ON MINIMAL BLOCKS[1,2]

BY

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1. **Introduction.**   A connected graph G is a *block* if there is no point v in G such that G-v is disconnected. A block G will be called *block-line-critical* (b.l.c.) if, for every line x in G, G-x is not a block. Such graphs occur repeatedly in the study of blocks—for example, when a proof by induction on the number of lines of a block is being attempted.

Clearly, a cycle of any length is a b.l.c. graph. There are, however, b.l.c. graphs with a much more complex structure. In this paper several structural characterizations of b.l.c. graphs are obtained as well as a number of additional properties of such graphs.

2. **Additional terminology.**   For the sake of completeness, we introduce the following additional definitions. A *graph G* is a finite nonempty set $V(G)$ of *points* together with a collection $E(G)$ of *lines* each of which is an unordered pair of points. If x is the line containing the points u and v, we write $x = uv$ and say that u and v are *adjacent*, x *joins* u and v, and that x is *incident* with points u and v. Two lines x and y which have a common point are also said to be *adjacent*. The *complete graph on p points*, K_p, is that graph with p points in which every two points are adjacent. A *subgraph* of G is a graph all of whose points and lines are also in G. The *subgraph of a graph G generated by a set of lines X* is that graph H whose set of lines is X and whose points are those points of G incident with a line of X.

A *path P* is an alternating sequence of distinct points and lines, beginning and ending with points (said to be *joined* by P) such that each line is incident with the points before and after it. The first and last points in this sequence are called the *endpoints* of P, and all other points are termed *intermediate*. We shall have occasion to refer to a path P by its sequence of points; e.g., $P = [u_1, u_2, \ldots, u_n]$. If $P = [u_1, u_2, \ldots, u_m]$ and $Q = [v_1, v_2, \ldots, v_n]$ are two paths where the intermediate points of P are all distinct from the intermediate points of Q, $u_m = v_1$, and either $v_n = u_1$ or v_n is distinct from every point of P, we define a new path $P + Q$, the *sum of P and Q*, to be that path with point sequence $[u_1, u_2, \ldots, u_m = v_1, v_2, \ldots, v_n]$. A path of length $\geqq 2$ together with a line joining the first and last points is called a *cycle*. A path or a

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cycle is *odd* (*even*) if the number of lines in it is odd (even). As for paths, we shall have need to designate a cycle by the sequence of its points; e.g., $C = [u_1, u_2, \ldots u_n, u_1]$. If C is a cycle in a graph G and x is a line of G which is not in C, but which joins two points of C, then x is called a *diagonal* of C. More generally, a line x is a *diagonal of a graph G* if it is a chord of some cycle in G.

The graph G is *connected* if every two points are joined by a path. If x is a line of G, G-x will denote the graph obtained from G by deleting x. Similarly, G-v will denote the graph obtained from G by deleting the point v and all lines incident with v. More generally, if N is any set of points or lines in G, G-N will denote the graph obtained by deleting each element of the set N. A point v is a *cutpoint* of the connected graph G if G-v is not connected, and the set N is called a *separating* (or *disconnecting*) *set* for the connected graph G if G-N is disconnected. A line x is called a *bridge* in a connected graph G if G-x is disconnected. The cardinality of any minimum separating set of lines in G is called the *line-connectivity* of G and is denoted $\lambda(G)$. If v is any point of a graph G, a *branch* of G at v is a maximal connected subgraph of G not having v as a cutpoint. A connected graph is a *block* if it has no cutpoints. A line x in a block B is said to be *critical* with respect to B if B-x is not a block. A connected graph containing no cycles is a *tree*.

The *degree* of a point v, $d(v)$, is the number of lines incident with it. A set of points is said to be *independent* if no two of its members are adjacent. Let $|M|$ denote the number of elements in a set M. If M is a maximum independent set of points in G, $|M|$ is called the *point independence number* of G and is denoted by $\beta_0(G)$. A set of points M is said to *cover* a graph G if every line of G has at least one point in M. If M is a point cover for G with a minimum number of elements, then $|M|$ is called the *point covering number* of G and is denoted by $\alpha_0(G)$. Finally, a graph G is *bipartite* if $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 so that every line of G joins a point of V_1 and a point of V_2. Let m and n be the number of points in V_1 and V_2, respectively. The *complete bipartite graph* $K(m, n)$ is the bipartite graph in which every point of V_1 is adjacent to every point of V_2.

