ON A CLASS OF STOCHASTIC PROCESSES AND ITS RELATIONSHIP TO INFINITE PARTICLE GASES⁽¹⁾

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Introduction. Consider the right continuous sample paths $w: [0, +\infty) \rightarrow E = \{+1, -1\}$ with coordinates $x(t) = x_t = x_t(w) \in E$ together with the σ -algebras $\mathcal{M}_t$ generated by $x(s)$, $s \leq t$, and assume that for each probability measure f on E and $e \in E$, there exist probability measures P_f and $P_{f|e}$ on $\mathcal{M}_\infty$ for which

$$(0.A) \quad P_{f|e}(\cdot) = P_f(\cdot | x(0) = e),$$

$$(0.B) \quad P_f(x(0) = e) = f(e),$$

$$(0.C) \quad P_{f|e}(x(t+h) \in A | \mathcal{M}_t) = P_{f_t|x_t}(x(h) \in A), \text{ [a.e. } P_{f|e}],$$

where A is a set of points in E and $f_t(A) = P_f(x(t) \in A)$. Such a stochastic process will be called a K -process. The expression $P_{f|e}(\Lambda)$ is to be thought of as the probability that, starting with $x(0)$ distributed according to f , the event Λ will take place conditional on $x(0) = e$. A K -process is a temporally homogeneous Markov process if and only if $P_{f|e}$ is independent of f , as the reader can easily check. If

$$\gamma_e(u) = \left. \frac{d}{dt} P_{f|e}(x(t) = 1) \right|_{t=0}, \quad u = f(+1),$$

then when γ_{+1} and γ_{-1} are real analytic on the closed interval $[0, 1]$, they uniquely determine the distribution of the K -process $x(t)$.

In this paper, I shall construct a model of an infinite particle gas with velocities ± 1 in which the motion of a tagged particle is a K -process with specific $\gamma_{\pm 1}$ and in which the sample paths of any two particles are independent. This will be accomplished by constructing a gas of n like particles, each of which has velocities ± 1 , and then letting $n \rightarrow \infty$. Each of the n particle gases will be a Markov jump process in which one waits an exponential holding time and then picks an index i according to the uniform distribution $1/n$ and lets the corresponding particle collide with one or more of the remaining particles. The effect of a collision between a single particle and a set of particles will be a change of state only for the single particle. H. P. McKean [3] has carried out this construction for the case when $\gamma_{\pm 1}(u) = \pm(u-1)$. In this paper, we generalize his results to the case when $-\gamma_{+1}$

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and γ_{-1} are both positive on the open interval $(0, 1)$ and real analytic on the closed interval $[0, 1]$.

The paper is arranged as follows. In §1 we derive a few basic properties of K -processes, including the fact that the functions γ_{+1} and γ_{-1} uniquely determine the distribution of the process. In §2, we define the n -molecule gases. In §3, we introduce the basic notions of convergence and equivalence that will be used throughout the paper, and we calculate the limit as $n \rightarrow \infty$ of the generators of the n -molecule gases. In §4 we state the main theorems and finally give their proofs in §5.

1. **K -processes.** Let x_t , $t \in [0, \infty)$ be a K -process. Then, defining

$$P_{fle}(t; A) = P_{fle}(x_t \in A),$$

we get a formula for the probabilities of joint observations reminiscent of the case of temporally homogeneous Markov processes.

THEOREM 1. *If x_t is a K -process, then for $0 < t_1 < \dots < t_n < \infty$,*

$$\begin{aligned} P_{fle}[x(t_1) \in A_1, \dots, x(t_n) \in A_n] \\ = \int_{A_1} P_{fle}(t_1; d\xi_1) \int_{A_2} P_{f_{t_1}|\xi_1}(t_2 - t_1; d\xi_2) \cdots \int_{A_n} P_{f_{t_{n-1}}|\xi_{n-1}}(t_n - t_{n-1}; d\xi_n). \end{aligned}$$

Proof. This is immediate from (0.C).

COROLLARY. *If x_t is a K -process, then*

$$P_{fle}(s+t; A) = \int P_{fle}(t; d\xi) P_{f_t|\xi}(s; A).$$

If $\gamma(u) = u\gamma_{+1}(u) + (1-u)\gamma_{-1}(u)$, then we have the following theorem.

THEOREM 2. *If x_t is a K -process and if $P_{fle}(t; +1)$ is differentiable in $t \geq 0$, then*

$$(2.A) \quad \frac{d}{dt} f_i(+1) = \gamma[f_i(+1)],$$

$$(2.B) \quad \frac{d}{dt} P_{fle}(t; +1) = P_{fle}(t; +1)\gamma_{+1}[f_i(+1)] + P_{fle}(t; -1)\gamma_{-1}[f_i(+1)].$$

Proof. Taking the equation in the Corollary to Theorem 1 and differentiating both sides with respect to s and letting $s=0$, we get (2.B). (2.A) follows from (2.B) if we notice that

$$f_i(+1) = \int f(de) P_{fle}(t; +1).$$

Equation (2.A) of Theorem 2, which is in general nonlinear, has a unique solution bounded by 0 and 1 if γ is real analytic in the closed interval $[0, 1]$ and if $\gamma(0) \geq 0$ and $\gamma(1) \leq 0$. Once the solution of (2.A) is known, (2.B) becomes a linear differential equation for $P_{fle}(t; +1)$. This equation, in turn, has a unique solution

bounded by 0 and 1 if γ_{+1} and γ_{-1} are continuous and $\mp \gamma_{\pm} \geq 0$. Having uniquely determined the transition function $P_{f|e}$, we can construct the K -process by defining probabilities on the cylinder sets in the manner suggested in Theorem 1:

$$P[x(t_1) \in A_1, x(t_2) \in A_2, \dots, x(t_n) \in A_n] \\ = \int_{A_1} P_{f|e}(t_1; d\xi_1) \int_{A_2} P_{f_{t_1|\xi_1}}(t_2 - t_1; d\xi_2) \cdots \int_{A_n} P_{f_{t_{n-1}|\xi_{n-1}}}(t_n - t_{n-1}; d\xi_n),$$

where $0 < t_1 < \cdots < t_n < \infty$. Thus to each pair of functions $-\gamma_{+1}$ and γ_{-1} which are positive on the open interval $(0, 1)$ and real analytic on the closed interval $[0, 1]$, there corresponds a unique K -process.

2. n -molecule gases. To define the n -molecule gases; let E be the set of integers ± 1 , let L be a fixed positive constant and let $C_N^e(k)$, $e \in E$ and $k \leq N$ be a family of nonnegative real numbers with the property that when $C_N = \max_{k \leq N, e = \pm 1} C_N^e(k)$ one has

$$\sum_{N=1}^{\infty} N^P C_N \leq P! L^P, \quad P \geq 1.$$

We construct the n -molecule gases as follows. Let $X_n(t) = [x_1^n(t), \dots, x_n^n(t)]$ be a Markov jump process on the n -dimensional space E^n with holding time distribution, in the state $[e_1, \dots, e_n]$, equal to

$$\exp \left[-t \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \right]$$

where $\sum_{j_0, j_1, \dots, j_N}^{(n)}$ denotes the sum taken over all sequences $(j_0, \dots, j_N)$, $1 \leq j_k \leq n$, $j_k \neq j_p$ when $k \neq p$; and where $C(e | e_1, \dots, e_N) = C_N$ (number of $+1$'s in the set $e_1, \dots, e_N$). Starting at the state $[e_1, \dots, e_n]$, the probability that the first jump is to the state $[e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n]$ will be given by

$$\frac{\sum_{N=1}^{n-1} n^{-N} \sum_{j_1, \dots, j_N}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N})}{\sum_{N=1}^{n-1} n^{-N} \sum_{k, j_1, \dots, j_N}^{(n)} C(e_k | e_{j_1}, \dots, e_{j_N})}.$$

The generator of the n -molecule gas will therefore be

$$G_n \phi(e_1, \dots, e_n) = \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \Delta_i \phi$$

where $\Delta_i \phi = \phi(\dots, -e_i, \dots) - \phi(\dots, e_i, \dots)$ or zero, depending on whether ϕ depends on the variable e_i or not.

