

(b) *Conversely, if $z \to A(z)$, $z \to B(z)$ are two complex one-parameter groups of unitary operators on H such that* (2.1.6) *holds with $\lambda \neq 0$, then there is a unique* (*to within equivalence, of course*) *representation T of Γ such that* (2.1.4) *and* (2.1.5) *are satisfied.*

Proof. (a) is an obvious consequence of (2.1.3). To prove (b) one notices that any element $\gamma(z_1, z_2, z_3) = \gamma$ of Γ has a unique decomposition as $\gamma = \zeta(z_3)\beta(z_2)\alpha(z_1)$. One then defines T by $T(\gamma) = [\exp i \operatorname{Re}(\bar{\lambda} z_3)] B(z_2) A(z_1)$, and makes the necessary verification using (2.1.6).

The operator equation (2.1.6) is known as the Stone-von Neumann equation, the solution to which is given by the following lemma. For its proof we refer to [3].

LEMMA 2.1.5 (STONE-VON NEUMANN). *Let A and B be complex one-parameter groups of unitary operators in H which satisfy* (2.1.6) *with $\lambda \neq 0$. Then there exists a separable Hilbert space K and a unitary map $\Phi: H \to L^2(\boldsymbol{C}, K)$ such that $\Phi A(s) \Phi^{-1} = \tau_s$ and $\Phi B(s) \Phi^{-1} = M_{\lambda s}$ for all $s \in \boldsymbol{C}$, where M and τ are complex one-parameter groups of unitary operators on $L^2(\boldsymbol{C}, K)$ defined for $a \in \boldsymbol{C}$ and $f \in L^2(\boldsymbol{C}, K)$ by*

$$\begin{aligned} (M_a f)(z) &= \{\exp i \operatorname{Re}(\bar{a} z)\} f(z), \\ (\tau_a f)(z) &= f(z+a), \quad a.e. \end{aligned} \tag{2.1.7}$$

As a combination of the two preceding lemmas we have:

LEMMA 2.1.6. *Let T be a representation of Γ in the space H satisfying* (2.1.5) *with $\lambda \neq 0$. Then there exists a separable Hilbert space K and a unitarily equivalent representation $\tilde{T}$ acting in $L^2(\boldsymbol{C}, K)$ such that $\tilde{T}(\gamma(t, 0, 0)) = \tau_t$ and $\tilde{T}(\gamma(0, t, 0)) = M_{\lambda t}$ for all $t \in \boldsymbol{C}$. Thus, for $f \in L^2(\boldsymbol{C}, K)$ one has*

$$(\tilde{T}(\gamma(t_1, t_2, t_3))f)(z) = \{\exp i \operatorname{Re}[\bar{\lambda}(t_3 + t_2 z)]\} f(z + t_1) \quad a.e.$$

2. **The irreducible representations.** In [12] Kirillov has shown how to compute all the irreducible representations of a simply-connected, connected, nilpotent real Lie group. One can slightly modify Kirillov's technique to get the same construction in the case of complex Lie groups. In this section we perform this construction for the nilpotent part of G_2, i.e., for the simply connected, connected, nilpotent complex Lie group V consisting of the matrices v of (1.5.2). There are in this classification five series of irreducible representations determined by the size of

the kernel; in particular, the representations in Series A have kernel of real dimension 0 or 1; those in Series B, real dimension either 2 or 3; Series C, 4 or 5; Series D, 6 or 7; Series E, 8, 9, 10, or 11. The complete classification is as follows:

THEOREM 2.2.1. *The irreducible representations of V, to within unitary equivalence, fall into the five series below.*

Series A. *These representations $T^{(\mu,\lambda)}$ are indexed by $(\mu, \lambda) \in \boldsymbol{C}^2$ with $\lambda \neq 0$, and act in $L^2(\boldsymbol{C}^2)$ by the formula*

$$(2.2.1)\quad \begin{aligned}&(T^{(\mu,\lambda)}(v(v_1, \ldots, v_6))f)(z, w)\\ &= \exp i \operatorname{Re} \{\bar{\mu}v_1 + \bar{\lambda}(v_6 - v_3v_4 - v_1v_3^2) + \bar{\lambda}z(v_5 - v_1v_4 - 2v_1^2v_3)\\ &\quad - 3\bar{\lambda}w(v_4 + v_1v_3) - \bar{\lambda}(v_1^3z^2 + 3v_1^2zw + 3v_1w^2)\}f(z+v_2, w+v_3+v_1z), \quad a.e.\end{aligned}$$

Series B. *These representations $T_B^{(\alpha,\beta,\gamma)}$ are indexed by $(\alpha, \beta, \gamma) \in \boldsymbol{C}^3$ with $\gamma \neq 0$, and act in $L^2(\boldsymbol{C})$ by the formula*

$$\begin{aligned}&(T_B^{(\alpha,\beta,\gamma)}(v(v_1, \ldots, v_6))f)(z)\\ &= \exp i \operatorname{Re} \{\bar{\alpha}v_2 + \bar{\beta}(v_3 - v_1v_2 - v_2z) + \bar{\gamma}[v_5 - v_1v_4 + v_1^2v_3 - v_1^3v_2\\ &\quad - 3z(v_4 - v_1v_3 + v_1^2v_2) + 3z^2(v_3 - v_1v_2) - z^3v_2]\}f(z+v_1), \quad a.e.\end{aligned}$$

Series C. *These representations $T_C^{(\alpha,\beta)}$ are indexed by $(\alpha, \beta) \in \boldsymbol{C}^2$ with $\beta \neq 0$, and act in $L^2(\boldsymbol{C})$ by the formula*

$$\begin{aligned}&(T_C^{(\alpha,\beta)}(v(v_1, \ldots, v_6))f)(z)\\ &= \exp i \operatorname{Re} \{\bar{\alpha}v_2 + \bar{\beta}(v_4 - v_1v_3 + v_1^2v_2) - 2\bar{\beta}z(v_3 - v_1v_2) + \bar{\beta}v_2z^2\}f(z+v_2), \quad a.e.\end{aligned}$$

Series D. *These representations $T_D^{(\gamma)}$ are indexed by complex numbers $\gamma \neq 0$ and act in $L^2(\boldsymbol{C})$ by the formula*

$$(T_D^{(\gamma)}(v(v_1, \ldots, v_6))f)(z) = \exp i \operatorname{Re} \{\bar{\gamma}(v_3 + zv_2)\}f(z+v_1), \quad a.e.$$

Series E. *These representations $T_E^{(\alpha,\beta)}$ are one-dimensional, are indexed by $(\alpha, \beta) \in \boldsymbol{C}^2$, and are given by the formula*

$$T_E^{(\alpha,\beta)}(v(v_1, \ldots, v_6)) = \exp i \operatorname{Re} (\bar{\alpha}v_1 + \bar{\beta}v_2).$$

REMARKS. The representations of Series A are sufficient to describe the Plancherel transform of V. It is constructive to note the explicit form of these representations on the generating complex one-parameter subgroups of V. These are the subgroups V_1 and V_2 consisting respectively of the elements $v_1 = v(v_1, 0, 0, 0, 0, 0)$ and $v_2 = v(0, v_2, 0, 0, 0, 0)$, and the operators of the representation $T^{(\mu,\lambda)}$ corresponding to these elements act on $f \in L^2(\boldsymbol{C}^2)$ by the formulae

$$(2.2.2)\quad \begin{aligned}&(T^{(\mu,\lambda)}(v_1)f)(z, w)\\ &= \exp i \operatorname{Re} \{\bar{\mu}v_1 + \bar{\lambda}z^{-1}[(w+v_1z)^3 - w^3]\}f(z, w+v_1z), \quad \text{a.e.};\end{aligned}$$

$$(2.2.2')\quad (T^{(\mu,\lambda)}(v_2)f)(z, w) = f(z+v_2, w), \quad \text{a.e.}$$

The fact that the operators (2.2.2) involve both multiplications and (generalized) translations is manifested in a complicated Plancherel transform and introduces significant difficulties in the analysis of the principal series of representations of G.

We shall outline below the procedure for computing the representations in Series A, the other series being computed by a repetition of the technique.

V has two generating complex one-parameter subgroups

$$V_1 = \{v_1 = v(v_1, 0, 0, 0, 0, 0) \in V \mid v_1 \in \boldsymbol{C}\}; \qquad (2.2.3)$$
$$V_2 = \{v_2 = v(0, v_2, 0, 0, 0, 0) \in V \mid v_2 \in \boldsymbol{C}\};$$

a center Z of complex dimension one given by

$$Z = \{v_6 = v(0, 0, 0, 0, 0, v_6) \in V \mid v_6 \in \boldsymbol{C}\}; \qquad (2.2.4)$$

a complex one-parameter subgroup

$$V_5 = \{v_5 = v(0, 0, 0, 0, v_5, 0) \in V \mid v_5 \in \boldsymbol{C}\} \qquad (2.2.5)$$

with the property that the centralizer of V_5 in V is a normal subgroup V_2' of complex codimension one given by

$$V_2' = \{v \in V \mid v_2 = 0\}. \qquad (2.2.6)$$

Then V is the semidirect of V_2' and V_2; i.e., if $v = v(v_1, \ldots, v_6) \in V$, then

$$v = v_2' v_2 \quad \text{with} \quad v_2' = v(v_1, 0, v_3, v_4, v_5, v_6) \in V_2'. \qquad (2.2.7)$$

The subgroup of V generated by V_2 and V_5 consists of the elements $v(0, v_2, 0, 0, v_5, v_6)$, and is isomorphic to the standard three-dimensional nilpotent group Γ under the map $v(0, v_2, 0, 0, v_5, v_6) \to \gamma(v_2, v_5, v_6)$.

