AN IDENTITY FOR ELLIPTIC EQUATIONS WITH APPLICATIONS

BY

C. A. SWANSON(1)

1. **Introduction.** An elementary identity involving a linear elliptic partial differential operator L and its associated hermitian form will be used to obtain new comparison theorems, oscillation theorems, and lower bounds for eigenvalues. Comparison theorems will be obtained for both subsolutions and complex-valued solutions in unbounded domains of Euclidean space, generalizing earlier results of Hartman and Wintner [4], Protter [8], and the author [11], [12]. Oscillation theorems of Kreith's type [6] will be extended to (i) unbounded domains; (ii) non-self-adjoint operators; and (iii) subsolutions.

Lower bounds for the eigenvalues of L arise naturally from the basic identity in the case of bounded domains, and are extended to unbounded domains when the coefficients of L satisfy suitable conditions. The form of the lower bounds is the same as that obtained by Protter and Weinberger [9], [10] for bounded domains.

2. **The main lemma.** The linear elliptic differential operator L defined by

$$Lv = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv)+2\sum_{i=1}^{n} B_iD_iv+Cv \tag{1}$$

will be considered on unbounded domains R in n-dimensional Euclidean space E^n. The boundary P of R is supposed to have a piecewise continuous unit normal vector at each point. As usual, points in E^n are denoted by $x=(x_1, x_2, \dots, x_n)$ and differentiation with respect to x_i is denoted by D_i, $i=1, 2, \dots, n$. The coefficients A_{ij}, B_i, and C are assumed to be real and continuous in $R \cup P$ and the matrix (A_{ij}) positive definite in R (ellipticity condition). The domain $\mathfrak{D}_L=\mathfrak{D}_L(R)$ of L is defined to be the set of all complex-valued functions $v \in C^1(R \cup P)$ such that all derivatives of v involved in Lv exist and are continuous at every point in R.

Let T_a denote the n-disk $\{x \in E^n : |x-x_0|<a\}$ and let S_a denote the bounding $(n-1)$-sphere, where x_0 is a fixed point in E^n. Define

$$R_a = R \cap T_a, \quad P_a = P \cap T_a, \quad C_a = R \cap S_a. \tag{2}$$

Clearly there exists a positive number a_0 such that R_a is a bounded domain with boundary $P_a \cup C_a$ for all $a \geq a_0$.

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Let $Q[z]$ be the hermitian form in $n+1$ variables $z_1, z_2, \ldots, z_{n+1}$ defined by

$$Q[z] = \sum_{i,j=1}^{n} A_{ij} z_i \bar{z}_j - \sum_{i=1}^{n} B_i (z_i \bar{z}_{n+1} + z_{n+1} \bar{z}_i) + G|z_{n+1}|^2 \tag{3}$$

where G is any continuous function in R satisfying the inequality

$$G \det (A_{ij}) \geqq \sum_{i=1}^{n} B_i B_i^*, \tag{4}$$

B_i^* denoting the cofactor of $-B_i$ in the matrix associated with $Q[z]$. Condition (4) is known to be necessary and sufficient for $Q[z]$ to be positive semidefinite [2], [12].

Let M_a be the quadratic functional defined by

$$M_a[u] = \int_{R_a} F[u]\, dx, \tag{5}$$

where

$$F[u] = \sum_{i,j} A_{ij} D_i u D_j \bar{u} - 2 \operatorname{Re} \left(u \sum_i B_i D_i \bar{u} \right) + (G - C)|u|^2. \tag{6}$$

Define $M[u] = \lim_{a \to \infty} M_a[u]$ (whenever the limit exists). The domain $\mathfrak{D}_M = \mathfrak{D}_M(R)$ of M is defined to be the set of all complex-valued functions $u \in C^1(R \cup P)$ such that $M[u]$ exists and u vanishes on P.