To calculate the joint probabilities of M molecules in the gas, notice that if

$$T_t \phi(e_1, \dots, e_n) = E[\phi(X_t^n) | X_0^n = (e_1, \dots, e_n)]$$

is the semigroup associated with the Markov process X^n , then

$$T_t \phi(e_1, \dots, e_n) = \exp [tG_n] \phi(e_1, \dots, e_n).$$

Thus for any two functions ϕ and ψ on E^n and $\xi \in E^n$, we have

$$\begin{aligned} E[\phi(X_s^n) \psi(X_{s+t}^n) \mid X_0^n = \xi] &= E[\phi(X_s^n) E[\psi(X_{s+t}^n) \mid X_s^n] \mid X_0^n = \xi] \\ &= E[\phi(X_s^n) T_t \psi(X_s^n) \mid X_0^n = \xi] \\ &= T_s(\phi T_t \psi)(\xi) \\ &= \exp [sG_n] \phi \exp [tG_n] \psi. \end{aligned}$$

Using this argument, it is easily shown that if the molecules are initially independent and indentially distributed with distribution f and if $E_k = (e_1^k, \dots, e_M^k, E, E, \dots, E)$, then the joint distribution of the first M particles is given by

$$\begin{aligned} (1) \quad &P[X^n(t_1) \in E_1, \dots, X^n(t_M) \in E_M] \\ &= \int f(d\xi_1) \cdots f(d\xi_n) \exp [t_1 G_n] \chi_{E_1} \exp [(t_2 - t_1) G_n] \chi_{E_2} \cdots \exp [(t_M - t_{M-1}) G_n] \chi_{E_M}, \end{aligned}$$

where χ_A is the indicator function of the set A .

3. Preliminaries and notation. We wish to calculate the limit of (1) as n goes to infinity. To do this, one might calculate $\lim_{n \rightarrow \infty} G_n \phi$ when ϕ has a finite number of variables. However, as is evident from (1), this is not necessary as we are only interested in the behavior of $G_n \phi$ modulo an integration. Instead, we introduce the following notion of equivalence and convergence.

DEFINITION 3. Let I be the set of all indices ij where i and j are nonnegative integers. Suppose that ϕ and ψ are functions whose variables are indexed by indices in I and suppose that $J \subset I$. Then we define

(3.A) If f is a probability measure on E , then

$$\int_J f^\infty \phi = \int \prod_{\alpha \in J} f(d\xi_\alpha) \phi.$$

(3.B) $\phi \equiv \psi \bmod J$ if and only if there exist functions ϕ_j and ψ_j , whose variables have indices in I , and one to one mappings δ_j of J onto J such that

$$\phi = \sum \phi_j, \quad \psi = \sum \psi_j$$

and $\phi_j^\delta = \psi_j$ where ϕ_j^δ is ϕ_j with the variables $e_{\delta(\alpha)}$ replacing e_α for $\alpha \in J$.

(3.C) $\|\phi\|$ is the sup norm of ϕ .

(3.D) $\|\phi\|_J = \inf_{\psi \equiv \phi \bmod J} \|\psi\|$.

(3.E) $\phi_n \rightarrow \phi \bmod J$ as $n \rightarrow \infty$ if and only if $\|\phi_n - \phi\|_J \rightarrow 0$ as $n \rightarrow \infty$.

One should note that if $\phi \equiv \psi \bmod J$, then

$$\int_J f^\infty \phi = \int_J f^\infty \psi.$$

Similarly, if $\phi_n \rightarrow \phi \pmod J$, then

$$\int_J f^\infty \phi_n \rightarrow \int_J f^\infty \phi.$$

It is also easily verified that if $\phi_1 \equiv \psi_1 \pmod J$, $\phi_2 \equiv \psi_2 \pmod J$ and ϕ has variables whose indices all lie in J^c , then

$$\phi_1 + \phi_2 \equiv \psi_1 + \psi_2 \pmod J$$

and

$$\phi \phi_1 \equiv \phi \psi_1 \pmod J.$$

These facts will be used throughout the paper without further comment.

Now suppose that ϕ is a function of M variables, M finite, which are indexed by an index set J of positive integers. We wish to evaluate $\lim_{n \rightarrow \infty} G_n \phi$ modulo J^c . Clearly we have

$$\begin{aligned} G_n \phi(e_1, \dots, e_n) &= \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \Delta_i \phi \\ (2) \quad &= \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N: j_k \in J^c}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \Delta_i \phi \\ (3) \quad &+ \sum_{N=1}^{n-1} n^{-N} \sum_{i, j_1, \dots, j_N: \text{some } j_k \in J}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \Delta_i \phi. \end{aligned}$$

Since we are only interested in evaluating these sums modulo J^c , we may rename the indices of variables which are not contained in J . Using the new indices $1j$, and adopting the convention that the indices $0i$ and i are the same, we find that the first sum (2) is equivalent to

$$(4) \quad \sum_{N=1}^{n-M} n^{-N} (n-M) \cdots (n-M-N+1) \sum_i C(e_{0i} | e_{11}, \dots, e_{1N}) \Delta_{0i} \phi$$

modulo J^c since

$$\begin{aligned} &\sum_{i, j_1, \dots, j_N: j_k \in J^c}^{(n)} C(e_i | e_{j_1}, \dots, e_{j_N}) \Delta_i \phi \\ &\equiv (n-M)(n-M-1) \cdots (n-M-N+1) \sum_i C(e_{0i} | e_{11}, \dots, e_{1N}) \Delta_{0i} \phi \pmod{J^c}, \\ &\quad M+N \leq n, \\ &\equiv 0 \text{ otherwise.} \end{aligned}$$

As $n \rightarrow \infty$, (4) converges absolutely since it is clearly bounded by

$$2M \|\phi\| \sum_{N=1}^{\infty} C_N.$$

The second sum (3) is bounded by

$$2M^2\|\phi\|n^{-1}\sum_{N=1}^{\infty}NC_N\leq 2M^2\|\phi\|n^{-1}L$$

and thus, using the dominated convergence theorem, we see that as $n\rightarrow\infty$, $G_n\phi(e_1,\dots,e_N)$ converges modulo J^c to

$$\sum_{N=1}^{\infty}\sum_iC(e_{0i}|e_{11},\dots,e_{1N})\Delta_{0i}\phi.$$

Now let the pairs ij of nonnegative integers be indices for variables $e_{ij}\in E$ and let ψ be a function of a subset of these variables. We shall call i the order of the index ij and the order of ψ will be defined as the maximum of the orders of its variables. Let ψ be a function of finite order less than or equal to p . Then we define

$$D_p\psi=\sum_{N=1}^{\infty}\sum_{i=0}^p\sum_{j=1}^{\infty}C(e_{ij}|e_{p+1,1},\dots,e_{p+1,N})\Delta_{ij}\psi.$$

Clearly the operator D_p introduces a new set of variables of order $p+1$. Remembering the convention that the indices $0i$ and i are the same, we have established for any function ϕ whose variables are indexed by a finite set of positive integers J that

$$G_n\phi\rightarrow D_p\phi\pmod{J^c}$$

for $p\geq 0$.

In general, we will write $D\phi$ instead of $D_p\phi$ with the understanding that D always adds variables whose indices are of order at least one higher than the order of the function on which it is operating. If ϕ has infinite order, then we will always be able to write $\phi=\sum\phi_k$ where ϕ_k has finite order and then let $D\phi=\sum D\phi_k$. Finally, we let

$$\exp[tD]\phi=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n\phi.$$

4. Main theorems. Our first theorem is the following:

THEOREM 4. *If $\phi_1,\dots,\phi_p$ are bounded functions whose variables have indices in a finite set J of M indices, then for $8LM(t_1+\dots+t_p)<1$ we have*

$$\exp[t_pG_n]\phi_p\cdots\exp[t_1G_n]\phi_1\rightarrow\exp[t_pD]\phi_p\cdots\exp[t_1D]\phi_1\pmod{J^c},$$

where the expression on the right is well defined and finite.