Let T be an irreducible representation of V in the space H. T acts by scalars on Z. Let us suppose T is not trivial on Z; i.e., there is a $\lambda \neq 0$ in $\boldsymbol{C}$ such that

$$T(v_6) = \exp i \operatorname{Re}(\bar{\lambda} v_6) I \qquad \text{for all } v_6 \in Z. \qquad (2.2.8)$$

(Note that if T is trivial on Z, T is determined by a representation of the lower-dimensional group V/Z.) Thus, by considering the restriction of T to the forementioned three-dimensional subgroup one sees from Lemma 2.1.6 that H may be taken to be $L^2(\boldsymbol{C}, K)$, K a separable Hilbert space, and that $T(v_2) = \tau_{v_2}$, $T(v_5) = M_{\lambda \bar{v}_5}$ for all $v_2 \in V_2$, $v_5 \in V_5$. (We are, of course, identifying v_2 and v_5 with complex numbers.) Since the elements of V_2' commute with those of V_5, Lemma 2.1.1 shows that $T(v_2')$ is the direct integral of an operator-valued function $z \to \alpha(v_2', z)$ on $\boldsymbol{C}$. From (2.2.7) one sees that T is a multiplier representation acting in $L^2(\boldsymbol{C}, K)$ by $(T(v)f)(z) = \alpha(v_2', z) f(z + v_2)$, a.e. Hence, T may be realized as a representation induced by a representation U of V_2'; i.e., according to formula (2.1.1) $T(v)$ acts on $f \in L^2(\boldsymbol{C}, K)$ as

$$(T(v)f)(z) = U(z v_2'(-z)) f(z + v_2), \quad \text{a.e.}; \qquad (2.2.9)$$

here we have written z for the element $v(0, z, 0, 0, 0, 0)$ of V_2. By Lemma 2.1.2, U is irreducible.

Thus, the problem of computing all the irreducible representations T of V satisfying (2.2.8) with $\lambda \neq 0$ reduces to the problem of computing all irreducible representations U of the subgroup V_2' which act on Z by formula (2.2.8) and induce mutually inequivalent representations of V.

Now, V_2' has its center generated by V_5 and Z. Thus, there is a $\mu \in C$ such that $U(v_5 v_6) = \exp i \operatorname{Re} \{\bar{\mu} v_5 + \bar{\lambda} v_6\} I$ for all $v_5 \in V_5$, $v_6 \in Z$. Next, apply Lemma 2.1.3 with $N = V_2'$, $B = V_2$ and $\tilde{b} = v(0, -\bar{\mu}/\bar{\lambda}, 0, 0, 0, 0) \in V_2'$, from which we see that, to within unitary equivalence, we may suppose $\mu = 0$. I.e., T is induced by an irreducible representation U of V_2' which is trivial on V_5 and nontrivial on Z. In Kirillov's terminology we call such representations U of V_2' *admissible*. If U and U' are inequivalent admissible representations, then the representations T and T' of V induced by U and U' are inequivalent. For, if not there is a unitary operator A such that $AT = T'A$, but T and T' agree on V_2 and V_5, so by Lemma 2.1.1 A is in fact the direct integral of a constant operator valued function $z \to A(z) = A_0$. Since A_0 intertwines U and U', we have established a contradiction.

Thus, it suffices to compute the inequivalent admissible representations U of V_2'. Let $\hat{V}$ denote the quotient group V_2'/V_5, and let ϕ denote the canonical map of V_2' onto $\hat{V}$. Since U is trivial on V_5, one can define a representation $\hat{U}$ of $\hat{V}$ by the formula

$$\hat{U} \circ \phi = U. \tag{2.2.10}$$

$\hat{V}$ is a group of complex dimension four and has a center $\phi(Z)$ of complex dimension one. Set $\hat{v} = \phi(v_2') = \hat{v}(v_1, v_3, v_4, v_6)$, where $v_2' = v(v_1, 0, v_3, v_4, v_5, v_6)$. v_1, v_3, v_4, v_5 are global complex coordinates on $\hat{V}$ (i.e., $\hat{V}$ is being thought of geometrically as a slice submanifold V_2'). Since $\hat{U}$ is irreducible and acts on the center of $\hat{V}$ by (2.2.8), one can apply to $\hat{U}$ the same procedure used to reduce T. Indeed, $\hat{V}$ is the semidirect product of the normal abelian subgroup $\hat{V}_3' = \{\hat{v} \in \hat{V} \mid v_3 = 0\}$ and the complex one-parameter subgroup $\hat{V}_3 = \{\hat{v}_3 = \hat{v}(0, v_3, 0, 0) \mid v_3 \in C\}$, and $\hat{U}$ is induced by a one-dimensional representation $S^{(\mu,\lambda)}$ of $\hat{V}_3'$ of the form

$$S^{(\mu,\lambda)}(\hat{v}(v_1, 0, v_4, v_6)) = \exp i \operatorname{Re} (\bar{\mu} v_1 + \bar{\lambda} v_4) \tag{2.2.11}$$

for some $\mu \in C$. Thus, $\hat{U} = \hat{U}^{(\mu,\lambda)}$ acts in $L^2(C)$ by the formula

$$(\hat{U}(\hat{v})f)(w) = S^{(\mu,\lambda)}(\hat{w}\hat{v}_3'\hat{w}^{-1}) f(w + v_3), \quad \text{a.e.}, \tag{2.2.12}$$

where $\hat{w} = \hat{v}(0, w, 0, 0) \in \hat{V}_3$ and

$$\hat{v}_3' = v(v_1, 0, v_4 + v_1 v_3, v_6 - v_3 v_4 - v_1 v_3^2) \in \hat{V}_3'.$$

The representations $\hat{U}^{(\mu,\lambda)}$ are inequivalent and the representations $U^{(\mu,\lambda)}$ of V_2' defined by (2.2.10) exhaust all the admissible representations. Substitution of (2.2.12) into (2.2.9) and the identification of $L^2(C, L^2(C))$ with $L^2(C^2)$ yield the representations $T^{(\mu,\lambda)}$ of Series A.

3. **The Plancherel transform.** As mentioned earlier, the Plancherel transform on V can be expressed in terms of the representations $T^{(\mu,\lambda)}$ in Series A. For convenience of notation let $\Omega=\{\omega=(\mu, \lambda)\in C^2 \mid \lambda\neq 0\}$. Speaking formally, the Plancherel formula for V takes the form

$$\int_V |\phi(v)|^2\, dv = \int_\Omega \operatorname{tr}\,(T^{(\omega)}(\phi)T^{(\omega)}(\phi)^*)\, d\omega \tag{2.3.1}$$

where $d\omega=9(2\pi)^{-4}|\lambda|^4\, d\lambda\, d\mu$. I.e., the map $\phi\to T(\phi)$ takes $L^2(V)$ unitarily onto $L^2(\Omega, \mathcal{H}, d\omega)$ where $\mathcal{H}$ denotes the Hilbert space of Hilbert-Schmidt operators on $L^2(C^2)$ and $(T(\phi))(\omega)=T^{(\omega)}(\phi)=\int_V \phi(v)T^{(\omega)}(v)\, dv$. This mapping is called the *Plancherel transform* on V and $d\omega$, the *Plancherel measure.*

The fact that the Plancherel transform for a simply-connected, connected nilpotent real Lie group can be exhibited explicitly was first shown by Dixmier [7], and reformulated by Kirillov [12]. Although one can develop the Plancherel transform for V by either of these general techniques, we choose a more classical computational approach here.

For convenience we rewrite (2.2.1) as

$$\begin{aligned}
&\text{(i)} && (T^{(\omega)}(v)f)(\zeta) = m(\omega; v; \zeta)f(\zeta\cdot v) \quad \text{a.e., where}\\
&\text{(ii)} && v = v(v_1,\ldots,v_6)\in V,\ \omega = (\mu,\lambda)\in\Omega,\ \zeta = (z, w)\in C^2;\\
(2.3.2)\quad &\text{(iii)} && m(\omega; v; \zeta) = \exp i\,\mathrm{Re}\,\{\bar\mu v_1+\bar\lambda(v_6-v_3v_4-v_1v_3^2)+\bar\lambda z(v_5-v_1v_4-2v_1^2v_3)\\
&&& \qquad -3\bar\lambda w(v_4+v_1v_3)-\bar\lambda(v_1^3z^2+3v_1^2zw+3v_1w^2)\};\ \text{and}\\
&\text{(iv)} && \zeta\cdot v = (z+v_2, w+v_3+v_1z).
\end{aligned}$$

We note that whenever f is a continuous function, $(\omega, v, \zeta)\to(T^{(\omega)}(v)f)(\zeta)$ is continuous.

As is well known, one extends $T^{(\omega)}$ to the "group ring" $L^1(V)$ by the formula

$$T^{(\omega)}(\phi) = \int_V \phi(v)T^{(\omega)}(v)\, dv$$

for $\phi\in L^1(V)$. $\phi\to T^{(\omega)}(\phi)$ is a $*$-representation of the algebra $L^1(V)$. It plays the role of the "Fourier transform of ϕ evaluated at ω." By an application of Fubini's theorem on the double integral $(T^{(\omega)}(\phi)f\,|\,g)$, and the dominated convergence theorem, one sees that

$$(\omega; z, w)\to(T^{(\omega)}(\phi)f)(z, w) = \int_V \phi(v)(T^{(\omega)}(v)f)(z, w)\, dv$$

is a continuous function whenever $f\in C_0(C^2)=\{$continuous functions on C^2 with compact support$\}$. Thus, for $f\in C_0(C^2)$, $\phi\in L^1(V)$,

$$(T^{(\omega)}(\phi)f)(z, w) = \int_V \phi(v)m(\omega; v; \zeta)f(\zeta\cdot v)\, dv. \tag{2.3.3}$$

If ϕ is a function on V, let us adopt the notation ϕ_* for the expression of ϕ in coordinates $v_1,\ldots,v_6$; and let us denote by $S=S(V)$ the class of functions ϕ on V

for which ϕ_* is in Schwartz' class of infinitely differentiable functions on $\boldsymbol{R}^{12}$, all of whose derivatives vanish at infinity faster than any rational function.