3. Characterization of block-line-critical graphs. The following theorem immediately yields a first characterization of b.l.c. graphs.

THEOREM 1. *Let G be a block and x a line of G. Then G-x is a block if and only if x is a diagonal in G.*

Proof. Clearly, if x is a diagonal of G, then G-x is still a block. On the other hand, let $x = uv$ be a line in G and assume that G-x is a block. Then in G-x there must be a cycle containing u and v and x is a diagonal of this cycle.

COROLLARY 1a. *The following three conditions are equivalent for any graph G:*
(1) *G is block-line-critical,*
(2) *G is a block containing no diagonals, and*
(3) *G is a block in which no point is adjacent to three points of the same cycle.*

COROLLARY 1b. *If G is a b.l.c. graph and $G \neq K_3$, then G contains no triangles.*

Proof. Suppose G has a triangle with points u, v, and w. At least one of these, say u, is incident with a line x not in the triangle. But then there must be a cycle containing lines x and vw, and clearly either uv or vw is a diagonal of this cycle.

A cycle is, of course, a b.l.c. graph in which every point has degree two. We next proceed to show that any b.l.c. graph must contain points of degree two and that such points have an important bearing upon the structure of the b.l.c. graph.

THEOREM 2. *Let $x = uv$ be any line of a b.l.c. graph G and let w be any cutpoint of G-x. Then*

(1) *w is on every cycle containing x, and*

(2) *every path containing x and w has an intermediate point of degree two in G.*

Proof. For (1), it is sufficient to show that u and v are in different blocks of G-x. But if u and v were in the same block of G-x, they would lie on a cycle in this block and such a cycle would have x as a diagonal in G, thus contradicting Corollary 1a.

We proceed to prove (2). Let B be the block of G-x containing v. Let w be the cutpoint of G-x contained in B. Let $P = [u, v, v_1, v_2, \ldots, v_n, w]$ be a path in G (cf. Figure 1).

We assume that the degree of each point $v = v_0, v_1, v_2, \ldots, v_n, v_{n+1} = w$ is at least three in G. Form a new graph A from B by deleting the lines of the path P. In A, each v_i must be joined to another v_j $(i \neq j)$ by a path Q which has no intermediate points in P. Moreover, Q cannot join v_i and v_{i+1} since the line $v_i v_{i+1}$ would be a diagonal of the cycle determined by Q, P-$v_i v_{i+1}$, and any path of G-x joining w and u and not containing v. In particular, v_0 is joined by a path in A to v_k, $k > 1$. Let
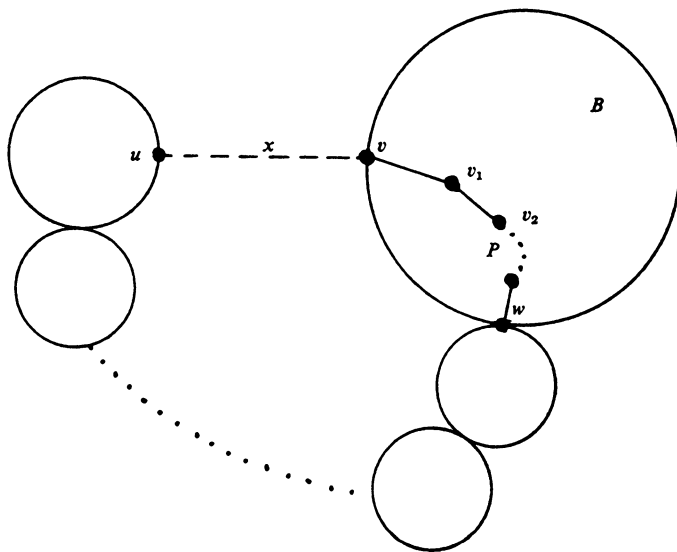