Thus as $n\rightarrow\infty$, (1) converges to

$$\int f^\infty\exp[t_1D]_{\chi_{E_1}}\exp[(t_2-t_1)D]_{\chi_{E_2}}\cdots\exp[(t_M-t_{M-1})D]_{\chi_{E_M}}.$$

This limiting distribution can be used to define a combined motion of M molecules for which:

- I. the paths of any two molecules are independent for $16Lt < 1$, and
 II. the distribution of a tagged particle is that of a K -process with

$$\gamma_{\pm 1}(u) = \mp \sum_{N=1}^{\infty} \sum_{k=0}^N C_N^{\pm 1}(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Finally, we are able to show that

III. any k -process for which $-\gamma_+$ and γ_- are positive on the open interval $(0, 1)$ and real analytic on the closed interval $[0, 1]$ can be constructed as the motion of a tagged particle in an infinite particle gas as described above.

The proof of I is contained in the following theorem.

THEOREM 5. *If $a_\nu, b_\nu \in E, \nu = 1, \dots, m$ and if $0 < t_1 < \dots < t_m$ are real and $16Lt_m < 1$, then for $i \neq j$ and any initial probability measure f on E ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} P[x_i^n(t_\nu) = a_\nu, x_j^n(t_\nu) = b_\nu, 1 \leq \nu \leq m] \\ = \lim_{n \rightarrow \infty} P[x_i^n(t_\nu) = a_\nu, 1 \leq \nu \leq m] \lim_{n \rightarrow \infty} P[x_j^n(t_\nu) = b_\nu, 1 \leq \nu \leq m]. \end{aligned}$$

The proof of this and the following theorem is based on the fact that if ϕ and ψ have variables whose indices form disjoint sets whose union is J , then their product $\phi\psi$, written $\phi \otimes \psi$ when the indices of their variables are disjoint, is such that

$$D(\phi \otimes \psi) \equiv \phi \otimes D\psi + \psi \otimes D\phi \pmod{J^c}.$$

Letting m be the maximum order of ϕ and ψ , this equation is easily extended to

$$(5) \quad D^p(\phi \otimes \psi) = \sum_{k=0}^p \binom{p}{k} (D_{p-1+m} \cdots D_{k+m}\phi) \otimes (D_{k-1+m} \cdots D_m\psi) \pmod{J^c}$$

(see M. Kac [2] for the terminology and another instance of this phenomenon).

If we let

$$(6) \quad \begin{aligned} \chi_a^i(\dots, e_i, \dots) &= 1 \quad \text{if } e_i = a, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

$$f_i(e) = \lim_{n \rightarrow \infty} P[x_n^1(t) = e] = \int f^\infty \exp [tD] \chi_e^1$$

and

$$P_{f|a}(t, b) = \lim_{n \rightarrow \infty} P[x_n^1(t) = b \mid x_n^1(0) = a] = \int_{\{1\}^c} f^\infty \exp [tD] \chi_b^1(a, \dots),$$

then II is proved in the following theorem which establishes the limiting motion of a fixed molecule in the gas as a K -process for which

$$\gamma_{\pm 1}(u) = \mp \sum_{N=1}^{\infty} \sum_{k=0}^N C_N^{\pm 1}(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

THEOREM 6. For $0 < t_1 < \dots < t_m$ with $8Lt_m < 1$ and $e_0, \dots, e_m \in E$, we have

$$\int_{\{1\}^c} f^\infty \exp [t_1 D] \chi_{e_1}^1 \cdots \exp [(t_m - t_{m-1}) D] \chi_{e_m}^1(e_0) \\ = P_{f|e_0}(t_1, e_1) P_{f|t_1|e_1}(t_2 - t_1, e_2) \cdots P_{f|t_{m-1}|e_{m-1}}(t_m - t_{m-1}, e_m)$$

and

$$\gamma_{\pm 1}(u) = \frac{d}{dt} P_{f|\pm 1}(t; 1)|_{t=0} = \mp \sum_{N=1}^{\infty} \sum_{k=0}^N C_N^{\pm 1}(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

COROLLARY. For $8Lt < 1$,

$$f_t(+1) = \int f^\infty \exp [tD] \chi_{+1}^1$$

is the solution of the differential equation

$$(d/dt)f_t(+1) = \gamma[f_t(+1)]$$

where $\gamma(u) = u\gamma_{+1}(u) + (1-u)\gamma_{-1}(u)$.

To prove III, we introduce the following definition.

DEFINITION 7. Let H be the class of functions γ , mapping $[0, 1]$ into the real numbers, for which there exist positive real numbers $C_N(k)$ and L such that if $C_N = \max_{k \leq N} C_N(k)$,

$$\gamma(u) = \sum_{N=1}^{\infty} B_N(u)$$

where

$$B_N(u) = \sum_{k=0}^N C_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

and

$$\sum_{N=1}^{\infty} N^p C_N \leq p! L^p, \quad p \geq 1.$$

Thus if $-\gamma_{+1}$ and γ_{-1} are both in H , we can construct an associated K -process for small t by taking the limiting motion of one coordinate in an n -dimensional Markov chain as $n \rightarrow \infty$. The class H can be described more simply as follows.

THEOREM 8. $F \in H$ if and only if F is positive on the open interval $(0, 1)$ and real analytic on the closed interval $[0, 1]$.

The proofs of Theorems 4 through 8 follow.

5. Proofs.

Proof of Theorem 4.

LEMMA 9. If $\tau_k(i_1, \dots, i_p)$ is the number of integers $i_1, \dots, i_p$ equal to k and if

$$A_p = \sum_{i_p=1}^p \sum_{i_{p-1}=1}^{p-1} \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)!,$$

then $A_p = 1 \cdot 3 \cdot \dots \cdot (2p-1) \leq 2^p p!$.

Proof. Use induction on p . The lemma is certainly true for $p=1$. Suppose that it also holds for p . Then

$$\begin{aligned}
 A_{p+1} &= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_{p+1}(i_1, \dots, i_{p+1})! \cdots \tau_1(i_1, \dots, i_{p+1})! \\
 &= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_{p+1})! \cdots \tau_1(i_1, \dots, i_{p+1})! \\
 &= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_p)! \cdots \tau_{i_{p+1}+1}(i_1, \dots, i_p)! \\
 &\quad \cdot [1 + \tau_{i_{p+1}}(i_1, \dots, i_p)]! \tau_{i_{p+1}-1}(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)! \\
 &= \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 [1 + \tau_{i_{p+1}}(i_1, \dots, i_p)] \tau_p(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)! \\
 &= (p+1) \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)! \\
 &\quad + \sum_{i_{p+1}=1}^{p+1} \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_{i_{p+1}}(i_1, \dots, i_p) \tau_p(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)! \\
 &= (p+1)A_p + \sum_{i_{p+1}=1}^{p+1} \cdots \sum_{i_1=1}^1 \left[\sum_{i_p=1}^{p+1} \tau_{i_{p+1}}(i_1, \dots, i_p) \right] \tau_p(i_1, \dots, i_p)! \\
 &\quad \cdots \tau_1(i_1, \dots, i_p)! \\
 &= (p+1)A_p + \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 p \tau_p(i_1, \dots, i_p)! \cdots \tau_1(i_1, \dots, i_p)! \\
 &= (p+1)A_p + pA_p = (2p+1)A_p = 1 \cdot 3 \cdot 5 \cdots (2p-1)(2p+1)
 \end{aligned}$$

and the lemma is proved.