LEMMA 2.3.1. *For* $\phi \in S, f \in C_0(\boldsymbol{C}^2)$,

$$(T^{(\omega)}(\phi)f)(z, w) = \int_{C^2} K_\phi^{(\omega)}(z, w; v_2, v_3)f(v_2, v_3)\, dv_2\, dv_3 \tag{2.3.4}$$

where

$$\begin{aligned} K_\phi^{(\omega)}(z, w; v_2, v_3) = \int_{C^4} & \phi_*(v_1, v_2-z, v_3-w-v_1z, v_4, v_5, v_6) \\ & \cdot \exp i \operatorname{Re} \{v_1[\bar{\mu}-\bar{\lambda}(v_3^2+w^2+v_3w)] \\ & -3\bar{\lambda}v_4(v_3+2w)+\bar{\lambda}(zv_5+v_6)\}\, dv_1\, dv_4\, dv_5\, dv_6. \end{aligned} \tag{2.3.5}$$

Furthermore, the function

$$(\omega; z, w; v_2, v_3) \to K_\phi(\omega; z, w; v_2, v_3) = K_\phi^{(\omega)}(z_1w; v_2, v_3)$$

is continuous.

Proof. Equations (2.3.4) and (2.3.5) follow immediately from (2.3.3) upon writing the integral over V as an iterated integral and making the substitution $v_2 \to v_2 - z$, $v_3 \to v_3 - w - v_1 z$. To see what equation (2.3.5) really means, let us introduce the partial Fourier transforms $\mathfrak{F}_j$ $(j=1, \ldots, 6)$; e.g., if $\phi \in S$, we have $(\mathfrak{F}_1\phi)(v) = \int_C \phi_*(x, v_2, \ldots, v_6) \exp i \operatorname{Re}(\bar{x}v_1)\, dx$. Also, we must consider a generalized translation operator $A_{z,w}$ defined on $\phi \in S$ by $(A_{z,w}\phi)(v) = \phi_*(v_1, v_2 - z, v_3 - w - v_1 z, v_4, v_5, v_6)$. The function

$$(z, w; v) \to \Phi(z, w; v) = (\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6 A_{z,w}\phi)(v)$$

is easily seen to be continuous (in fact, it is much smoother), and indeed, since

$$K_\phi(\omega; z, w; v_2, v_3) = \Phi(z, w; \mu - \lambda\overline{(v_3^2+w^2+v_3w)}, v_2, v_3, -3\lambda\overline{(v_3+2w)}, \lambda\bar{z}, \lambda)$$

one sees that K_ϕ is continuous.

LEMMA 2.3.2. *For* $\phi \in S$, $\omega \in \Omega$, $T^{(\omega)}(\phi)$ *is of trace class.*

Proof. By (2.3.4), $T^{(\omega)}(\phi)$ is an integral operator on $L^2(\boldsymbol{C}^2)$ with kernel $K_\phi^{(\omega)}$. We first show $K_\phi^{(\omega)}$ is square integrable. Indeed (2.3.5) shows that

$$|K_\phi^{(\omega)}(z, w; v_2, v_3)| \leqq \int_C |\psi(v_1, v_2-z, v_3-w-v_1z, -3\lambda\overline{(v_3+2w)}, \lambda\bar{z}, \lambda)|\, dv$$

where $\psi = \mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\phi$, which is rapidly decreasing at infinity. By the obvious change of variables and Minkowski's integrated inequality,

$$\begin{aligned} & \left\{\int_{C^4} |K_\phi^{(\omega)}(z, w; v_2, v_3)|^2\, dz\, dw\, dv_2\, dv_3\right\}^{1/2} \\ & \qquad \leqq \int_C \left\{\int_{C^4} |\psi(v_1, v_2, v_3, -3\lambda\bar{w}, \lambda\bar{z}, \lambda)|^2\, dv_2\, dv_3\, dz\, dw\right\}^{1/2} dv_1 < \infty. \end{aligned}$$

Thus, $T^{(\omega)}(\phi)$ is a Hilbert-Schmidt operator. Moreover,

$$|K_\phi^{(\omega)}(z, w; z, w)| \leqq \int_C |\psi(v_1, 0, -v_1 z, -9\lambda\bar{w}, \lambda\bar{z}, \lambda)|\, dv_1;$$

from which it follows that $\int_{C^2} |K_\phi^{(\omega)}(z, w; z, w)|\, dz\, dw < \infty$, so $T^{(\omega)}(\phi)$ has finite trace.

It is now clear how to make rigorous sense out of formula (2.3.1). Indeed, for $\phi \in S$, the function $\omega \to \|K_\phi^{(\omega)}\|_2^2$ is measurable on Ω and

$$\operatorname{tr}(T^{(\omega)}(\phi)T^{(\omega)}(\phi)^*) = \|K_\phi^{(\omega)}\|_2^2.$$

Thus, one must show that the measure $d\omega$ on Ω has the property that (2.3.1) holds for $\phi \in S$ and that the subspace $\mathcal{M} = \{T(\phi) \mid \phi \in S\}$ is dense in $L^2(\Omega, \mathcal{K}, d\omega)$. We begin with the following standard lemma.

LEMMA 2.3.3. *If the inversion formula*

$$\phi(e) = \int_\Omega \operatorname{tr}(T^{(\omega)}(\phi))\, d\omega \tag{2.3.6}$$

holds for all $\phi \in S$ (where $e = v(0, \ldots, 0)$ is the identity element of V), then the Plancherel formula (2.3.1) *holds for all $\phi \in S$.*

Proof. For $\phi \in S$ define ϕ^* by $\phi^*(v) = \bar{\phi}(v^{-1})$. Then, assuming (2.3.6), we get

$$\int_V |\phi(v)|^2\, dv = (\phi * \phi^*)(e) = \int_\Omega \operatorname{tr}(T^{(\omega)}(\phi * \phi^*))\, d\omega = \int_\Omega \operatorname{tr}(T^{(\omega)}(\phi)T^{(\omega)}(\phi)^*)\, d\omega,$$

which is (2.3.1).

LEMMA 2.3.4. *Let $d\omega = 9(2\pi)^{-4}|\lambda|^4\, d\lambda\, d\mu$. Then for any $\phi \in S$ the function $\omega \to \operatorname{tr}(T^{(\omega)}(\phi))$ is absolutely integrable with respect to $d\omega$ and* (2.3.6) *holds.*

Proof. Let $\phi \in S$. Then $\operatorname{tr}(T^{(\omega)}(\phi)) = \int_{C^2} K_\phi^{(\omega)}(z, w; z, w)\, dz\, dw$ with $K_\phi^{(\omega)}$ as in (2.3.5). Define ψ by $\psi(v_1, z, v_4, v_5, v_6) = \phi_*(v_1, 0, -v_1 z, v_4, v_5, v_6)$. As a function of the five displayed variables ψ is in Schwartz's class on C^5, and

$$\operatorname{tr}(T^{(\omega)}(\phi)) = \int_{C^2} (\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi)(\mu - 3\lambda\bar{w}^2, z, -9\lambda\bar{w}, \lambda\bar{z}, \lambda)\, dz\, dw.$$

Thus,

$$\int_\Omega |\operatorname{tr}(T^{(\omega)}(\phi))|\, d\omega \leqq \int_\Omega \int_{C^2} |\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi(\mu - 3\lambda\bar{w}^2, z, -9\lambda\bar{w}, \lambda\bar{z}, \lambda)|\, dz\, dw\, |\lambda|^2\, d\lambda\, d\mu$$

which is finite because $\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi$ can be majorized by the multiplicative inverse of a polynomial in z, w, λ, μ of sufficiently high degree. Thus, $\omega \to \operatorname{tr}(T^{(\omega)}(\phi))$ is absolutely integrable. The remainder of the proof involves an iteration of the Plancherel formula on the additive group of C; i.e., the formula

$$\phi_*(v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_6) = (2\pi)^{-1} \int_C (\mathfrak{F}_j\phi_*)(v_1, \ldots, v_{j-1}, t, v_{j+1}, \ldots, v_6)\, dt,$$

which holds for $\phi \in S$. Let ψ be as above. The following computation, which completes the proof of the lemma, hinges on the fact that $\psi(0, z, v_4, v_5, v_6) = \psi(0, 0, v_4, v_5, v_6)$ for all $z, v_4, v_5, v_6 \in C$. Thus, with $\alpha = 9(2\pi)^{-4}$,

$$\begin{aligned}
\int_\Omega \operatorname{tr}(T^{(\omega)}(\phi))\, d\omega &= \alpha \int_{C^2} \operatorname{tr}(T^{(\mu,\lambda)}(\phi))\, |\lambda|^4\, d\lambda\, d\mu \\
&= \alpha \int_{C^4} (\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi)(\mu - 3\lambda\bar{w}^2, z, -9\lambda\bar{w}, \lambda\bar{z}, \lambda)\, |\lambda|^4\, dz\, dw\, d\mu\, d\lambda \\
&= \frac{\alpha}{9} \int_{C^4} (\mathfrak{F}_1\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi)(\mu, \lambda^{-1}\bar{z}, w, z, \lambda)\, dz\, dw\, d\mu\, d\lambda \\
&= \frac{1}{9}(2\pi\alpha) \int_{C^3} (\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi)(0, \lambda^{-1}\bar{z}, w, z, \lambda)\, dz\, dw\, d\lambda \\
&= \frac{1}{9}(2\pi\alpha) \int_{C^3} (\mathfrak{F}_4\mathfrak{F}_5\mathfrak{F}_6\psi)(0, 0, w, z, \lambda)\, dz\, dw\, d\lambda \\
&= \psi(0, 0, 0, 0, 0) = \phi(e).
\end{aligned}$$

THEOREM 2.3.1 (PLANCHEREL THEOREM FOR V). *For each $\phi \in S$ and $\omega \in \Omega$, $T^{(\omega)}(\phi) \in \mathcal{K}$, the Hilbert space of Hilbert-Schmidt operators on $L^2(C^2)$, $T^{(\omega)}(\phi)$ is of trace class, and the function $T(\phi)$ defined by $T(\phi)(\omega) = T^{(\omega)}(\phi)$ is in $L^2(\Omega, \mathcal{K}, d\omega)$ where $d\omega = \alpha\, |\lambda|^4\, d\lambda\, d\mu$ and $\alpha = 9(2\pi)^{-4}$. Furthermore, the mapping $\phi \to T(\phi)$ is an isometry of S (as a subspace of $L^2(V)$) onto a dense subspace $\mathcal{M}$ of $L^2(\Omega, \mathcal{K}, d\omega)$, and thus extends by continuity to a unitary map, the Plancherel transform, of $L^2(V)$ onto $L^2(\Omega, \mathcal{K}, d\omega)$.*

REMARK. For computational reasons it will often be convenient to alter the form of the Plancherel transform by making the canonical identification of $\mathcal{K}$ with $L^2(C^4)$. This identifies the Hilbert-Schmidt operator $T^{(\omega)}(\phi)$ with its Hilbert-Schmidt kernel $K_\phi^{(\omega)}$ given by Lemma 2.3.1. It is often a further convenience to identify $L^2(\Omega, \mathcal{K}, d\omega)$ with $L^2(\Omega \times C^4, d\omega \times dt)$ where dt denotes Lebesgue measure on C^4. This latter convention identifies $T(\phi)$ with the function K_ϕ on $\Omega \times C^4$. Thus, we set $\hat{\phi} = K_\phi$ and allow the ambiguity of calling the map $\phi \to \hat{\phi}$ the Plancherel transform. One can then interpret the Plancherel transform as a map from functions on C^6 to functions on C^6; indeed,

$$\hat{\phi}(\mu, \lambda; z, w; \xi, \eta) = K_\phi^{(\mu,\lambda)}(z, w; \xi, \eta) \tag{2.3.7}$$

with $K_\phi^{(\mu,\lambda)}$ given by (2.3.5).