FIGURE 1

α denote the greatest subscript i such that i is joined by a path Q in A to some v_m, $m > i$. Clearly, $0 \leqq \alpha \leqq n$. But then $v_{\alpha+1}$ must be joined by a path R in A to some v_β, $\beta < \alpha$. Furthermore, Q and R are disjoint since otherwise there would be a path from v_α to $v_{\alpha+1}$ in A. Thus one can form a cycle from Q, R, and $P\text{-}v_\alpha v_{\alpha+1}$ (cf. Figure 2) which has $v_\alpha v_{\alpha+1}$ as a diagonal, contradicting Corollary 1a and completing the proof of the theorem.
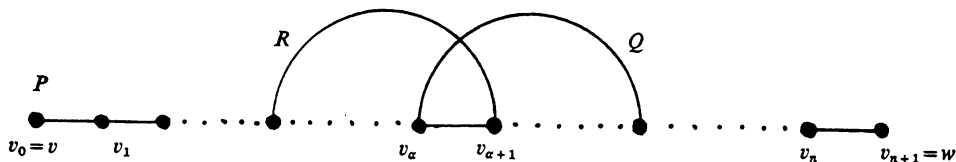

FIGURE 2

The following corollary is immediate.

COROLLARY 2a. *If G is a b.l.c. graph $\neq K_3$, and C is any cycle in G, then C contains at least one pair of independent points whose degree in G is two.*

According to Theorem 2, every cycle in a b.l.c. graph is separated by points of degree two. We next proceed to show that the graph as a whole is separated by these points.

Let G' be the subgraph of G generated by all points of degree greater than two and let $T_1, T_2, \ldots, T_s$ denote the components of G'. By Theorem 2, G' contains no cycles. Hence each T_i is a tree. Denote by S the set of points in G of degree two. A path P will be called a *S-path* if each of its intermediate points is of degree two in G.

THEOREM 3. *If G is a b.l.c. graph and if S and T_i are as defined above, then there is no S-path joining two points of the same tree T_i.*

Proof. Suppose the conclusion to be false; i.e., suppose that $[v_0, v_1, \ldots, v_n]$ is an S-path with v_0 and v_n in the same tree T of $G\text{-}S$. Clearly v_n is a cutpoint of $G\text{-}v_0 v_1$. But the path of T joining v_0 and v_n has no points of degree 2, contradicting Theorem 2.

We then have the following result.

COROLLARY 3a. *A b.l.c. graph G is separated by its points of degree 2.*

Proof. Suppose $G\text{-}S$ is connected. Then by Theorem 2, it contains no cycles and thus must be a tree. But then any S-path must join two points of $G\text{-}S$, contradicting Theorem 3.

To summarize then, a b.l.c. graph consists of at least two mutually point-disjoint trees $T_1, T_2, \ldots, T_s$ and a collection of paths joining pairs of T_i's, where the degree of all intermediate points of such a path is two.

Next it is shown that the array of S-paths in a b.l.c. graph is not completely arbitrary, but rather their location must be such so as to avoid the formation of cycles of a certain type. More specifically, we have the next theorem.

THEOREM 4. *Let G be a b.l.c. graph and let $T_1, T_2, \ldots, T_s$ ($s \geq 2$) be the component trees of G-S. Let C be any cycle in G. Then for each j, either $C \cap T_j = \varnothing$, or $C \cap T_j$ is connected.*

Proof. Suppose there is a cycle C in G and a component tree T_k of G-S such that $C \cap T_k$ is disconnected. Let $D_1, \ldots, D_r$ be the components of $C \cap T_k$ ($r > 1$). Now given i and j, there is a path P_{ij} in T_k joining a point of D_i and a point of D_j. Suppose P_0 is a path of minimum length joining two components of $C \cap T_k$, where the minimum is taken over all cycles C and all trees T_k, $k = 1, \ldots, s$, such that $C \cap T_k$ is disconnected. For the sake of argument, suppose P_0 joins two components of the intersection of the cycle C_0 and the tree T_1.

Observe first that since G contains no diagonal, P_0 consists of at least two lines. Let $P_0 = [u = u_0, u_1, \ldots, u_t = v]$, where u and v are in $V(C_0)$. Now since $d(u_1) > 2$, there is a line $y = u_1 w_1$ where $u_0 \neq w_1 \neq u_2$. If $w_1 \in V(C_0) - \{v\}$, then one may obtain a cycle with $x - u_0 u_1$ as a diagonal contrary to Theorem 1. Hence $w_1 \notin V(C_0) - \{v\}$. Also, if $w_1 \in V(P_0)$, then the length of P_0 would not be minimal, hence $w_1 \notin V(P_0)$.