LEMMA 10. If p, q and M are positive integers and $L \geq 0$, then

$$\sum_{N_1, \dots, N_p} (M + N_1 + \cdots + N_p)^q C_{N_p} \cdots C_{N_1} \leq q! (2L)^q \exp [M/L] 2^p.$$

Proof. If B_n^m equals the number of ways of picking $0 \leq k_m \leq \cdots \leq k_1 \leq n$, then $B_n^m = B_{n-1}^m + B_{n-1}^{m-1}$ and hence $B_n \leq 2^{m+n}$. Thus

$$\begin{aligned}
 &\sum_{N_1, \dots, N_p} (M + N_1 + \cdots + N_p)^q C_{N_p} \cdots C_{N_1} \\
 &\leq \sum_{k=0}^q \binom{q}{k} \left[\sum_{N_1, \dots, N_{p-1}} C_{N_{p-1}} \cdots C_{N_1} (M + N_1 + \cdots + N_{p-1})^k \right] \left[\sum_{N_p=1}^{\infty} N_p^{q-k} C_{N_p} \right] \\
 &\leq \sum_{k=0}^q \frac{q!}{k!} L^{q-k} \sum_{N_1, \dots, N_{p-1}} C_{N_{p-1}} \cdots C_{N_1} (M + N_1 + \cdots + N_{p-1})^k \\
 &\leq \sum_{k_1=0}^q \frac{q!}{k_1!} L^{q-k_1} \sum_{k_2=0}^{k_1} \frac{k_1!}{k_2!} L^{k_1-k_2} \cdots \sum_{k_p=0}^{k_{p-1}} \frac{k_{p-1}!}{k_p!} L^{k_{p-1}-k_p} M^{k_p} \\
 &= q! L^q \sum_{k_1=0}^q \sum_{k_2=0}^{k_1} \cdots \sum_{k_p=0}^{k_{p-1}} \frac{L^{-k_p}}{k_p!} M^{k_p} \\
 &\leq q! L^q \exp [M/L] B_{q+1}^{p-1} \leq q! (2L)^q \exp [M/L] 2^p.
 \end{aligned}$$

LEMMA 11. *There exists a sequence a_n with $\lim_{n \rightarrow \infty} a_n = 0$ for which*

$$\sum_{N=n}^{\infty} N^p C_N \leq (p+2)! L^{p+2} a_n, \quad p \geq 0.$$

Proof. Let

$$a_n(p) = \frac{\sum_{N=n}^{\infty} N^p C_N}{(p+2)! L^{p+2}}$$

and

$$a_n = \sum_{q=1}^{\infty} a_n(q) < \infty.$$

We need only check that $\lim_{n \rightarrow \infty} a_n = 0$. But this follows from the monotone convergence theorem, since the functions $a_n(\cdot)$ are nonnegative monotone decreasing with $\lim_{n \rightarrow \infty} a_n(p) = 0$.

Let $E_N^j = (e_{i_1}, \dots, e_{i_N})$ be an N -tuple of variables for which $i_k \neq i_p$ when $k \neq p$; the N -tuples may be different for different j . Using this notation, we have

$$G_n \phi_p \cdots G_n \phi_1 = \sum_{N_p=1}^{n-1} n^{-N_p} \sum_{(n)}^{(n)} C(e_{i_p} | E_{N_p}^p) \Delta_{i_p} \phi_p \\ \sum_{N_{p-1}=1}^{n-1} n^{-N_{p-1}} \sum_{(n)}^{(n)} C(e_{i_{p-1}} | E_{N_{p-1}}^{p-1}) \Delta_{i_{p-1}} \cdots \Delta_{i_1} \phi_1.$$

We can break this sum into two parts; the first part having each choice of $E_{N_1}^1, \dots, E_{N_p}^p$ such that no two sets have a variable in common, and the second being bounded by

$$(7) \quad 2^p \|\phi_1\| \cdots \|\phi_p\| \sum_{\max(N_1, \dots, N_p) \leq n-1} n^{-N_1 - \cdots - N_p} \\ \cdot M(M+N_1) \cdots (M+N_1 + \cdots + N_p) W C_{N_1} \cdots C_{N_p}$$

where $M(M+N_1) \cdots (M+N_1 + \cdots + N_p)$ is the number of ways of choosing $e_{i_1}, \dots, e_{i_p}$ and

$$W = N_1(N_2 + \cdots + N_p) n^{N_1 + \cdots + N_p - 1} + N_2(N_3 + \cdots + N_p) n^{N_1 + \cdots + N_p - 1} \\ + \cdots + N_{p-1} N_p n^{N_1 + \cdots + N_p - 1};$$

$N_k(N_{k+1} + \cdots + N_p) n^{N_1 + \cdots + N_p - 1}$ bounds the number of ways of having one of the paired variables in E_k and one in $E_{k+1} \cup \cdots \cup E_p$. Clearly we have

$$|W| \leq (M+N_1 + \cdots + N_p)^2 n^{N_1 + \cdots + N_p - 1}.$$

Using this bound for $|W|$ in (7) and using Lemma 10, we have (7) bounded by

$$(8) \quad n^{-1} 2^p \|\phi_1\| \cdots \|\phi_p\| \sum_{N_1, \dots, N_p} (M+N_1 + \cdots + N_p)^{p+2} C_{N_1} \cdots C_{N_p} \\ \leq n^{-1} \|\phi_1\| \cdots \|\phi_p\| \exp [M/L] (p+2)! (8L)^{p+2}.$$

In the first sum, the sets of variables $E_{N_1}^1, \dots, E_{N_p}^p$ are such that no two sets have a variable in common. Thus, since we will be integrating over all of the variables whose indices are outside of J , we shall rename those variables by letting $E_{N_i}^i = (e_{i1}, \dots, e_{iN_i})$. Furthermore, we shall rename the variables whose indices are in J as $e_{01}, e_{02}, \dots, e_{0M}$. With this in mind, we have modulo J^c ,

$$(9) \quad G_n \phi_p \cdots G_n \phi_1 + \text{error} = \sum_{N_p=1}^{n-1} \sum_{i_{p-1}, j_{p-1} \leq n} C(e_{i_{p-1}j_{p-1}} | E_{N_p}^p) \\ \cdot \Delta_{i_{p-1}j_{p-1}} \phi_p \sum_{N_{p-1}=1}^{n-1} \cdots \sum_{N_1=1}^{n-1} \sum_{i_0, j_0 \leq n} C(e_{i_0j_0} | E_{N_1}^1) \Delta_{i_0j_0} \phi_1.$$

If (9) converges as $n \rightarrow \infty$, it clearly converges to $D\phi_p \cdots D\phi_1$. To see that it does converge, we use Lemma 9 to show that the tail of the series (9) is bounded by

$$\begin{aligned} & \sum_{q=1}^p \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} 2^p \|\phi_1\| \cdots \|\phi_p\| M(M+N_1) \cdots (M+N_1+\cdots+N_p) C_{N_1} \cdots C_{N_p} \\ &= \sum_{q=1}^p (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} \left(\sum_{i_p=1}^p N_{i_p} \right) \left(\sum_{i_{p-1}=1}^{p-1} N_{i_{p-1}} \right) \\ & \quad \cdots \left(\sum_{i_1=1}^1 N_{i_1} \right) C_{N_1} \cdots C_{N_p} \\ &= (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^p \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \sum_{N_p=1}^{\infty} \cdots \sum_{N_q=n}^{\infty} \cdots \sum_{N_1=1}^{\infty} N_{i_1} \\ & \quad \cdots N_{i_p} C_{N_1} \cdots C_{N_p} \\ &= (2M)^p M \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^p \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \left(\sum_{N_p=1}^{\infty} N_p^{i_p(i_1, \dots, i_p)} C_{N_p} \right) \\ (10) \quad & \quad \cdots \left(\sum_{N_q=n}^{\infty} N_q^{i_q(i_1, \dots, i_p)} C_{N_q} \right) \cdots \left(\sum_{N_1=1}^{\infty} N_1^{i_1(i_1, \dots, i_p)} C_{N_1} \right) \\ & \leq (2M)^p M \|\phi_1\| \cdots \|\phi_p\| a_n \sum_{q=1}^p \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_p)! L^{i_p(i_1, \dots, i_p)} \\ & \quad \cdots [\tau_q(i_1, \dots, i_p) + 2]! L^{i_q(i_1, \dots, i_p) + 2} \cdots \tau_1(i_1, \dots, i_p)! L^{i_1(i_1, \dots, i_p)} \\ & \leq a_n (2ML)^p ML^2 \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^p (p+2)(p+1) \sum_{i_p=1}^p \cdots \sum_{i_1=1}^1 \tau_p(i_1, \dots, i_p)! \\ & \quad \cdots \tau_1(i_1, \dots, i_p)! \\ & \leq a_n (4ML)^p ML^2 \|\phi_1\| \cdots \|\phi_p\| \sum_{q=1}^p (p+2)! \\ & \leq a_n \|\phi_1\| \cdots \|\phi_p\| (p+3)! (4LM)^{p+3}. \end{aligned}$$