Define

$$[u, v]_a = \int_{C_a} u \sum_{i,j} A_{ij} n_i D_j v\, ds, \tag{7}$$

where (n_i) denotes the unit normal to C_a, and define

$$[u, v] = \lim_{a \to \infty} [u, v]_a, \tag{8}$$

whenever the limit on the right side exists. The notation $M[u; R]$ will be used for $M[u]$ and $[u, v; R]$ will be used for $[u, v]$ in §5 when different domains are under consideration.

An *L-subsolution* (*-supersolution*) is a real-valued function $v \in \mathfrak{D}_L(R)$ which satisfies $Lv \leqq 0$ ($Lv \geqq 0$) at every point in R.

The following are extensions of results in [12] to subsolutions and supersolutions, and to complex-valued functions $u \in \mathfrak{D}_M(R)$.

LEMMA 1. *For every $u \in C^1(R)$ and every real $v \in \mathfrak{D}_L(R)$ which does not vanish in R, the following identity is valid at each point in R:*

$$\sum_{i,j} A_{ij} X_i \bar{X}_j - 2 \operatorname{Re} \left(u \sum_i B_i \bar{X}_i \right) + G|u|^2 + \sum_i D_i(|u|^2 Y_i) = F[u] + |u|^2 v^{-1} Lv, \tag{9}$$

where

$$X_i = v D_i(u/v), \qquad Y_i = v^{-1} \sum_{j=1}^{n} A_{ij} D_j v, \qquad i = 1, 2, \ldots, n.$$

The proof is a direct calculation similar to that given in [12].

THEOREM 1. *If there exists $u \in \mathfrak{D}_M(R)$ not identically zero such that $M[u]<0$, then there does not exist an L-subsolution (-supersolution) v satisfying $[|u|^2/v, v] \geqq 0$ which is positive (negative) everywhere in $R \cup P$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geqq 0$ must vanish at some point of $R \cup P$. In the self-adjoint case $B_i=0$, $i=1, 2, \ldots, n$, and $G=0$, the same conclusions are valid when the hypothesis $M[u]<0$ is weakened to $M[u] \leqq 0$.*

Proof. Suppose to the contrary that there exists such a positive L-subsolution. Then integration of (9) over R_a yields

$$\int_{R_a} F[u]\,dx \geqq \int_{R_a} \sum_i D_i(|u|^2 Y_i)\,dx \tag{10}$$

since the first three terms on the left side of (9) constitute a positive semidefinite form by the hypothesis (4). Since $u=0$ on P_a, by the definition of $\mathfrak{D}_M$, it follows from Green's formula that the right side of (10) is equal to

$$\int_{P_a \cup C_a} \sum_i |u|^2 n_i Y_i\,ds = \int_{C_a} \frac{|u|^2}{v} \sum_{i,j} A_{ij} n_i D_j v\,ds = [|u|^2/v, v]_a.$$

Thus (7), (10), and the hypothesis $[|u|^2/v, v] \geqq 0$ imply that

$$M[u] = \lim_{a\to\infty} \int_{R_a} F[u]\,dx \geqq 0.$$

The contradiction proves that a positive L-subsolution satisfying $[|u|^2/v, v] \geqq 0$ cannot exist. The analogous statement for a negative L-supersolution v follows from the fact that $-v$ would then be a positive L-subsolution.

To prove the second statement of Theorem 1, suppose to the contrary that there exists a real solution $v \neq 0$ in $R \cup P$. Then v would be either a positive L-subsolution or a negative L-supersolution in $R \cup P$.

The proof in the self-adjoint case is similar to that given in [12, p. 281] and will be omitted.

We remark that the condition $[|u|^2/v, v] \geqq 0$ of Theorem 1 is a mild "boundary condition at ∞" generalizing the usual condition $v \neq 0$ on the boundary of bounded domains.