Since G is a block, there is a cycle C_1 containing x and y. Let B_1 be the branch of C_1-x-y traversed by starting at w_1 and terminating upon the first encounter with a point p_0 of $V(C_0) \cup V(P_0)$. Once again a diagonal may be obtained if $p_0 \in V(C_0) - \{v\}$. Hence we may assume that $p_0 \in V(P_0) - \{u\}$ (cf. Figure 3).

Now not all lines of $B_1 + y$ are in T_1, for if they were, $B_1 + y + [u_1, u_2, \ldots, p_0]$ would be a cycle in T_1, contrary to the assumption that T_1 is a tree. Let C_L be one of the two paths in C_0 determined by the points u and v. Then $E(C_L) \not\subseteq E(T_1)$.
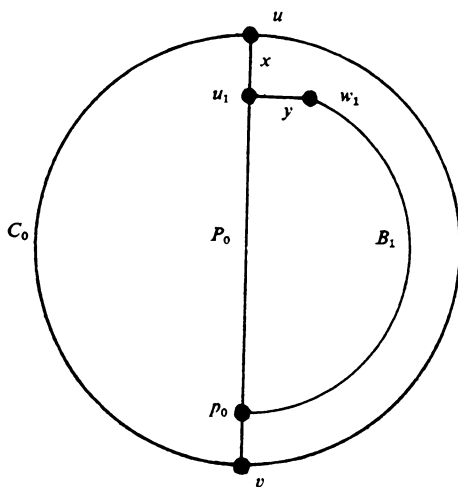

FIGURE 3

Hence, if P_0' is the subpath of P_0 with endpoints p_0 and v, then $C' = C_L + x + y + B_1 + P_0'$ is a cycle and $C' \cap T_1$ is disconnected. But then $P_0'' = [u_1, u_2, \ldots, p_0]$ is a path in T_1 joining two components of $C' \cap T_1$ and the length of P_0'' is less than that of P_0. This contradicts the minimality of the length of P_0 and completes the proof of the theorem.

The following construction will prove useful to us later. If G is a b.l.c. graph, we define its *tree-condensation graph* G^* as follows: the points of G^* are the component trees of G-S, $T_1, T_2, \ldots, T_s$. Two points T_i and T_j are adjacent in G^* if and only if there is some S-path joining them in G. We may apply Theorem 4 to obtain the following result.

COROLLARY 4a. *If G is a b.l.c. graph, then G^* is a block.*

Proof. Suppose T_i is a cutpoint in G^*. Let T_j and T_k be two points of G^* adjacent to T_i, but which lie in different components of G^*-T_i. Let P_{ij} and P_{ik} be S-paths joining T_i and T_j and T_i and T_k respectively. Let x be a line of P_{ij} and y, a line of P_{ik}. Now x and y lie on a cycle C in G. But then $C \cap T_j \neq \varnothing$ and is disconnected, contrary to Theorem 4. Hence G^* has no cutpoints and the corollary is proved.

We may now use the properties of b.l.c. graphs derived in Theorems 2, 3, and 4 to characterize this family of graphs.

THEOREM 5. *Let G be a block. Then G is a b.l.c. graph if and only if:*
(1) *G is a cycle, or*
(2) *if S denotes the set of points of degree two in G, then there are at least two components T_j in G-S, each component of G-S is a tree, and if C is any cycle in G, for each j either $C \cap T_j = \varnothing$ or $C \cap T_j$ is connected.*

Proof. If G is a b.l.c. graph, then (1) or (2) holds by Theorems 2, 3, and 4.