Thus combining the inequalities (8) and (10), we have

$$(11) \quad \|G_n \phi_p \cdots G_n \phi_1 - D \phi_p \cdots D \phi_1\|_{J^c} \leq (n^{-1} + a_n) \|\phi_1\| \cdots \|\phi_p\| \exp [M/L](p+3)!(8LM)^{p+3}.$$

Using (10) with $n=1$, we see that

$$\exp [t_m D] \phi_m \cdots \exp [t_1 D] \phi_1$$

converges since

$$\begin{aligned} & \|\exp [t_m D] \phi_m \cdots \exp [t_1 D] \phi_1\| \\ &= \left\| \sum_{p_m=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \frac{t_1^{p_1} \cdots t_m^{p_m}}{p_1! \cdots p_m!} D^{p_m} \phi_m \cdots D^{p_1} \phi_1 \right\| \\ &\leq \sum_{p_m=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \frac{t_1^{p_1} \cdots t_m^{p_m}}{p_1! \cdots p_m!} a_1 \|\phi_1\| \cdots \|\phi_m\| (p_1 + \cdots + p_m + 3)!(4LM)^{p_1 + \cdots + p_m + 3} \\ &= a_1 \|\phi_1\| \cdots \|\phi_m\| (4LM)^3 \sum_{q=0}^{\infty} \sum_{p_1 + \cdots + p_m = q} (4LM t_1)^{p_1} \cdots (4LM t_m)^{p_m} \frac{(q+3)!}{p_1! \cdots p_m!} \\ &= a_1 \|\phi_1\| \cdots \|\phi_m\| (4LM)^3 \sum_{q=0}^{\infty} (q+3)(q+2)(q+1)(4LM t_1 + \cdots + 4LM t_m)^q \\ &= 6a_1 \|\phi_1\| \cdots \|\phi_m\| (4LM)^3 (1 - 4LM t_1 - \cdots - 4LM t_m)^{-4} \end{aligned}$$

for $|4LM t_1 + \cdots + 4LM t_m| < 1$.

Finally, using (11) we have

$$\begin{aligned} & \|\exp [t_m G_n] \phi_m \cdots \exp [t_1 G_n] \phi_1 - \exp [t_m D] \phi_m \cdots \exp [t_1 D] \phi_1\|_{J^c} \\ &\leq \sum_{p_1=0}^{\infty} \cdots \sum_{p_m=0}^{\infty} \frac{t_1^{p_1} \cdots t_m^{p_m}}{p_1! \cdots p_m!} \|G_n^{p_m} \phi_m \cdots G_n^{p_1} \phi_1 - D^{p_m} \phi_m \cdots D^{p_1} \phi_1\|_{J^c} \\ &\leq (n^{-1} + a_n) \|\phi_1\| \cdots \|\phi_m\| \exp [M/L] \sum_{p_1=0}^{\infty} \cdots \sum_{p_m=0}^{\infty} \frac{t_1^{p_1} \cdots t_m^{p_m}}{p_1! \cdots p_m!} \\ &\quad \cdot (p_1 + \cdots + p_m + 3)!(8LM)^{p_1 + \cdots + p_m + 3} \\ &\leq (n^{-1} + a_n) \|\phi_1\| \cdots \|\phi_m\| e^{M/L} (1 - 8LM t_1 - \cdots - 8LM t_m)^{-4}. \end{aligned}$$

Letting $n \rightarrow \infty$, the theorem is proved.

Proof of Theorem 5. Using equation (5), we have

$$\lim_{n \rightarrow \infty} P[x_i^n(t_\nu) = a_\nu, x_j^n(t_\nu) = b_\nu, 1 \leq \nu \leq m]$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \int f^n \exp [t_1 G_n] \chi_{a_1}^i \chi_{b_1}^j \cdots \exp [(t_m - t_{m-1}) G_n] \chi_{a_m}^i \chi_{b_m}^j \\ &= \lim_{n \rightarrow \infty} \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \int f^n G_n^{p_1} \chi_{a_1}^i \chi_{b_1}^j \cdots G_n^{p_m} \chi_{a_m}^i \chi_{b_m}^j \end{aligned}$$

$$\begin{aligned}
&= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \int f^\infty D^{p_1} \chi_{a_1}^i \chi_{b_1}^j \cdots D^{p_m} \chi_{a_m}^i \chi_{b_m}^j \\
&= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \int f^\infty D^{p_1} \chi_{a_1}^i \chi_{b_1}^j \cdots D^{p_{m-1}} \chi_{a_{m-1}}^i \chi_{b_{m-1}}^j \\
&\quad \sum_{k_m=0}^{p_m} \binom{p_m}{k_m} (D_{k_m} \cdots D_1 \chi_{a_m}^i) \otimes (D_p \cdots D_{k_m+1} \chi_{b_m}^j) \\
&= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \sum_{k_m=0}^{p_m} \binom{p_m}{k_m} \int f^\infty D^{p_1} \chi_{a_1}^i \chi_{b_1}^j \\
&\quad \cdots D^{p_{m-1}} (\chi_{a_{m-1}}^i D^{k_m} \chi_{a_m}^i) \otimes (\chi_{b_{m-1}}^j D^{p_m-k_m} \chi_{b_m}^j) \\
&= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{p_1! \cdots p_m!} \sum_{k_m=0}^{p_m} \cdots \sum_{k_1=0}^{p_1} \binom{p_m}{k_m} \cdots \binom{p_1}{k_1} \\
&\quad \cdot \int f^\infty (D^{k_1} \chi_{a_1}^i \cdots D^{k_m} \chi_{a_m}^i) \otimes (D^{p_1-k_1} \chi_{b_1}^j \cdots D^{p_m-k_m} \chi_{b_m}^j) \\
&= \sum \frac{t_1^{p_1} \cdots (t_m - t_{m-1})^{p_m}}{k_1! \cdots k_m! (p_1 - k_1)! \cdots (p_m - k_m)!} \\
&\quad \cdot \int f^\infty (D^{k_1} \chi_{a_1}^i \cdots D^{k_m} \chi_{a_m}^i) \otimes (D^{p_1-k_1} \chi_{b_1}^j \cdots D^{p_m-k_m} \chi_{b_m}^j) \\
&= \left(\int f^\infty \exp [t_1 D] \chi_{a_1}^i \cdots \exp [(t_m - t_{m-1}) D] \chi_{a_m}^i \right) \\
&\quad \cdot \left(\int f^\infty \exp [t_1 D] \chi_{b_1}^j \cdots \exp [(t_m - t_{m-1}) D] \chi_{b_m}^j \right) \\
&= \lim_{n \rightarrow \infty} P[x_i^n(t_\nu) = a_\nu, 1 \leq \nu \leq m] \lim_{n \rightarrow \infty} P[x_j^n(t_\nu) = b_\nu, 1 \leq \nu \leq n].
\end{aligned}$$