3. **Lower bounds for eigenvalues.** Let $\mathfrak{H}$ be the Hilbert space $\mathscr{L}^2(R)$, with inner product $\langle u, v\rangle = \int_R u(x)\bar{v}(x)\,dx$ and norm $\|u\| = \langle u, u\rangle^{1/2}$. Let $\mathfrak{D}$ be the set of all complex-valued functions $u \in \mathfrak{D}_L \cap \mathfrak{H}$ such that u vanishes on P. In this section the elliptic operator (1), with domain $\mathfrak{D}$, is assumed to have the self-adjoint form

$$Lv = \sum_{i,j} D_i(A_{ij} D_j v) - Cv,$$

under the conditions described below (1). In the case of the Schrödinger operator $-L = -\Delta + C(x)$, it is well-known [1], [3, p. 146] that the lower part of the spectrum contains only eigenvalues of finite multiplicity if $C(x)$ is bounded from below.

In the self-adjoint elliptic case, an assumption on the coefficients A_{ij} is needed as well.

Let $A^+(x)$ denote the largest eigenvalue of $(A_{ij}(x))$ and define

$$\alpha(r) = \max_{1 \leqq |x| \leqq r} A^+(x),$$

$$\alpha_0(r) = \max\left[\alpha(1), \max_{1 \leqq |x| \leqq r} |x|^{-2} A^+(x)\right],$$

which are nondecreasing functions of r. The following assumptions are special cases of those given by Ikebe and Kato [5].

ASSUMPTIONS. (i) $C(x)$ is bounded from below;

(ii) $\int_1^\infty [\alpha(r)\alpha_0(r)]^{-1/2} = \infty$.

It follows in particular from (i) and (ii) that the conditions $u \in \mathfrak{H}$, $Lu \in \mathfrak{H}$ imply that $[u, u] = 0$ [5].

Our purpose is to obtain a useful lower bound for the eigenvalues (if any) of $-L$. In the case of bounded domains, Protter and Weinberger [10] recently obtained results of this type by using a general form of the maximum principle. It will be shown here in the case of unbounded domains that a lower bound is available as an easy consequence of Lemma 1.

THEOREM 2. *Let λ be the lowest eigenvalue and u be an associated normalized eigenfunction of the problem $-Lu = \lambda u$, $u \in \mathfrak{D}$. If v is any function in $\mathfrak{D}_L$ such that $v(x) > 0$ for $x \in R \cup P$ and $[|u|^2/v, v] \geqq 0$, then*

$$\lambda \geqq \inf_{x \in R} [-Lv(x)/v(x)]. \tag{11}$$

Proof. With $B_i = 0$, $i = 1, 2, \ldots, n$ and $G = 0$, integration of (9) over R_a yields

$$M_a[u] + \int_{R_a} |u|^2 v^{-1} Lv \; dx \geqq \int_{R_a} \sum_i D_i(|u|^2 Y_i) \, dx \tag{12}$$

where the positive-definiteness of (A_{ij}) has been taken into account. Since $u = 0$ on P_a, it follows from Green's formula that

$$M_a[u] = -\int_{R_a} \bar{u} Lu \, dx + [u, u]_a$$

$$= \lambda \int_{R_a} |u|^2 \, dx + [u, u]_a.$$

However, $\lim\, [u, u]_a = 0$ $(a \to \infty)$ is a general consequence of $u \in \mathfrak{H}$ and $Lu \in \mathfrak{H}$ under the above assumptions [5], and therefore

$$M[u] = \lim_{a \to \infty} M_a[u] = \lambda \|u\|^2 = \lambda.$$

As in the proof of Theorem 1, the right member of (12) has the limit $[|u|^2/v, v]$ as $a \to \infty$, which is nonnegative by hypothesis. Thus

$$\lambda + \int_R |u|^2 v^{-1} Lv \, dx \geqq 0,$$

which implies (11).