As for the proof of Theorem 2.3.1, all that remains is to prove that $\mathcal{M}$ is dense in $L^2(\Omega, \mathcal{K}, d\omega)$. This follows from the general result for nilpotent groups (see [7, Theorem 4]), but we shall in the next section prove directly that the operator in $L^2(\Omega, \mathcal{K}, d\omega)$ of projection onto the closure of $\mathcal{M}$ is in fact the identity operator. For now, let us simply proceed as if $\phi \to \hat{\phi}$ is unitary.

4. **Some specific transforms.** It is, of course, well known that the Plancherel transform "decomposes" the left and right regular representations of a group. In the next theorem we determine the explicit form of this direct integral decomposition of the regular representations of V. Toward this end we regard the range of the Plancherel transform $\phi \to \hat{\phi}$ as $H = L^2(\Omega; L^2(C^2, dz\, dw) \otimes L^2(C^2, d\xi\, d\eta); d\omega)$; i.e., we think of the function $\hat{\phi}$ of (2.3.7) as a vector-valued function on Ω whose values are functions on the product space $C^2 \times C^2$.

The left and right regular representations L and R of V are defined on $L^2(V)$ by

$$(L(v)\phi)(w) = \phi(v^{-1}w) \quad (R(v)\phi)(w) = \phi(wv), \quad \text{a.e.} \tag{2.4.1}$$

Let $\mathscr{F}$ denote the Plancherel transform and set $\hat{L}(v) = \mathscr{F}L(v)\mathscr{F}^{-1}$ and $\hat{R}(v) = \mathscr{F}R(v)\mathscr{F}^{-1}$.

Proposition 2.4.1. *With the notation as above, the representations $\hat{L}$ and $\hat{R}$ of V act in the Hilbert space H by the formulae:*

$$\hat{L} = \int_\Omega^\oplus (T^{(\omega)} \otimes I_2)\, d\omega, \tag{2.4.2}$$

$$\hat{R} = \int_\Omega^\oplus (I_1 \otimes T^{(-\omega)})\, d\omega, \tag{2.4.3}$$

where I_1 is the identity operator on $L^2(C^2; dz\, dw)$ and I_2 the identity on $L^2(C^2; d\xi\, d\eta)$, $T^{(\omega)}$ is given by (2.3.2), *and* $-\omega = (-\mu, -\lambda)$.

Proof. We prove (2.4.3), the other formula being similarly derived. Let $\phi \in S$, $v = v(v_1, \ldots, v_6) \in V$, and $f \in C_0(C^2)$. As is well known $T^{(\omega)}(R(v)\phi) = T^{(\omega)}(\phi)T^{(\omega)}(v^{-1})$. Thus, by Lemma 2.3.1,

$$\begin{aligned}
&(T^{(\omega)}(R(v)\phi)f)(z, w)\\
&\quad = \int_{C^2} K_\phi^{(\omega)}(z, w; t_2, t_3)(T^{(\omega)}(v^{-1})f)(t_2, t_3)\, dt_2\, dt_3\\
&\quad = \int_{C^2} K_\phi^{(\omega)}(z, w; t_2, t_3)m(\omega; v^{-1}; t_2, t_3)f(t_2 - v_2, t_3 - v_3 - t_2 v_1)\, dt_2\, dt_3\\
&\quad = \int_{C^2} K_\phi^{(\omega)}(z, w; t_2 + v_2, t_3 + v_3 + v_1 t_2)m(\omega; v^{-1}; t_2 + v_2, t_3 + v_3 + v_1 t_2)\\
&\qquad \cdot f(t_2, t_3)\, dt_2\, dt_3.
\end{aligned}$$

From (2.3.2)(iii) and (1.5.4) one verifies that $m(\omega; v^{-1}; t_2 + v_2, t_3 + v_3 + v_1 t_2) = m(-\omega; v; t_2, t_3)$. Hence, it follows from (2.3.4) and (2.3.7) that

$$\begin{aligned}
&(R(v)\phi)^\wedge(\mu, \lambda; z, w; \xi, \eta)\\
&\qquad = m(\omega; v; z, w)\hat{\phi}(-\mu, -\lambda; z, w; \xi + v_2, \eta + v_3 + v_1\xi).
\end{aligned} \tag{2.4.4}$$

I.e., $[\hat{R}(v)\hat{\phi}](\omega) = [I_1 \otimes T^{(-\omega)}(v)](\hat{\phi}(\omega))$. This proves (2.4.3).

Next we consider a representation $c \to D_c$ of the Cartan subgroup C of G which acts in $L^2(V)$. This representation may be thought of as the analogue for V of the representation of the additive group of $\boldsymbol{C}$ by multiplicative dilations. More specifically, the elements $c = c(c_1, c_2)$ of C are given by matrices (1.5.6) and those $v = v(v_1, \ldots, v_6)$ of V by (1.5.2). C acts on V by conjugation; i.e., for $c \in C$, $v \in V$, $c^{-1}vc \in V$. In fact,

$$c^{-1}vc = v(v_1 c_1^{-1}, v_2 c_1 c_2^{-1}, v_3 c_2^{-1}, v_4 c_3, v_5 c_3 c_1^{-1}, v_6 c_3 c_2^{-1}); \tag{2.4.5}$$

and

$$\mu(c) = d(c^{-1}vc)/dv = |c_1^2 c_2^3|^{-4}. \tag{2.4.6}$$

Thus, we define the unitary representation D of C in the space $L^2(V)$ by

$$(D_c\phi)(v) = \mu(c)^{1/2}\phi(c^{-1}vc) \quad \text{a.e.} \tag{2.4.7}$$

Let $\mathscr{F} D_c \mathscr{F}^{-1} = \hat{D}_c$, the Plancherel transform of D_c.

LEMMA 2.4.1. *Consider the unitary representation $c \to E_c$ of C acting in $L^2(\boldsymbol{C}^2)$ by the formula*

$$(E_c f)(z_1, z_2) = |c_1 c_2^{-2}| f(z_1 c_1 c_2^{-1}, z_2 c_2^{-1}), \quad \text{a.e.} \tag{2.4.8}$$

For any representation $T^{(\mu,\lambda)}$ of V in Series A *we have*

$$T^{(\mu,\lambda)}(cvc^{-1}) = E_c T^{(\mu\bar{c}_1, \lambda\bar{c}_1\bar{c}_2^2)}(v) E_c^{-1} \tag{2.4.9}$$

for all $c = c(c_1, c_2) \in C$, $v \in V$.

Proof. The identity (2.4.9) is easily verified from (2.3.2).

Lemma 2.4.1 allows us to compute the representation $\hat{D}$. We first, however, develop some notation. Let $H_1 = L^2(\Omega, d\omega)$, $H_2 = L^2(\boldsymbol{C}^2, dz\, dw)$, $H_3 = L^2(\boldsymbol{C}^2, d\xi\, d\eta)$, and identify $L^2(\Omega \times \boldsymbol{C}^4, d\omega \times dz\, dw\, d\xi\, d\eta)$ with $H_1 \otimes H_2 \otimes H_3$.

PROPOSITION 2.4.2. *Consider the range of the Plancherel transform $\phi \to \hat{\phi}$ to be $H_1 \otimes H_2 \otimes H_3$. Let D be the representation of C defined by* (2.4.7) *and let $\hat{D}$ be its Plancherel transform. Then*

$$\hat{D}_c = F_c \otimes E_c \otimes E_c \tag{2.4.10}$$

where E is the representation of C in $L^2(\boldsymbol{C}^2)$ defined by (2.4.8), *and F is the representation of C in $L^2(\Omega, d\omega)$ defined by*

$$(F_c f)(\mu, \lambda) = |c_1^2 c_2^3|^2 f(\mu\bar{c}_1, \lambda\bar{c}_1\bar{c}_2^2) \quad \text{a.e.} \tag{2.4.11}$$

Proof. It follows immediately from the previous lemma that for $\phi \in L^1(V)$,

$$T^{(\mu,\lambda)}(D_c\phi) = |c_1^2 c_2^3|^2 E_c T^{(\mu\bar{c}_1, \lambda\bar{c}_1\bar{c}_2^2)}(\phi) E_c^{-1}.$$

From (2.3.4) and (2.3.7) one computes the formula

$$\begin{aligned}(\hat{D}_c\hat{\phi})(\mu, \lambda; z, w; \xi, \eta) \\ = |c_1^3 c_2|^2 \hat{\phi}(\mu\bar{c}_1, \lambda\bar{c}_1\bar{c}_2^2; z c_1 c_2^{-1}, w c_2^{-1}; \xi c_1 c_2^{-1}, \eta c_2^{-1}),\end{aligned} \tag{2.4.12}$$

which implies (2.4.10).

Let us now return to the proof of Theorem 2.3.1 in which we left hanging the proof of the ontoness of the Plancherel transform, or equivalently, the denseness of $\mathscr{M}$ in $H=L^2(\Omega, \mathscr{K}, d\omega)$. Let P denote the projection operator in H with the closure of $\mathscr{M}$ as range. The representations L, R, and D leave S invariant. Thus, the representations $\hat{L}$, $\hat{R}$, and $\hat{D}$ on H (rather Hilbert spaces canonically identified with H) commute with P. We shall see that this forces P to be the identity operator on H; or in other words, $\mathscr{M}$ is dense.