Proof of Theorem 6. Clearly

$$\exp [tD] \phi \otimes \psi \equiv \left(\sum \frac{t^p}{p!} D_{2p} D_{2p-2} \cdots D_2 \phi \right) \otimes \left(\sum \frac{t^p}{p!} D_{2p-1} D_{2p-3} \cdots D_1 \psi \right)$$

modulo the indices of variables of ϕ and ψ and thus $\exp [tD] \phi \otimes \psi$ converges modulo the indices of variables of ϕ and ψ whenever $\exp [tD] \phi$ and $\exp [tD] \psi$ converge. Using this fact, we have for $8Lt < 1$ and χ_e^i defined as in (6),

$$\begin{aligned}
&\int_{(1)^c} f^\infty \exp [tD] \chi_{\xi_1}^1 \otimes \cdots \otimes \chi_{\xi_m}^m (e_0, \dots) \\
&= \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_{(1)^c} f^\infty D^p \chi_{\xi_1}^1 \otimes \cdots \otimes \chi_{\xi_m}^m (e_0, \dots) \\
&= \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_{(1)^c} f^\infty \sum_{k=0}^p \binom{p}{k} [D_p \cdots D_{k+1} \chi_{\xi_1}^1 (e_0, \dots)] \otimes D_k \cdots D_1 \chi_{\xi_2}^2 \cdots \chi_{\xi_m}^m
\end{aligned}$$

$$\begin{aligned}
 (12) \quad &= \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{k=0}^p \binom{p}{k} \left[\int_{\{1\}^c} f^{\infty} D_p \cdots D_{k+1} \chi_{\xi_1}^1(e_0, \dots) \right] \int f^{\infty} D_k \cdots D_1 \chi_{\xi_2}^2 \cdots \chi_{\xi_m}^m \\
 &= \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{t^k t^{p-k}}{k!(p-k)!} \left[\int_{\{1\}^c} f^{\infty} D^{p-k} \chi_{\xi_1}^1(e_0, \dots) \right] \int f^{\infty} D^k \chi_{\xi_2}^2 \cdots \chi_{\xi_m}^m \\
 &= \left[\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_{\xi_1}^1(e_0, \dots) \right] \int f^{\infty} \exp [tD] \chi_{\xi_2}^2 \cdots \chi_{\xi_m}^m \\
 &= \left[\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_{\xi_1}^1(e_0, \dots) \right] \prod_{k=2}^m \int f^{\infty} \exp [tD] \chi_{\xi_k}^k.
 \end{aligned}$$

Now suppose that ϕ is a function on the product space E^m . Then for $a, b \in E$ and $8Lt < 1$ we have, using (12),

$$\begin{aligned}
 &\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_b^1 \phi(a, \dots) \\
 &= \sum_{\xi_1, \dots, \xi_m} \left[\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_{\xi_1}^1 \otimes \cdots \otimes \chi_{\xi_m}^m(a, \dots) \right] \\
 &\quad \cdot \chi_b^1(\xi_1, \dots, \xi_m) \phi(\xi_1, \dots, \xi_m) \\
 (13) \quad &= \sum_{\xi_1, \dots, \xi_m} \left[\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_{\xi_1}^1(a, \dots) \right] \left(\prod_{k=2}^m \int f^{\infty} \exp [tD] \chi_{\xi_k}^k \right) \\
 &\quad \cdot \chi_b^1(\xi_1, \dots, \xi_m) \phi(\xi_1, \dots, \xi_m) \\
 &= \sum_{\xi_1} \left[\int_{\{1\}^c} f^{\infty} \exp [tD] \chi_{\xi_1}^1(a, \dots) \right] \int_{\{1\}^c} f_t^{\infty} \chi_b^1(\xi_1, \dots) \phi(\xi_1, \dots) \\
 &= \int_{\{1\}^c} f^{\infty} \exp [tD] \chi_b^1(a, \dots) \int_{\{1\}^c} f_t^{\infty} \phi(b, \dots).
 \end{aligned}$$

We can now easily prove the theorem if we notice that (13) holds whenever ϕ can be written as an absolutely converging series $\sum \phi_k$, where each ϕ_k has a finite number of variables. Since $\phi = \exp [t_2 D] \chi_{e_2}^1 \cdots \exp [t_m - t_{m-1} D] \chi_{e_m}^1$ is such a function, we have

$$\begin{aligned}
 &\int_{\{1\}^c} f^{\infty} \exp [t_1 D] \chi_{e_1}^1 \cdots \exp [(t_m - t_{m-1}) D] \chi_{e_m}^1 \\
 &= \left[\int_{\{1\}^c} f^{\infty} \exp [t_1 D] \chi_{e_1}^1(e_0, \dots) \right] \int_{\{1\}^c} f_{t_1}^{\infty} \exp [(t_2 - t_1) D] \chi_{e_2}^1 \\
 &\quad \cdots \exp [(t_m - t_{m-1}) D] \chi_{e_m}^1(e_1, \dots) \\
 &= \left[\int_{\{1\}^c} f^{\infty} \exp [t_1 D] \chi_{e_1}^1(e_0, \dots) \right] \left[\int_{\{1\}^c} f_{t_1}^{\infty} \exp [(t_2 - t_1) D] \chi_{e_2}^1(e_1, \dots) \right] \\
 &\quad \cdot \int_{\{1\}^c} (f_{t_1})_{t_2 - t_1}^{\infty} \exp [(t_3 - t_2) D] \chi_{e_3}^1 \cdots \exp [(t_m - t_{m-1}) D] \chi_{e_m}^1(e_2, \dots).
 \end{aligned}$$

Proceeding in this manner and noting that $(f_t)_s = f_{t+s}$, we prove the first part of Theorem 6.

Finally,

$$\begin{aligned}\gamma_{\pm 1}(u) &= \frac{d}{dt} P_{f|\pm 1}(t; 1) \Big|_{t=0} \\ &= \frac{d}{dt} \int_{\{1\}^c} f^\infty e^{tD} \chi_{\pm 1}^1(\pm 1, \dots) \Big|_{t=0} \\ &= \int_{\{1\}^c} f^\infty D \chi_{\pm 1}^1(\pm 1, \dots) \\ &= \int_{\{1\}^c} f^\infty \sum_{N=1}^{\infty} C(\pm 1 | e_{2,1}, \dots, e_{2,N}) [\chi_{\pm 1}(\mp 1) - \chi_{\pm 1}(\pm 1)] \\ &= \mp \sum_{N=1}^{\infty} \sum_{k=0}^N C_N^{\pm 1}(k) \binom{N}{k} u^k (1-u)^{N-k},\end{aligned}$$

the differentiation being justified by the absolute convergence of the resulting sum. This completes the proof.

Proof of Theorem 8. A necessary condition for a function to be in H is that it be real analytic, as the following lemma demonstrates.

LEMMA 12. *If $F \in H$, then F has derivatives of all orders and*

$$|(d/du)^p F(u)| \leq p!(2L)^p.$$

Proof.

$$\begin{aligned}\left| \left(\frac{d}{du} \right)^p F(u) \right| &= \left| \left(\frac{d}{du} \right)^p \sum_{N=1}^{\infty} \sum_{k=0}^N C_N(k) \binom{N}{k} u^k (1-u)^{N-k} \right| \\ &= \left| \sum_{q=0}^p \binom{p}{q} \sum_{N=p-q}^{\infty} \sum_{k=q}^{N-p+q} C_N(k) \binom{N}{k} k \cdots (k-q+1) u^{k-q} \right. \\ &\quad \cdot (N-k) \cdots (N-k-p+q+1) (-1)^{p-q} (1-u)^{N-k-p+q} \Big| \\ &= \left| \sum_{q=0}^p \binom{p}{q} \sum_{N=p-q}^{\infty} \sum_{k=0}^{N-p} C_N(k+q) \binom{N}{k+q} (k+q) \cdots (k+1) \right. \\ &\quad \cdot (N-k-q) \cdots (N-k-p+1) (-1)^{p-q} u^k (1-u)^{N-p-k} \Big| \\ &\leq \sum_{q=0}^p \binom{p}{q} \sum_{N=p-q}^{\infty} \frac{N!}{(N-p)!} C_N \sum_{k=0}^{N-p} \binom{N-p}{k} u^k (1-u)^{N-p-k} \\ &\leq \sum_{q=0}^p \binom{p}{q} \sum_{N=p-q}^{\infty} N^p C_N \leq \sum_{q=0}^p \binom{p}{q} p! L^p = p!(2L)^p.\end{aligned}$$

Notice that the term-wise differentiation is justified by the convergence of the resulting sums.

LEMMA 13. If $F, G \in H$, then $FG \in H$.