On the other hand, if G is a cycle, it is clearly b.l.c.. So suppose G-S has components $T_1, \ldots, T_s$, $s \geq 2$, where each T_j is a tree and each has the cycle intersection property stated in the hypotheses. Let $x = uv$ be any line of G. If either $d(u) = 2$ or $d(v) = 2$, clearly G-x is not a block. Otherwise, x is a line in T_j, for some j. Now let the components of T_j-x be T_j' and T_j'' where $u \in V(T_j')$ and $v \in V(T_j'')$. Suppose G-x is a block. Then u and v lie on a cycle C in G-x. Hence $C' = C \cap T_j' \neq \varnothing$ and $C'' = C \cap T_j'' \neq \varnothing$. But $T_j' \cap T_j'' = \varnothing$, hence $C' \cap C'' = \varnothing$ and $C \cap T_j$ is disconnected, contrary to assumption. This completes the proof of the theorem.

4. Further properties of block-line-critical graphs. We next turn our attention to some additional properties of b.l.c. graphs. The *chromatic number* of a graph G, $\chi(G)$, is the minimum number of colors needed to color the points of G so that no two adjacent points have the same color. Clearly, any bipartite graph has chromatic number two. This observation, together with the next theorem, completely determines $\chi(G)$ for any b.l.c. graph G.

THEOREM 6. *If G is a b.l.c. graph, and G contains an odd cycle, then $\chi(G) = 3$.*

Proof. This is clear from our previously gained knowledge of the structure of any b.l.c. graph G. For if G is a cycle, the theorem is immediate. Otherwise, let the component trees of G-S be $T_1, T_2, \ldots, T_s$, $s \geq 2$. Since these trees are bipartite, each can be colored with two colors. Moreover, if P is any S-path joining two trees T_i and T_j, then it requires at most three colors to color the points of P, if the endpoints are already colored. On the other hand, since G contains an odd cycle, at least three colors are required to color the points of G. Hence $\chi(G) = 3$ and the Theorem is proved.

We may combine the result of Theorem 6 with a well-known inequality involving the point independence number of a b.l.c. graph G, $\beta_0(G)$.

COROLLARY 6a. *If G is a b.l.c. graph, then $\beta_0(G) \geq |V(G)|/3$.*

Proof. It is well known (cf. Ore [2]) that $\beta_0(G)\chi(G) \geq |V(G)|$, for any graph. This, combined with the result of Theorem 6, yields the desired inequality.

One may use a well-known result of Gallai [1], which says that $\alpha_0(G) + \beta_0(G) = |V(G)|$, to state the conclusion of Corollary 6a as: $\alpha_0(G) \leq 2|V(G)|/3$.

Since any b.l.c. graph G must contain a point of degree two, the line-connectivity of G, $\lambda(G)$, must be equal to two. Let G_0 denote the subgraph of G generated by all lines having at least one endpoint of degree two in G. We now proceed to investigate the possible locations of minimum disconnecting sets of two lines. There are, a priori, four possible types:

(A) both lines in G_0,
(B) one line in G_0, the other in T_i for some i,
(C) one line in T_i, the other in T_j, for $i \neq j$, or
(D) both lines in T_i, for some i.

If x and y are lines on the same S-path in G_0, then $\{x, y\}$ is a set of type (A) and every b.l.c. graph possesses minimum disconnecting sets of this kind. There may also be sets $\{x, y\}$ of type (A), where x is on an S-path joining T_i and T_j, y is on an S-path joining T_k and T_m, and where $T_i, T_j, T_k,$ and T_m are all distinct. The next theorem shows that these are the only two possibilities for sets of type (A).

THEOREM 7. *If $\{x, y\}$ is a minimum disconnecting set of type* (A) *for a b.l.c. graph G which is not a cycle, if x lies on an S-path joining T_i and T_j and if y lies on an S-path Q joining T_k and T_m, then either i, j, k, and m are all distinct, or $i=k$ and $j=m$ (or $i=m$ and $j=k$) and then P and Q are identical.*

Proof. By Theorem 4, $i \neq j$ and $k \neq m$. Suppose $i = k$. Then if $j \neq m$, T_i is a cutpoint of G^*, contradicting Corollary 4a. Thus $j = m$. Then, if P and Q are not identical, the line of G^* containing P and Q is a bridge in G^*. But then $G^* = K_2$, or else Corollary 4a is again contradicted. But if $G^* = K_2$, since $\{x, y\}$ is a disconnecting set, there are no other S-paths joining T_i and T_j and hence no other S-paths in G. But then since G is a block, it follows that T_i and T_j each have but one point which

therefore must be of degree two, contradicting the definitions of T_i and T_j. The proof is complete.