First, identify H with $H_1 \otimes H_2$ where $H_1=L^2(\boldsymbol{C}^3, |\lambda|^4\, d\lambda\, d\xi\, d\eta)$, $H_2= L^2(\boldsymbol{C}^3, d\mu\, dz\, dw)$. Since P commutes with the operators $\hat{R}(v(0,0,0,v_4,v_5,v_6))$ for all $v_4, v_5, v_6 \in \boldsymbol{C}$, it follows from Lemma 2.1.1, (a), that P decomposes as follows: There is an operator-valued function $(\lambda, \xi, \eta) \to P(\lambda, \xi, \eta)$ such that for $\phi_1 \in H_1$, $\phi_2 \in H_2$ one has

$$(P(\phi_1 \otimes \phi_2))(\lambda, \xi, \eta; \mu, z, w) = \phi_1(\lambda, \xi, \eta)(P(\lambda, \xi, \eta)\phi_2)(\mu, z, w).$$

Next, one uses the fact that P commutes with $\hat{R}(v(0, v_2, v_3, 0, 0, 0))$ for all $v_2, v_3 \in \boldsymbol{C}$, together with a slight variant of Lemma 2.1.1, (b), to see that the operator-valued function $\lambda \to P(\lambda, \xi, \eta)$ is essentially constant; say $P(\lambda, \xi, \eta)=P(\lambda)$, a.e. (ξ, η). I.e., for $\phi_1 \in H_1$, $\phi_2 \in H_2$, one has that

$$(P(\phi_1 \otimes \phi_2))(\lambda, \xi, \eta; \mu, z, w) = \phi_1(\lambda, \xi, \eta)(P(\lambda)\phi_2)(\mu, z, w), \quad \text{a.e.}$$

Thus, we may consider P as acting in the space $K_1 \otimes K_2$ where $K_1=L^2(\boldsymbol{C}^2, d\xi\, d\eta)$ and $K_2=L^2(\boldsymbol{C}^4, |\lambda|^4\, d\lambda\, d\mu\, dz\, dw)$, and as we have just seen

$$P = I \otimes P_0 \tag{2.4.13}$$

where I denotes the identity on K_1 and P_0 the projection operator on K_2 defined as multiplication by the operator-valued function $\lambda \to P(\lambda)$. At this point we can repeat the same procedure used above with $\hat{R}$ replaced by $\hat{L}$. This yields the fact that

$$P_0 = I \otimes P_{00} \tag{2.4.14}$$

where I is the identity on $L^2(\boldsymbol{C}^2, dz\, dw)$ and P_{00} is a projection operator on $L^2(\boldsymbol{C}^2, |\lambda|^4\, d\mu\, d\lambda)$ defined as multiplication by an operator-valued function $\lambda \to P_{00}(\lambda)$. I.e., if $\phi_1 \in L^2(\boldsymbol{C}, |\lambda|^4\, d\lambda)$ and $\phi_2 \in L^2(\boldsymbol{C}, d\mu)$, we have

$$(P_{00}(\phi_1 \otimes \phi_2))(\mu, \lambda) = \phi_1(\lambda)(P_{00}(\lambda)\phi_2)(\mu), \quad \text{a.e.} \tag{2.4.15}$$

It remains to show that $\lambda \to P_{00}(\lambda)$ is essentially constant. Because P and $\hat{D}$ commute, it follows from the form (2.4.10) of $\hat{D}_c$ that P_{00} and F_c commute for all $c \in C$. In other words, $P_{00}(\lambda)=P_{00}(\lambda\bar{c}_1\bar{c}_2^2)$, a.e. λ. Since c_1 and c_2 are arbitrary nonzero numbers, it follows from the multiplicative version of Lemma 2.1.1, (a), that there is a fixed operator P' on $L^2(\boldsymbol{C}, d\mu)$ such that $P_{00}(\lambda)=P'$ a.e. Hence, from (2.4.13), (2.4.14), and (2.4.15) we see that $P=I \otimes P'$, where I denotes the identity on $L^2(\boldsymbol{C}^5, |\lambda|^4\, d\lambda\, d\xi\, d\eta\, dz\, dw)$ and P' is some projection operator on $L^2(\boldsymbol{C}, d\mu)$. So, in order to complete the proof of Theorem 2.3.1 it is sufficient to show that P' is the identity operator. To see that this is the case, one observes that since P commutes with $\hat{R}(v(v_1, 0, 0, 0, 0, 0))$ for all $v_1 \in C$, it follows from Lemma 2.1.1, (a), that P' is multiplication by a scalar-function $\mu \to p(\mu)$. On the other hand, since P commutes with $\hat{D}$, for each nonzero $c_1 \in C$ we have $p(\mu) = p(\mu\bar{c}_1)$, a.e. μ. Again it follows from the multiplicative version of Lemma 2.1.1, (b), that $\mu \to p(\mu)$ is essentially constant; say $p(\mu) = a$ a.e. However, since P' is a projection operator, p must have the two point set $\{0, 1\}$ as its essential range. It follows that $p(\mu) = 1$ a.e., and that P' is the identity operator on $L^2(C, d\mu)$.

CHAPTER III. REPRESENTATION THEORY ON G

1. **The principal series.** The principal series of representations on a complex semisimple Lie group can be formulated as the family of representations induced from unitary characters of a maximal solvable subgroup. As we have already seen, it is rewarding to realize induced representations as multiplier representations; so we now briefly explain the general setting for this realization.

Let G be a separable locally compact group with two closed subgroups B and V such that $B \cap V = \{e\}$ and $BV = X$ is an open subset of G whose complement has Haar measure zero. (According to Theorem 1.5.3 these conditions hold for the group G of type G_2.) Then the map $(b, v) \to bv$ is a homeomorphism of $B \times V$ onto X, V may be identified with an open dense subset of the homogeneous space $B\backslash G$, and right Haar measure $d_r x$ on G is absolutely continuous with respect to a product of right Haar measures on B and V. We make this latter point more specific. Define left and right Haar measures $d_l b$ and $d_r b$ on B related by

$$\int_B f(b)\, d_l b = \int_B f(b^{-1})\, d_r b. \tag{3.1.0}$$

The *modular function* δ_B of B is the Radon-Nikodym derivative of $d_l b$ with respect to $d_r b$. Similarly, define left and right Haar measures $d_l x$ and $d_r x$ of G related in the same way, and let δ_G be the modular function of G. One then considers the positive character μ of B defined for $b \in B$ by

$$\mu(b) = \delta_G(b)/\delta_B(b), \tag{3.1.1}$$

and shows that Haar measure $d_r v$ on V can be chosen so that $d_r x = \mu^{-1}(b) d_r b \times d_r v$; i.e., for any nonnegative Borel measurable function f on G

$$\int_G f(x)\, d_r x = \int_V d_r v \int_B f(bv)\mu^{-1}(b)\, d_r b. \tag{3.1.2}$$

For proofs of the above facts, as well as a more detailed explanation of the material to follow, we refer to [13].

Let λ be a unitary character of B. We explain how the representation of G induced by λ can be realized as a multiplier representation. Let π_B and π_V denote the coordinate projections of $X=BV$ onto B and V respectively. For each $a \in G$, the set

$$V^a = \{v \in V \mid va \in X\} \tag{3.1.3}$$

is open in V and its complement has Haar measure zero. Thus, for $v \in V^a$ we have $va=\pi_B(va)\pi_V(va)$. For $v \in V^a$ set

$$v\tilde{a} = \pi_V(va). \tag{3.1.3$'$}$$

Define a "multiplier" function $m_\lambda(v, a)$ by

$$\begin{aligned} m_\lambda(v, a) &= \lambda(\pi_B(va))\mu^{1/2}(\pi_B(va)), \qquad v \in V^a, \\ &= 0, \qquad v \in V - V^a. \end{aligned} \tag{3.1.4}$$

Then define the operator $T(a, \lambda)$ on $L^2(V)$ by

$$(T(a, \lambda)f)(v) = m_\lambda(v, a)f(v\tilde{a}) \quad \text{a.e.} \tag{3.1.5}$$

$T(a, \lambda)$ is a unitary operator on $L^2(V)$ and $a \to T(a, \lambda)$ is a unitary representation of G unitarily equivalent to the representation of G induced by λ.

We now specialize the preceding discussion to the group G of type G_2. As we mentioned above, the role of B is played by the group B of (1.5.7) and that of V by the nilpotent part of G. The following proposition particularizes the above general facts.

PROPOSITION 3.1.1. (i) *Let C^* denote the multiplicative group of C. The Cartan subgroup C of elements* (1.5.6) *can be identified with* $(C^*)^2$ *and has Haar measure* $dc=|c_1c_2|^{-2}\,dc_1\,dc_2$ *where dc_1 and dc_2 are Lebesgue measures.*

(ii) *V is unimodular and has Haar measure $dv=dv_1\cdots dv_6$, where $v=v(v_1, \ldots, v_6)$ as in* (1.5.2).

(iii) *$N={}^tV$ has Haar measure $dn=dv$ where $n={}^tv$ as in* (1.5.5).

(iv) *The group C acts on V and N as a group of automorphisms by $v \to c^{-1}vc$ and $n \to c^{-1}nc$ respectively. With μ given by* (2.4.6) *one has the formulae*

$$\begin{aligned} \int_V f(c^{-1}vc)\,dv &= \mu(c^{-1})\int_V f(v)\,dv, \\ \int_N f(c^{-1}nc)\,dn &= \mu(c)\int_N f(n)\,dn. \end{aligned} \tag{3.1.6}$$

(v) *The group B is the semidirect product of C and N. With the notation $b=cn$ for the generic element of B we have Haar measures on B given by product measures in the form*

$$d_lb = dc\,dn, \quad d_rb = \mu(c)\,dc\,dn. \tag{3.1.7}$$

These measures are related by formula (3.1.0).