Proof. Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^N C_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

$$G(u) = \sum_{N=1}^{\infty} \sum_{k=0}^N d_N(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Then

$$F(u)G(u) = \sum_{N=1}^{\infty} \sum_{k=0}^N e_N(k) \binom{N}{k} u^k (1-u)^{N-k}$$

where

$$e_N(k) = \binom{N}{k}^{-1} \sum_{N_1+N_2=N} \sum_{k_1+k_2=k; k_1 \leq N_1; k_2 \leq N_2} C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1}{k_1} \binom{N_2}{k_2}.$$

Letting $e_N = \max_{k \leq n} e_N(k)$, we have

$$\begin{aligned} \sum_{N=1}^{\infty} N^p e_N &\leq \sum_{N=1}^{\infty} \sum_{k=0}^N N^p e_N(k) \\ &= \sum_{N=1}^{\infty} \sum_{k=0}^N N^p \binom{N}{k}^{-1} \sum_{N_1+N_2=N} \sum_{k_1+k_2=k; k_1 \leq N_1; k_2 \leq N_2} C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1}{k_1} \binom{N_2}{k_2} \\ &= \sum_{N_1, N_2} \sum_{k_1 \leq N_1; k_2 \leq N_2} (N_1 + N_2)^p C_{N_1}(k_1) d_{N_2}(k_2) \binom{N_1 + N_2}{k_1 + k_2}^{-1} \binom{N_1}{k_1} \binom{N_2}{k_2} \\ &\leq \sum_{N_1, N_2} \sum_{k_1 \leq N_1; k_2 \leq N_2} (N_1 + N_2)^p C_{N_1}(k_1) d_{N_2}(k_2) \\ &\leq \sum_{N_1, N_2} N_1 N_2 \sum_{q=0}^p \binom{p}{q} N_1^q N_2^{p-q} C_{N_1} d_{N_2} \\ &\leq \sum_{q=0}^p \binom{p}{q} (\sum N_1^{q+1} C_{N_1}) (\sum N_2^{p-q+1} d_{N_2}) \\ &\leq \sum_{q=0}^p \binom{p}{q} (q+1)! L^{q+1} (p-q+1)! L^{p-q+1} \leq (4L)^{p+2} p!. \end{aligned}$$

LEMMA 14. If $F \in H$, then $\exp(F) \in H$.

Proof. Let

$$F(u) = \sum_{N=1}^{\infty} \sum_{k=0}^N C_N(k) \binom{N}{k} u^k (1-u)^{N-k}.$$

Then

$$\exp[F(u)] = \sum_{M=1}^{\infty} \sum_{j=0}^M \binom{M}{j} \hat{C}_M(j) u^j (1-u)^{M-j},$$

where

$$\hat{C}_M(j) = \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1 + \dots + N_n = M} \sum_{k_1 + \dots + k_n = j} \binom{N_1}{k_1} \dots \binom{N_n}{k_n} C_{N_1}(k_1) \dots C_{N_n}(k_n);$$

in this last expression, $k_1 \leq N_1, \dots, k_n \leq N_n$. Thus

$$\begin{aligned} \sum_{M=1}^{\infty} M^p \hat{C}_M &\leq \sum_{M=1}^{\infty} \sum_{j=0}^M M^p \hat{C}_M(j) \\ &= \sum_{M=1}^{\infty} \sum_{j=0}^M M^p \binom{M}{j}^{-1} \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1 + \dots + N_n = M} \sum_{k_1 + \dots + k_n = j} \binom{N_1}{k_1} \dots \binom{N_n}{k_n} C_{N_1}(k_1) \dots C_{N_n}(k_n) \quad (k_1 \leq N_1, \dots, k_n \leq N_n) \\ &= \sum_{N_1, \dots, N_n} \sum_{k_1 \leq N_1, \dots, k_n \leq N_n} \sum_{n=0}^{\infty} (n!)^{-1} (N_1 + \dots + N_n)^p \cdot \binom{N_1 + \dots + N_n}{k_1 + \dots + k_n}^{-1} \binom{N_1}{k_1} \dots \binom{N_n}{k_n} C_{N_1}(k_1) \dots C_{N_n}(k_n) \\ &\leq \sum_{N_1, \dots, N_n} \sum_{k_1 \leq N_1, \dots, k_n \leq N_n} \sum_{n=0}^{\infty} (n!)^{-1} (N_1 + \dots + N_n)^p C_{N_1} \dots C_{N_n} \\ &\leq \sum_{N_1, \dots, N_n} \sum_{n=0}^{\infty} (n!)^{-1} N_1 \dots N_n (N_1 + \dots + N_n)^p C_{N_1} \dots C_{N_n} \\ &= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{N_1, \dots, N_n} \sum_{s_1 + \dots + s_n = p} \frac{p!}{s_1! \dots s_n!} \cdot N_1^{s_1} \dots N_n^{s_n} N_1 \dots N_n C_{N_1} \dots C_{N_n} \\ &= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{s_1 + \dots + s_n = p} p! L^p(s_1 + 1) \dots (s_n + 1) \\ &= \sum_{n=0}^{\infty} (n!)^{-1} L^p 2n(2n+1) \dots (2n+p-1) \\ &\leq (2L)^p p! \sum_{n=0}^{\infty} (n!)^{-1} \binom{n+p-1}{p} \\ &\leq p! (2L)^p \sum_{n=0}^{\infty} (n!)^{-1} 2^n 2^{p-1} \leq p! (4L)^p e^2. \end{aligned}$$

LEMMA 15 (HAUSDORFF [1]). *If a polynomial F is positive (>0) on the open interval $(0, 1)$, then it can be expressed as*

$$F(u) = \sum_{m=0}^N a_m \chi_{N,m}(u), \quad a_m \geq 0$$

where

$$\chi_{N,m}(u) = \binom{N}{m} u^m (1-u)^{N-m},$$

provided that N is sufficiently large.

LEMMA 16. If F is a complex valued function on a complex disc $|z| \leq 1 + \delta$, $\delta > 0$, real on the real numbers and analytic on the closed disc $|z| \leq 1$; then for any sufficiently large real constant C , $F(z) + C \in H$.

Proof. Let F be real on the real line and analytic in the closed disc $|z| \leq 1$. Then there exists a $\delta > 0$ such that $F(z) = \sum_{N=0}^{\infty} \alpha_N z^N$ for $|z| \leq 1 + \delta$ where

$$|\alpha_N| = |F^{(N)}(0)|/N! \leq (1 + \delta)^{-N} A, \quad A \text{ a positive constant.}$$

Now let

$$b_N = \alpha_N \text{ if } \alpha_N \geq 0, \quad a_N = -\alpha_N \text{ if } \alpha_N < 0, \quad C = d - b_0 + \sum_{N=0}^{\infty} a_N, \quad d \geq 0, \\ = 0 \text{ otherwise,} \quad = 0 \text{ otherwise,}$$

Then

$$F(z) + C = d + \sum_{N=1}^{\infty} b_N(z^N) + \sum_{N=1}^{\infty} a_N(1 - z^N).$$

But

$$1 - z^N = \sum_{k=0}^{N-1} \binom{N}{k} z^k (1 - z)^{N-k}, \quad N \geq 1,$$

and hence, letting

$$B_N(z) = \sum_{k=0}^N C_N(k) \binom{N}{k} z^k (1 - z)^{N-k}$$

where $C_1(0) = a_1 + d$, $C_1(1) = b_1 + d$ and

$$C_N(k) = a_N \text{ for } 0 \leq k < N, \quad N > 1, \\ = b_N \text{ for } k = N,$$

we get

$$F(z) + C = \sum_{N=1}^{\infty} B_N(z).$$

Thus we have represented $F + C$ as a sum of Bernstein polynomials when $C \geq -b_0 + \sum_{N=0}^{\infty} a_N$. We therefore need only show that there exists an $L > 0$ such that

$$\sum_{N=1}^{\infty} N^p C_N \leq p! L^p, \quad p \geq 1.$$

But

$$C_N = \max_{k \leq N} C_N(k) \leq (1 + \delta)^{-N} A.$$

Hence

$$\sum_{N=0}^{\infty} N^p C_N \leq A \sum_{N=0}^{\infty} N^p \exp[-N \ln(1 + \delta)]$$

which corresponds to

$$\int_0^{\infty} t^p \exp[-t \ln(1 + \delta)] dt = \frac{p!}{[\ln(1 + \delta)]^{p+1}} \leq p! L^p$$

for a suitable L .