We have the following restrictions on sets of type (B).

THEOREM 8. *If G is a b.l.c. graph, $\{x, y\}$ is a set of type (B), x is a line in T_i, and y lies on an S-path P joining T_j and T_k, then $i \neq j$ and $i \neq k$.*

Proof. Suppose $\{x, y\}$, T_i, T_j, T_k, and P are as given in the hypothesis. Assume $i = j$. Now let T_i' and T_i'' be the components of T_i-x. Let $x = uv$, where $u \in V(T_i')$ and $v \in V(T_i'')$. Let z be a line of G_0 with exactly one endpoint in T_i', $z \neq x$, and suppose z is not on the path P. (Such a line must exist by definition of T_i'.) Now y and z must lie on a cycle C, which must necessarily contain x. But then C is a cycle in G, $C \cap T_i \neq \varnothing$, and $C \cap T_i$ is disconnected, contradicting Theorem 4. Hence $i \neq j$, and similarly $i \neq k$, completing the proof.

Finally, we show that type (D) sets do not exist.

THEOREM 9. *If G is a b.l.c. graph, $\{x, y\}$ is a minimum disconnecting set, x is in T_i, and y is in T_j, then $i \neq j$.*

Proof. Suppose x and y, as given in the hypothesis, both lie in T_i, for some i. Now T_i-x-y must consist of exactly three components T_1, T_2, and T_3, where, say, x joins T_1 and T_2 and y joins T_2 and T_3. For $k = 1, 2$, and 3, let G_k denote the maximal connected subgraph of G-x-y containing T_k, respectively. By definition, the G_k's are either identical or point-disjoint.

Suppose $G_2 \neq G_3$. Then G_2 and G_3 must be point-disjoint and thus y is a bridge of G, contradicting the assumption that G is a block. Thus $G_2 = G_3$. Similarly, $G_1 = G_2$. Thus $G_1 = G_2 = G_3$ and G-x-y is connected, contrary to the definition of the set $\{x, y\}$. This completes the proof.

We next establish upper and lower bounds for the number of lines in any b.l.c. graph in terms of the number of points.

THEOREM 10. *If G is a b.l.c. graph with $|V(G)| > 3$, and if L denotes the number of lines in G, then $|V(G)| \leqq L \leqq 2|V(G)| - 4$, and these bounds are best possible.*

Proof. The lower bound is clear, since G is in particular a block. If G is a cycle, then the lower bound for L is assumed.

If G is not a cycle, G must contain at least two points of degree greater than two. Let S, as before, denote the set of points of degree two, and let k be the number of components T_i of G-S. Then $2 \leqq |S| \leqq |V(G)| - 2$ and $2 \leqq k \leqq |S|$. Thus we have

$$
\begin{aligned}
L &\leqq 2|S| + \sum_{i=1}^{k} (|T_i| - 1) = 2|S| + \left(\sum_{i=1}^{k} |T_i|\right) - k \\
&= 2|S| + |V(G)| - |S| - k \\
&= |S| + |V(G)| - k \leqq |S| + |V(G)| - 2 \\
&\leqq |V(G)| - 2 + |V(G)| - 2 \\
&= 2|V(G)| - 4.
\end{aligned}
$$

To show that this upper bound is best possible, we note that if $G = K(2, |V(G)| - 2)$, the complete bipartite graph with two points of one color and $|V(G)| - 2$ points of another color, then G is a b.l.c. graph with exactly $2|V(G)| - 4$ lines.

Our final task is to determine which b.l.c. graphs are planar. To treat this problem we introduce some terminology. Let G be any graph. Denote by G_0 the graph obtained from G by replacing every path in which each intermediate point has degree two by a single line. G_0 is then said to be the *contraction* of G. The reader will recall that Kuratowski's Theorem states that a graph G is planar if and only if it contains no subgraph which can be contracted to K_5 or to $K(3, 3)$.

THEOREM 11. *A b.l.c. graph G is planar if and only if its tree-condensation graph G^* is planar.*

Proof. Clearly if G is planar, then so is G^*. To prove the converse, assume G to be nonplanar. Hence G contains a subgraph H which can be contracted to K_5 or to $K(3, 3)$. Suppose first that K_5 is the contraction of H. We shall call those points of H which remain points in K_5 *principal*. Since the degree in G of each principal point of H is three, each principal point must lie in a component tree of G-S.