(vi) *G is unimodular, and with $X=BV=CNV$ as at the outset of this section formula* (3.1.2) *takes the form*

$$\int_G f(x)\,dx = \int_C\int_N\int_V f(cnv)\,dc\,dn\,dv; \tag{3.1.8}$$

i.e., $dx=dc\,dn\,dv$ is a Haar measure on G.

Proof. (i), (ii), and (iii) are obvious, and (iv) follows from (2.4.6). Let us prove (v). Let $b_0=c_0n_0$ with $c_0\in C$, $n_0\in N$ and let $f\in C_0(B)$. Then with $d_lb=dc\,dn$ we have

$$\begin{aligned}\int_B f(b_0b)\,d_lb &= \int_C\int_N f(c_0n_0cn)\,dc\,dn = \int_C dc\int_N f(c_0c(c^{-1}n_0c)n)\,dn\\ &= \int_C dc\int_N f(c_0cn)\,dn = \int_N dn\int_C f(c_0cn)\,dc\\ &= \int_N\int_C f(cn)\,dc\,dn = \int_B f(b)\,d_lb.\end{aligned}$$

Thus, d_lb is a left Haar measure. Similarly, one uses (3.1.6) to show that $d_rb=\mu(c)\,dc\,dn$ is a right Haar measure.

To prove (vi) we first remark that semisimple groups are well known to be unimodular. Thus from (3.1.1), (3.1.2), and (3.1.6) we have

$$\begin{aligned}\int_G f(x)\,d_rx &= \int_G f(x)\,dx = \int_V dv\int_B f(bv)\,\delta_B(b)\,d_rb\\ &= \int_V dv\int_B f(bv)\,d_lb = \int_V dv\int_C\int_N f(cnv)\,dc\,dn\\ &= \int_C\int_N\int_V f(cnv)\,dc\,dn\,dv,\end{aligned}$$

which shows that $dx = dc\,dn\,dv$ is a Haar measure on G.

Next we elucidate the form of (3.1.5) when G is the group of type G_2. As a first step we set down the action of a set of generators of G on the space V. According to Proposition 1.5.2, G is generated by the subgroups C and V together with the simple Weyl reflections p and q, and the action on V is expressed by (3.1.3) and (3.1.3)′. We summarize this action in the following proposition, the validity of which is verified by straightforward multiplication of the matrices involved (viz., (1.5.2), (1.5.6), and (1.5.10)).

PROPOSITION 3.1.2. (i) *If* $v_0\in V$, *then* $V^{v_0}=V$ *and* $v\tilde{v}_0=vv_0$.

(ii) *For* $c=c(c_1,c_2)\in C$, $V^c=V$ *and with* $v=v(v_1,\dots,v_6)\in V$ *one has*

$$\begin{aligned}&\pi_B(vc) = c \quad \textit{and}\\ &v\tilde{c} = v(v_1c_1^{-1}, v_2c_1c_2^{-1}, v_3c_2^{-1}, v_4c_1^{-1}c_2^{-1}, v_5c_1^{-2}c_2^{-1}, v_6c_1^{-1}c_2^{-2}).\end{aligned}\tag{3.1.9}$$

(iii) *With p and q given by* (1.5.10) *we have*: $V^p=\{v\in V\mid v_1\neq 0\}$ *and for* $v\in V^p$,

$$\begin{aligned}\pi_B(vp) &= c(-v_1^2, v_1^{-1})n(-v_1,0,0,0,0,0), \quad \textit{and}\\ v\tilde{p} &= v(w_1,\dots,w_6), \quad \textit{where}\\ w_1 &= v_1^{-1},\ w_2 = -(v_5+2v_1v_4+v_3v_1^2),\\ w_3 &= -(v_4+v_1v_3),\ w_4 = v_4v_1^{-1},\\ w_5 &= -v_2+v_3v_1^{-1}-v_4v_1^2,\\ w_6 &= -v_6+v_2v_5-v_3v_4+2v_1v_2v_4+v_1^2v_2v_3-v_3^2v_1.\end{aligned}\tag{3.1.10}$$

$V^q = \{v \in V \mid v_2 \neq 0\}$ *and for* $v \in V^q$,

$$\pi_B(vq) = c(v_2^{-1}, v_2)n(0, v_2, 0, 0, 0, 0), \quad \text{and} \tag{3.1.11}$$
$$v\tilde{q} = v(v_1v_2 - v_3, -v_2^{-1}, v_3v_2^{-1}, v_4, v_2v_5 - v_6, v_6v_2^{-1}).$$

Now that we have described explicitly the action on V, we focus on the multiplier m_λ in (3.1.5). Since B is the semidirect product of C and the normal subgroup N, the commutator subgroup of B is contained in N. Conversely, since it is easily verified that the generating complex one-parameter subgroups of N are in the commutator subgroup, one sees that N is precisely the commutator subgroup of B. Thus, if ρ is any character of B, N is contained in the kernel of ρ; i.e., the characters of B are uniquely determined by the characters of the Cartan subgroup C. One can choose very convenient homogeneous coordinates on C and the character group of C as follows:

Proposition 3.1.3. *Choose "homogeneous" coordinates* c_1, c_2, c_3 *on* C *by* $c = c(c_1, c_2, c_3) = c(c_1, c_2)$ *as in* (1.5.6), *where* $c_1c_2c_3 = 1$. *Then any character* χ *of* C *has the form*

$$\chi(c) = |c_1|^{s_1}|c_2|^{s_2}|c_3|^{s_3}[c_1]^{n_1}[c_2]^{n_2}[c_3]^{n_3} \tag{3.1.12}$$

for some triples s_1, s_2, s_3 *of complex numbers and* n_1, n_2, n_3 *of integers such that* $s_1+s_2+s_3=0$ *and* $0 \leqq n_1+n_2+n_3 < 3$, *where* $[z]=z/|z|$ *denotes the amplitude of the complex number* z. *Set* $\hat{s}=(s_1, s_2, s_3) \in \boldsymbol{C}^3$, $\hat{n}=(n_1, n_2, n_3) \in \boldsymbol{Z}^3$, *and* $\chi=\chi(\hat{s}, \hat{n})$. *Then the map* $(\hat{s}, \hat{n}) \to \chi(\hat{s}, \hat{n})$ *identifies* $\{(\hat{s}, \hat{n}) \in \boldsymbol{C}^3 \times \boldsymbol{Z}^3 \mid s_1+s_2+s_3=0$ *and* $0 \leqq n_1+n_2+n_3 < 3\}$ *with the character group of* C, *and the unitary characters correspond to those pairs for which* $\operatorname{Re}(s_1) = \operatorname{Re}(s_2) = \operatorname{Re}(s_3) = 0$.

One can now adapt the general formula (1.3.5) to the group at hand by means of the previous two propositions. Such a procedure yields the (nondegenerate) principal series of representations of G:

The principal series of representations of G. The principal series of unitary representations $a \to T(a, \chi)$ of G are indexed by the unitary characters $\chi=\chi(\hat{s}, \hat{n})$ of C given by Proposition 3.1.3, and act in the space $L^2(V)$ according to formulae (3.1.4) and (3.1.5). For $\phi \in L^2(V)$, the operators of the principal series corresponding to a set of generators for G have the explicit form:

(3.1.13)

(i) For $v_0 \in V$, $(T(v_0, \chi)\phi)(v) = \phi(vv_0)$, a.e.

(ii) For $c = c(c_1, c_2) \in C$, $(T(c, \chi)\phi)(v)$
$= |c_1|^{s_1-s_3-4}|c_2|^{s_2-s_3-6}[c_1]^{n_1-n_3}[c_2]^{n_2-n_3}\phi(v\tilde{c})$, a.e.;
where $v\tilde{c}$ is defined by (3.1.9).

(iii) For $p \in G$ given by (1.5.10) and with $v = (v_1, \ldots, v_6)$ one has $(T(p, \chi)\phi)(v) = (-1)^{3n_1+n_2}|v_1|^{3s_1-2}[v_1]^{3n_1}\phi(v\tilde{p})$, a.e.;
where $v\tilde{p}$ is given by (3.1.10).

(iv) For $q \in G$ given by (1.5.10),
$(T(q, \chi)\phi)(v) = |v_2|^{s_2-s_1-2}[v_2]^{n_2-n_1}\phi(v\tilde{q})$, a.e.,
where $v\tilde{q}$ is given by (3.1.11).

REMARKS. Since for any character χ the restriction of $T(\cdot, \chi)$ to V is just the right regular representation, it makes sense to look at the Plancherel transform of the principal series. Thus, we want to consider the operators

$$\hat{T}(a, \chi) = \mathscr{F}T(a, \chi)\mathscr{F}^{-1} \tag{1.3.14}$$

at least for the generating elements a of G considered above. In point of fact the computations for the restriction of $\hat{T}(\cdot, \chi)$ to the maximal solvable subgroup $H = CV$ were performed in §4 of Chapter II. In fact, $\hat{T}(v, \chi) = \hat{R}(v)$ is the operator of (2.4.3), and

$$\hat{T}(c, \chi) = |c_1|^{s_1-s_3}|c_2|^{s_2-s_3}[c_1]^{n_1-n_3}[c_2]^{n_2-n_3}\hat{D}_c \tag{3.1.15}$$

where D_c is given by (2.4.7) and $\hat{D}_c$ by (2.4.10).

The remaining two operators we want to know about are $\hat{T}(p, \chi)$ and $\hat{T}(q, \chi)$, and it is here that we run into trouble. Specifically, the action $v \to v\tilde{p}$ in (3.1.10) seems to be too unwieldy to computationally handle; and although the action $v \to v\tilde{q}$ seems to be somewhat more tractable, we have as yet been unable to evaluate $\hat{T}(q, \chi)$. Thus, at this point we can only analyze the restriction of the principal series to H, but it is hoped that a more penetrating look at the operators $\hat{T}(q, \chi)$ and $\hat{T}(p, \chi)$ will answer the question of whether *all* the representations in the principal series are irreducible and whether their restrictions to a maximal subgroup of G are irreducible.