We can now give sufficient conditions that F be contained in H .

LEMMA 17. If F is analytic on the closed disc $|z| \leq 1$, real on the reals and positive on $[0, 1]$, then $F \in H$.

Proof. Since F is analytic on $|z| \leq 1$, real on the reals and positive on $[0, 1]$, it can be written as $\exp [-C](z-z_1)(z-z_1^*) \cdots (z-z_n)(z-z_n^*) \exp [C+G(z)]$ where z^* is the conjugate of z , G is analytic on $|z| \leq 1$ and real on the reals. For C sufficiently large, $C+G(z) \in H$ and hence $\exp [C+G(z)] \in H$. Since, according to Hausdorff's lemma, $(z-z_1)(z-z_1^*) \cdots (z-z_n)(z-z_n^*) \in H$ and since H is closed under products, the proof is complete.

LEMMA 18. If $F, G \in H$ and G is a polynomial, $0 < G(u) < 1$ on $(0, 1)$, then $F[G(\cdot)] \in H$.

Proof. Let

$$\begin{aligned} F(u) &= \sum_{N=1}^{\infty} \sum_{k=0}^N C_N(k) \binom{N}{k} u^k (1-u)^{N-k} \\ G(u) &= \sum_{p=0}^M d(p) u^p (1-u)^{M-p} \\ 1-G(u) &= \sum_{q=0}^M e(q) u^q (1-u)^{M-q}. \end{aligned}$$

Then

$$\begin{aligned} F[G(u)] &= \sum_{N=1}^{\infty} \sum_{k=0}^N C_N(k) \binom{N}{k} \left[\sum_{p=0}^M d(p) u^p (1-u)^{M-p} \right]^k \left[\sum_{q=0}^M e(q) u^q (1-u)^{M-q} \right]^{N-k} \\ &= \sum_{N=1}^{\infty} \sum_{j=0}^{NM} \hat{C}_{NM}(j) \binom{NM}{j} u^j (1-u)^{MN-j}, \end{aligned}$$

where

$$\begin{aligned} \hat{C}_{NM}(j) &= \binom{NM}{j}^{-1} \sum_{k=0}^N C_N(k) \binom{N}{k} \sum_{p_1 + \cdots + p_k + q_1 + \cdots + q_{N-k} = j; 0 \leq p, q \leq M} \\ &\quad \cdot d(p_1) \cdots d(p_k) e(q_1) \cdots e(q_{N-k}). \end{aligned}$$

Using Stirling's formula, we have

$$\begin{aligned} \binom{NM}{j}^{-1} &\sim (2\pi MN)^{1/2} \left(\frac{j}{MN} \right)^{1/2} \left(1 - \frac{j}{MN} \right)^{1/2} \left(\frac{j}{MN} \right)^j \left(1 - \frac{j}{MN} \right)^{MN-j} \\ &\leq 2\pi MN \left(\frac{j}{MN} \right)^j \left(1 - \frac{j}{MN} \right)^{MN-j} \end{aligned}$$

and thus

$$\begin{aligned} \sum_{N=1}^{\infty} (NM)^p \hat{C}_{NM} &\leq \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \hat{C}_{NM}(j) \\ &= \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \sum_{k=0}^N C_N(k) \binom{N}{k} \binom{NM}{j}^{-1} \sum_{p_1 + \cdots + p_k + q_1 + \cdots + q_{N-k} = j; 0 \leq p, q \leq M} \\ &\quad \cdot d(p_1) \cdots d(p_k) e(q_1) \cdots e(q_{N-k}) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{N=1}^{\infty} (NM)^p \sum_{j=0}^{NM} \sum_{k=0}^N C_N(k) \binom{N}{k} 2\pi MN \sum_{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j; 0 \leq p, q \leq M} \\
&\quad \cdot d(p_1) \cdots d(p_k) e(q_1) \cdots e(q_{N-k}) \left(\frac{j}{NM}\right)^j \left(1 - \frac{j}{NM}\right)^{NM-j} \\
&\leq 2\pi M^{p+1} \sum_{N=1}^{\infty} N^{p+1} C_N \sum_{m=0}^{MN} \sum_{k=0}^N \binom{N}{k} \sum_{j=0}^{NM} \sum_{p_1+\dots+p_k+q_1+\dots+q_{N-k}=j; 0 \leq p, q \leq M} \\
&\quad \cdot d(p_1) \cdots d(p_k) e(q_1) \cdots e(q_{N-k}) \left(\frac{m}{NM}\right)^j \left(1 - \frac{m}{NM}\right)^{NM-j} \\
&= 2\pi M^{p+1} \sum_{N=1}^{\infty} N^{p+1} C_N \sum_{m=0}^{MN} \sum_{k=0}^N \binom{N}{k} \left[G\left(\frac{m}{NM}\right)\right]^k \left[1 - G\left(\frac{m}{NM}\right)\right]^{N-k} \\
&= 2\pi M^{p+2} \sum_{N=1}^{\infty} N^{p+2} C_N \\
&\leq 2\pi M^{p+2} (p+2)! L^{p+2} \leq p! \hat{L}^p
\end{aligned}$$

for suitable $\hat{L}$.

We are now in a position to prove Theorem 8. If $F \in H$, then F is certainly positive on the open interval $(0, 1)$ and, by Lemma 12, it is real analytic on the closed interval $[0, 1]$. Therefore assume that F is positive and real analytic on the closed interval $[0, 1]$; if F had roots at 0 or 1, we could divide through by them. Then we can find a domain D , symmetric about and containing the interval $[0, 1]$ on which F is analytic. If there exists a polynomial G mapping D conformally onto a domain containing the unit disc; and if G is real on the reals with $G(0)=0$ and $G(1)=1$; then $F[G^{-1}(w)]$ is analytic on the closed unit disc; real on the reals and positive on $[0, 1]$. Thus, by Lemma 17, $F[G^{-1}(\cdot)] \in H$ and by Lemma 18 $F[G^{-1}(G(\cdot))]=F(\cdot) \in H$. Therefore, to complete the proof we need only show that G exists.

Let $G_1(z)$ be the unique conformal mapping of D onto the disc $|z| < 1$, where $G_1(0)=0$ and $G_1'(0) > 0$. Since D is symmetric about the interval $[0, 1]$, $[G_1(z^*)]^*$ also maps D conformally onto $|z| < 1$ and hence $[G_1(z^*)]^* = G_1(z)$ and G_1 is real on the real axis. Let $G_2(z) = [G_1(1)]^{-1} G_1(z)$. Then G_2 maps D conformally onto the disc $|z| < [G_1(1)]^{-1}$. Clearly G_2 is real on the real axis with $G_2(0)=0$ and $G_2(1)=1$. Let $G_3(z)$ be a polynomial for which $G_3(0)=0$ and $|G_2(z) - G_3(z)| < \varepsilon$ for all $z \in D$, and define $G(z)$ as $a[G_3(z) + G_3(z^*)]^*$ where $a = [G_3(1) + G_3(1)^*]^{-1}$. Clearly we have $(2+2\varepsilon)^{-1} \leq |a| \leq (2-2\varepsilon)^{-1}$. If D' is the inverse image of the disc

$$|w| < (2-2\varepsilon)^{-1} [1 + [G_1(1)]^{-1} - 2\varepsilon] - \varepsilon(1-\varepsilon)^{-1}$$

under the mapping G , and C is the inverse image of the circle $|w| = 2^{-1} [1 + [G_1(1)]^{-1}]$ under the mapping G_2 ; then $G_2 \neq 0, \infty$ on C and

$$2aG_2(z) - w = G(z) - w + h(z)$$

where $|G(z) - w| > \varepsilon(1 - \varepsilon)^{-1} > |h(z)|$ for all $z \in C$ and $w \in G(D')$. Thus, by Rouché's theorem, $G(z)$ takes on the value w only once. Therefore, since $G(D')$ is a disc containing the unit disc for ε sufficiently small and since G is a 1-1 mapping of D' onto the disc $G(D')$ with $G(0) = 0$, $G(1) = 1$ and G real on the real axis; the proof is complete.

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