We proceed to show that no component tree of G-S may contain more than one principal point. Let T_0 be such a component tree and let the principal points of H be u_1, u_2, u_3, u_4, and u_5. Further, let P_{ij} denote the path in H joining u_i and u_j and having all intermediate points of degree two. Suppose T_0 contains more than one of the u_i. To be precise, there are four possible cases:

(1) Suppose T_0 contains exactly two principal points, say u_1 and u_2. Then $C = P_{13} + P_{32} + P_{24} + P_{45} + P_{51}$ is a cycle in G and $C \cap T_0$ is nonempty and disconnected, contradicting Theorem 4.

(2) Suppose T_0 contains exactly three principal points, say u_1, u_2, and u_3. Then the cycle $C = P_{14} + P_{42} + P_{23} + P_{35} + P_{51}$ and T_0 contradict Theorem 4.

(3) Suppose T_0 contains exactly four principal points, say, u_1, u_2, u_3, and u_4. Further suppose that P_{12}, P_{23}, and P_{34} all lie in T_0. Then P_{24} does not lie entirely in T_0, for if it did, T_0 would contain the cycle $P_{23} + P_{34} + P_{24}$. Thus the cycle $C = P_{12} + P_{24} + P_{45} + P_{51}$ and T_0 violate Theorem 4. Hence one of P_{12}, P_{23}, and P_{34} fails to be entirely within T_0. Then the cycle $C = P_{12} + P_{23} + P_{34} + P_{45} + P_{51}$ and T_0 contradict Theorem 4.

(4) Suppose all principal points of H lie in T_0. Then one of P_{12}, P_{23}, P_{34}, P_{45}, and P_{51} must not lie in T_0, say P_{12}. But now if any other of these five paths also fails to lie in T_0, $C = P_{12} + P_{23} + P_{34} + P_{45} + P_{51}$ and T_0 violate Theorem 4. Thus assume that each of the five paths, other than P_{12}, lies in T_0. Then in particular, P_{24} cannot lie in T_0, so that the cycle $C = P_{12} + P_{24} + P_{45} + P_{51}$ and T_0 contradict Theorem 4.

Since all four cases lead to contradictions, the proof that each principal point of H lies in a different component tree of G-S is complete.

Next assume that there is a component tree T_0 of G-S such that $P_{ij} \cap T_0 \neq \varnothing$ and

$P_{km} \cap T_0 \neq \varnothing$, where P_{ij} and P_{km} are different. Then clearly P_{ij} and P_{km} must have a common endpoint, for if not, then $C = P_{ij} + P_{jm} + P_{mk} + P_{ki}$ is a cycle such that $C \cap T_0$ is nonempty and disconnected, again contradicting Theorem 4. Suppose that u_i is this common endpoint; i.e., suppose that $u_i = u_k$. Then $u_i \in V(T_0)$, for if not, then the cycle $C = P_{ij} + P_{jm} + P_{mi}$ and T_0 contradict Theorem 4.

The above results suffice to show that if G contains a subgraph which may be contracted to K_5, then so does G^*. A similar result holds for subgraphs contractible to $K(3, 3)$. The details are left to the reader. We have thus shown that if there is a subgraph of G contractible to K_5 or to $K(3, 3)$, then there is a subgraph of G^* with this same property. The proof of the theorem is thus complete.

The preceding theorem is in a sense the best result possible concerning the planarity of b.l.c. graphs. To realize this, one need only note that the next obvious question is: "What graphs can occur as tree-condensation graphs of b.l.c. graphs?" The answer is that any block B_0 can be thought of as the tree-condensation graph of some b.l.c. graph B. To find one such B, just insert one new point on each line of B_0. The resulting graph B then has the property that every line is incident with a point of degree two. Such a graph is clearly block-line-critical.

Added in proof. The author wishes to acknowledge the appearance of a paper on this subject by G. A. Dirac (*Minimally 2-connected graphs*, J. Reine Angew. Math. **228** (1967), 204–216) while our paper was in press. In particular, he obtains a structural result similar to our Theorem 2.

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