THEOREM 3.1.1. *Consider the restriction of $\hat{T}(\cdot, \chi)$ to the subgroup $H = CV$. The operators of the representation act in the space $L = L^2(C^6, |\lambda|^4\, d\mu\, d\lambda\, dz\, dw\, d\xi\, d\eta)$ which we identify with $L^2(C^2; L_1 \otimes L_2, |\lambda|^4\, d\mu\, d\lambda)$ where $L_1 = L^2(C^2, dz\, dw)$ and $L_2 = L^2(C^2, d\xi\, d\eta)$. Let $c \to E_c$ be the representation of C defined by* (2.4.8) *and acting in the space L_1. Define an operator-valued function $(\mu, \lambda) \to A(\mu, \lambda)$, the values being operators on L_1, according to the rule*

$$A(\mu, \lambda) = E_{c(\bar{\mu}^{-1}, (\bar{\mu}\bar{\lambda}^{-1})^{1/2})}\, A(1, 1) E^{-1}_{c(\bar{\mu}^{-1}, (\bar{\mu}\bar{\lambda}^{-1})^{1/2})} \tag{3.1.16}$$

where $\mu, \lambda \neq 0$, we have chosen the principal branch of the square root, and where $A(1, 1)$ is any unitary operator on L_1. Then the operator

$$A = \int_{C^2}^{\oplus} (A(\mu, \lambda) \otimes I_2)|\lambda|^4\, d\mu\, d\lambda, \tag{3.1.17}$$

where I_2 denotes the identity operator on L_2, is a commuting operator for the restriction of $\hat{T}(\cdot, \chi)$ to H; i.e., $A\hat{T}(cv, \chi) = T(cv, \chi)A$ for all $c \in C$, $v \in V$. Furthermore, these are all the commuting operators.

Proof. Suppose A is a unitary operator which commutes with $\hat{T}(cv, \chi)$ for all $c \in C$, $v \in V$. By the same arguments used to prove the ontoness of the Plancherel transform in §4 of Chapter II, it follows that A must be of the form (3.1.17) for some operator valued function $(\mu, \lambda) \to A(\mu, \lambda)$. It is evident from the form (2.4.3) of $\hat{R}$ that any such operator A commutes with $\hat{R}(v)=\hat{T}(v, \chi)$ for all $v \in V$. On the other hand, from the form (2.4.10) of $\hat{D}_c$ (and (3.1.15)) it is clear that $\hat{T}(c, \chi)A = A\hat{T}(c, \chi)$ for $c \in C$ if and only if

$$E_c A(\mu\bar{c}_1, \lambda\bar{c}_1\bar{c}_2^2)E_c^{-1} = A(\mu, \lambda) \tag{3.1.18}$$

for a.e. (μ, λ). It is easily verified that the operator-valued function defined by (3.1.16) satisfies (3.1.18), so the corresponding operator A is indeed a commuting operator for the restriction to H of the principal series. To see that this gives us all the commuting operators we interpret (3.1.18) as the statement that a certain measurable function on $C \times \Omega$ is zero a.e. It then follows that $(\mu, \lambda) \to A(\mu, \lambda)$ is essentially continuous. Hence, we may suppose that by a possible redefinition of $A(\mu, \lambda)$ on a set of measure zero, (3.1.18) holds everywhere. Evaluation of (3.1.18) at $\mu=1$, $\lambda=1$ shows that (3.1.16) is satisfied and completes the proof of the theorem.

REMARK. Let A_1 be any operator on L_1 commuting with the representation E of C, and think of $\hat{T}(\cdot, \chi)$ as acting in the space $L_0 \otimes L_1 \otimes L_2$, where L_1 and L_2 are as in the theorem and $L_0 = L^2(\Omega, d\omega)$. The theorem then shows that the operator $A = I_0 \otimes A_1 \otimes I_2$, where I_0 and I_2 are the identity operators on L_0 and L_2 respectively, commutes with the restriction of $\hat{T}(\cdot, \chi)$ to H.

2. **Intertwining relations.** The point of this section is to formulate a problem connected with the intertwining operators of Kunze and Stein [13]. We shall omit the computational details as well as some of the limiting arguments which make some formally correct statements rigorous.

It is a result of Bruhat that the representations $T(\cdot, \chi_1)$ and $T(\cdot, \chi_2)$ of the principal series on a complex semisimple group are unitarily equivalent if and only if there is some element r of the Weyl group such that $r\chi_1 = \chi_2$. (One defines the character $r\chi$ by $(r\chi)(c) = \chi(r^{-1}cr)$ for all $c \in C$.) Kunze and Stein [13] showed how to obtain explicit intertwining operators in the case of the simple Weyl reflections, and then obtain the general result by the composition of these operators. We outline this theory in the case of the group G of type G_2.

For notational convenience set $p_1 = p$ and $p_2 = q$, where p and q are the simple Weyl reflections previously considered. Corresponding to each p_j $(j=1, 2)$ there is a semidirect product decomposition $V = V_j V_j'$ where V_j and V_j' are defined as in (2.2.3), (2.2.6), and (2.2.7). Thus, we write an element v of V as $v = v_j v_j'$, where $v_j' \in V_j'$ and $v_j \in V_j$. We allow the ambiguity of identifying v_j with the complex number corresponding to its jth coordinate. Next, we recall the definition of the characters $\chi(\hat{s}, \hat{n})$ in Proposition 3.1.3. We have $\chi(\hat{s}, \hat{n}) = \chi(\hat{s})\chi(\hat{n})$ where we have set $\chi(\hat{s}) = \chi(\hat{s}, 0)$ and $\chi(\hat{n}) = \chi(\hat{n}, 0)$. Since $\chi(\hat{s})$ and $\chi(\hat{n})$ are readily identified with elements of the dual $\mathfrak{H}^*$ of the Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ (see the very end of §3 of Chapter I), formulae (1.3.10) make sense when ξ is replaced by $\chi(\hat{s})$ or $\chi(\hat{n})$.

Let p_j be, as above, the simple Weyl reflection corresponding to the root α_j ($\alpha_1 = \alpha$, $\alpha_2 = \beta$, as in (1.3.8)) and let $\chi = \chi(\hat{s}, \hat{n})$ be a unitary character of C (i.e., Re $(s_k) = 0$,

$k=1, 2, 3$). Kunze and Stein consider the unitary operator $A(p_j, \chi)$ defined on $L^2(V)$ (thought of as $L^2(V_j)\otimes L^2(V_j')$ with V_j identified with $\boldsymbol{C}$) by the following formal expression: For $f\in L^2(V)$,

$$(A(p_j, \chi)f)(z, v_j') = \frac{1}{\gamma(\chi)}\int_C |w|^{s-2}[w]^n f(z+w, v_j')\,dw, \tag{3.2.1}$$

where

$$s = \frac{2(\chi(\hat{s})|\alpha_j)}{(\alpha_j|\alpha_j)}, \quad n = \frac{2(\chi(\hat{n})|\alpha_j)}{(\alpha_j|\alpha_j)} \tag{3.2.2}$$

and

$$\gamma(\chi) = \gamma(s, n) = \frac{2^{s_i|n|}\Gamma(\frac{1}{2}(|n|+s))}{\Gamma(\frac{1}{2}(|n|+2-s))}. \tag{3.2.3}$$

The numbers s and n depend of course on $j=1, 2$. If we denote this dependence by the notation $s=s^{(j)}$, $n=n^{(j)}$, it follows from (1.3.10) that

$$\begin{aligned} s^{(1)} &= -3s_1, \quad s^{(2)} = s_1 - s_2, \\ n^{(1)} &= -3n_1, \quad n^{(2)} = n_1 - n_2. \end{aligned} \tag{3.2.4}$$

We also define

$$A(p_j, \chi) = A_j(s, n) \tag{3.2.5}$$

where $\chi=\chi(\hat{s}, \hat{n})$, and $s=s^{(j)}, n=n^{(j)}$ are as above. (We remark that $A_j(s, n)$ is rigorously defined as the strong limit of the unbounded operators $A_j(t, n)$ with $t=s+\sigma$, σ real, and $\sigma\to 0^+$.) The important properties of these operators are the following (see [13] for the proof):

PROPOSITION 3.2.1. *With the notation as above we have*:

(i) $(s, n)\to A_j(s, n)$ *is a unitary representation of* $i\boldsymbol{R}\times\boldsymbol{Z}$; *i.e.*, $A_j(s, n)A_j(t, m)= A_j(s+t, n+m)$ *and* $A_j(0, 0)=I$.

(ii) $A(p_j, \chi)$ *intertwines* $T(\cdot, \chi)$ *and* $T(\cdot, p_j\chi)$; *i.e.*, $A(p_j, \chi)T(a, \chi)=T(a, p_j\chi)\cdot A(p_j, \chi)$ *for all* $a\in G$.

One desires to show that the relations in the Weyl group are mirrored by the intertwining operators. This can be done providing the second of relations (1.3.8) is reflected in the following operator equation, called the *intertwining equation*:

$$I = A(p_1, q_{11}\chi)A(p_2, q_{10}\chi)\cdots A(p_1, q_1\chi)A(p_2, \chi), \tag{3.2.6}$$

where $q_{2k}=(p_1p_2)^k$, $q_{2k+1}=p_2(p_1p_2)^k$ for $k=0, 1, 2, 3, 4, 5$. This equation is more symmetrical when written out in terms of the parameters s and n by means of (3.2.5) and (3.2.4), and (1.3.11). In fact (3.2.6) is equivalent to the equation

$$C(s, n)C(s+t, n+m)C(t, m) = C(t, m)C(s+t, n+m)C(s, t) \tag{3.2.7}$$

for all $s, t\in i\boldsymbol{R}$, $n, m\in\boldsymbol{Z}$; where we have defined $C(s, n)$ by

$$C(s, n) = B(s, n)A(3s, 3n)B(s, n) \tag{3.2.8}$$

and

$$A(-3s_1, -3n_1) = A(p_1, \chi), \quad B(s_1-s_2, n_1-n_2) = A(p_2, \chi). \tag{3.2.9}$$

In [13] it is shown that (3.2.6) can be made to hold by means of "normalizing" $A(p_j, \chi)$. It is, however, of considerable interest to find out if the operators $C(s, n)$ form a *commutative family*; i.e., if

$$C(s, n)C(t, m) = C(t, m)C(s, n) \tag{3.2.10}$$

for all $s, t \in i\boldsymbol{R}$, $n, m \in \boldsymbol{Z}$. (Notice that (3.2.10) implies (3.2.7).)

We have not yet been able to prove, or disprove, (3.2.10); however, an application of the Plancherel transform makes the question more explicit. In fact we have the following proposition which is formally true by straightforward computation and can be made rigorous by the type of "regularization" argument used in [13].

PROPOSITION 3.2.2. *For each* $\omega = (\mu, \lambda) \in \Omega$ *define the unitary operator* $M(\mu, \lambda)$ *on* $L^2(\boldsymbol{C}^2; dz\, dw)$ *by*

$$(M(\mu, \lambda)f)(z, w) = \exp i \operatorname{Re}\{wz^{-1}(\bar{\lambda}w^2 - \bar{\mu})\}f(z, w) \tag{3.2.11}$$

a.e., and define unbounded operators $R_1(s, n)$ *and* $R_2(s, n)$ *for* $s \in \boldsymbol{C}$, $n \in \boldsymbol{Z}$, *with* $0 < \operatorname{Re}(s) < 1$ *such that for* $f \in C_0(\boldsymbol{C}^2)$

$$\begin{aligned} (R_1(s, n)f)(z, w) &= \frac{1}{\gamma(s, n)} \int_{\boldsymbol{C}} |t|^{s-2}[t]^n f(z-t, w)\, dt \\ (R_2(s, n)f)(z, w) &= \frac{1}{\gamma(s, n)} \int_{\boldsymbol{C}} |t|^{s-2}[t]^n f(z, w-t)\, dt \end{aligned} \tag{3.2.12}$$

where $\gamma(s, n)$ *is given by* (3.2.3). *Then*:

(i) *The map* $s \to R_j(s, n)f$ $(j = 1, 2)$, *for fixed* $f \in C_0(\boldsymbol{C}^2)$, *has a unique extension to* $\operatorname{Re}(s) = 0$ *such that it is continuous in* $0 \leqq \operatorname{Re}(s) < 1$. *Furthermore, for* $\operatorname{Re}(s) = 0$ *the operators* $R_j(s, n)$ *extend by continuity to unitary operators* $R_j(s, n)$ *on* $L^2(\boldsymbol{C}^2)$ *and* $(s, n) \to R_j(s, n)$ *are unitary representations of* $i\boldsymbol{R} \times \boldsymbol{Z}$.

(ii) *Let* $\hat{A}(s, n) = \mathscr{F}A(s, n)\mathscr{F}^{-1}$ *and* $\hat{B}(s, n) = \mathscr{F}B(s, n)\mathscr{F}^{-1}$ *where* $\mathscr{F}$ *denotes the Plancherel transform on V and A and B the intertwining operators* (3.2.9). *For* $0 \leqq \operatorname{Re}(s) < 1$,

$$\begin{aligned} \hat{A}(s, n) &= \int_{\Omega}^{\oplus} (\hat{A}^{(\omega)}(s, n) \otimes I)\, d\omega, \\ \hat{B}(s, n) &= \int_{\Omega}^{\oplus} (\hat{B}^{(\omega)}(s, n) \otimes I)\, d\omega, \end{aligned} \tag{3.2.13}$$

where I denotes the identity on $L^2(\boldsymbol{C}^2, d\xi\, d\eta)$, *and*

$$(\hat{A}^{(\mu,\lambda)}(s, n)f)(z, w) = |z|^{-s}[z]^{-n}(M(\mu, \lambda)R_2(s, n)M(\mu, \lambda)^{-1}f)(z, w) \tag{3.2.14}$$

for all $f \in C_0(\boldsymbol{C}^2)$, *and*

$$\hat{B}^{(\mu,\lambda)}(s, n) = R_1(s, n) \tag{3.2.15}$$

for all μ, λ.

The commutative family problem (3.2.10) can now be given the explicit form: Is it true that for any fixed $s, t \in i\boldsymbol{R}$ and $n, m \in \boldsymbol{Z}$ one has

$$\hat{C}^{(\omega)}(s, n)\hat{C}^{(\omega)}(t, m) = \hat{C}^{(\omega)}(t, m)\hat{C}^{(\omega)}(s, n) \tag{3.2.16}$$

for a.e. $\omega \in \Omega$, where

$$\hat{C}^{(\omega)}(s, n) = \hat{B}^{(\omega)}(s, n)\hat{A}^{(\omega)}(3s, 3n)\hat{B}^{(\omega)}(s, n) \text{ ?} \tag{3.2.17}$$

According to the above proposition

$$\hat{C}^{(\mu,\lambda)}(s, n) = R_1(s, n)T(s, n)M(\mu, \lambda)R_2(3s, 3n)M(\mu, \lambda)R_1(s, n) \tag{3.2.18}$$

where $T(s, n)$ is the operator on $L^2(\boldsymbol{C}^2, dz\, dw)$ given by multiplication by the function $z \to |z|^{-s}[z]^{-n}$.

REMARK. It is shown in [13] that for the two classical rank two complex simple groups there do exist commutative families. The proofs for those groups do not seem to generalize to the group G of type G_2. The complications seem to be two-fold: First, the L^2-spaces for G are of *two* variables as opposed to *one* variable for the classical groups, and secondly, operators $M(\mu, \lambda)$ do not appear in the classical cases.

3. **Complementary series.** In this section we construct two *complementary series* of representations of G. There are, in fact, five more series which seem to have formal validity but for which the rigorous development is quite delicate. We postpone the presentation of these latter series for another time.

We shall call a unitary representation of G *complementary* if it is "induced" from a nonunitary character of B. Such a representation acts in a Hilbert space contrived by means of a suitable inner product on $C_0(V)$. Let us first notice that the operator $T(a, \chi)$ of (3.1.5) makes sense for nonunitary χ providing we interpret it as an unbounded operator with dense domain. Define the contragredient character χ' of χ by $\chi'(c) = \bar{\chi}(c^{-1})$ for all $c \in C$. Then the adjoint of $T(a, \chi)$ has the form

$$T(a, \chi)^* = T(a^{-1}, \chi')$$

for all $a \in G$ and characters χ of C. Furthermore, for $0 < \operatorname{Re}(s) < 1$ the operators $A(p_j, \chi) = A_j(s, n)$ defined by (3.2.1) make sense as densely-defined unbounded operators; and it is shown in [13] that, as operators in $L^2(V)$,

$$A(p_j, \chi)T(a, \chi) = T(a, p_j\chi)A(p_j, \chi)$$

(for $j = 1, 2$) for all $a \in G$, $\chi = \chi(\hat{s}, \hat{n})$ where $0 < \operatorname{Re}(s) < 1$. As a final preliminary point we know that the operator $A_j(s, n)$ is transformed by the (partial) Fourier transform on V_j to the operator on $L^2(V_j) \otimes L^2(V_j')$ of the form $\tilde{A}_j(s, n) \otimes I$ where $(\tilde{A}_j(s, n)f)(z) = (-1)^n|z|^{-s}[z]^n f(z)$, a.e.

We construct the first complementary series. Consider the characters $\chi=\chi(\hat{s}, \hat{n})$ of C, where $\hat{s}$ and $\hat{n}$ are of the form

$$\begin{aligned} \hat{s} &= (\sigma, s_2, s_3), \qquad \sigma \in \boldsymbol{R}, 0 > \sigma > -\tfrac{1}{3}, \\ \hat{n} &= (0, n_2, n_3). \end{aligned} \tag{3.3.1}$$

Then $A(p, \chi)$ is defined on $f \in C_0(V)$ by

$$(A(p, \chi)f)(z, v_j') = \frac{1}{\gamma(\chi)} \int_C |w|^{-3\sigma-2} f(z+w, v_j')\, dw$$

(as in (3.2.1)); and in fact $A(p, \chi)$ is a positive operator; i.e., $(A(p, \chi)f \mid f) > 0$ for all $f \in C_0(V), f \neq 0$. Thus, we define an inner product $(\cdot \mid \cdot)_{p,\sigma}$ on $C_0(V)$ by

$$(f \mid g)_{p,\sigma} = (A(p, \chi)f \mid g),$$

and call $H(p, \sigma)$ the Hilbert space completion. From the first of equations (1.3.11) one sees that $p\chi=\chi'$ (contragredient), so $T(a, p\chi)^*=T(a^{-1}, \chi)$. Thus, for $f, g \in C_0(V^a)$,

$$\begin{aligned} (T(a, \chi)f \mid T(a, \chi)g)_{p,\sigma} &= (A(p, \chi)T(a, \chi)f \mid T(a, \chi)g) \\ &= (T(a, p\chi)A(p, \chi)f \mid T(a, \chi)g) \\ &= (A(p, \chi)f \mid T(a^{-1}, \chi)T(a, \chi)g) \\ &= (f \mid g)_{p,\sigma}. \end{aligned}$$

Since uniform convergence in $C_0(V)$ implies convergence in $H(p, \sigma)$ it follows that $T(a, \chi)$ extends to a unitary operator on $H(p, \sigma)$. It is a simple matter to verify that $T(a, \chi)T(b, \chi)=T(ab, \chi)$ and that $T(a, \chi) \to I$ strongly as $a \to e$ (see [13]). Thus, we have established the following:

Complementary series I. With χ as in (3.3.1), $H(p, \sigma)$ as above, and $T(a, \chi)$ the unitary operator on $H(p, \sigma)$ constructed above, we have that $a \to T(a, \chi)$ is a unitary representation of G.

The second series we consider is the analog of the above with p replaced by q. Let $\chi=\chi(\hat{s}, \hat{n})$ now be defined by

$$\begin{aligned} \hat{s} &= (s_1, -\bar{s}_1, s_3), \qquad 0 < \sigma = s_1+\bar{s}_1 < 1, \\ \hat{n} &= (n_1, n_1, n_3). \end{aligned} \tag{3.3.2}$$

Then $q\chi=\chi'$, $A(q, \chi)$ is a positive operator, $(A(q, \chi)f \mid g)=(f \mid g)_{q,\sigma}$ defines an inner product on $C_0(V)$ whose completion we call $H(q, \sigma)$. By the same arguments as above, we have:

Complementary series II. Let χ be defined by (3.3.2). The operators $T(a, \chi)$ extend to unitary operators on $H(q, \sigma)$ and $a \to T(a, \chi)$ is a unitary representation of G in the space $H(q, \sigma)$.

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