In the bounded case, the condition $[|u|^2/v, v] \geqq 0$ is vacuous and Theorem 2 reduces to a well-known result [9]. However, the proof given here is especially easy. We remark that the extra condition $[|u|^2/v, v] \geqq 0$ in the unbounded case is a condition on the asymptotic behavior of v as $|x| \to \infty$; it is roughly equivalent to the usual hypotheses for bounded domains that $u=0$ on the boundary, $v>0$ in $R \cup P$, and $v \in C^1(R \cup P)$. In the case of the Schrödinger operator $-\Delta + C(x)$, it is known [3, p. 179] that $|u(x)| < Ke^{-\mu|x|}$, where K and μ are constants, for every eigenfunction u, and hence various exponential functions can serve as the test functions v. As an easy example, consider the one-dimensional harmonic oscillator problem

$$-\frac{d^2u}{dx^2} + x^2 u = \lambda u, \qquad 0 \leqq x < \infty,$$

$$u(0) = 0.$$

The test function $v = \exp(-x^2/2)$ yields the lower bound 1 whereas the exact lowest eigenvalue is known to be 3.

4. **Comparison theorems.** Consider, in addition to (1), a second elliptic operator l defined by

$$lu = \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + 2 \sum_{i=1}^{n} b_i D_i u + cu \tag{13}$$

in which the coefficients satisfy the same conditions as the coefficients in (1). In addition to (5) consider the quadratic functional defined by

$$m_a[u; Q] = \int_{Q \cap T_a} \left[\sum_{i,j} a_{ij} D_i u D_j \bar{u} - 2 \operatorname{Re}\left(u \sum_i b_i D_i \bar{u} \right) - c|u|^2 \right] dx$$

for every subdomain $Q \subset R$, and let $m[u; Q] = \lim m_a[u; Q]$ $(a \to \infty)$. The domain $\mathfrak{D}_m(Q)$ of m is the analogue of $\mathfrak{D}_M(Q)$ (defined in §2). The *variation* of L relative to the domain Q is defined as $V[u; Q] = m[u; Q] - M[u; Q]$, that is

$$V[u; Q] = \int_Q \left[\sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \bar{u} - 2 \operatorname{Re}\left\{ u \sum_i (b_i - B_i) D_i \bar{u} \right\} + (C - c - G)|u|^2 \right] dx, \tag{14}$$

with domain $\mathfrak{D}_V(Q) = \mathfrak{D}_m(Q) \cap \mathfrak{D}_M(Q)$.

The analogues of (7), (8) for the operator l relative to the domain Q are

$$\{u, v; Q\}_a = \int_{Q \cap S_a} \sum_{i,j} a_{ij} n_i \operatorname{Re}(u D_j \bar{v})\, ds; \tag{15}$$

$$\{u, v; Q\} = \lim_{a \to \infty} \{u, v; Q\}_a. \tag{16}$$

When $Q = R$ is the only domain under consideration, the abbreviations $V[u]$, $\{u, v\}$ will be used for $V[u; R]$, $\{u, v; R\}$, respectively.

The following comparison theorems of Sturm's type are easy extensions of those in [12] to L-subsolutions (-supersolutions) and to complex-valued solutions of $lu=0$.

THEOREM 3. *Suppose G is a continuous function in R satisfying the inequality* (4). *If there exists a nontrivial solution $u \in \mathfrak{D}_V(R)$ of $lu=0$ such that $\{u, u\} \leqq 0$ and $V[u]>0$ then there does not exist an L-subsolution* (*-supersolution*) *which is positive* (*negative*) *everywhere in $R \cup P$ and satisfies $[|u|^2/v, v] \geqq 0$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geqq 0$ must vanish at some point of $R \cup P$. The same conclusions hold if the hypotheses $V[u]>0$, $[|u|^2/v, v] \geqq 0$ are replaced by $V[u] \geqq 0$, $[|u|^2/v, v]>0$, respectively.*

THEOREM 4. *With G as in Theorem* 3, *if there exists a positive l-supersolution $u \in \mathfrak{D}_V(R)$ such that $\{u, u\} \leqq 0$ and $V[u]>0$, then the conclusions of Theorem* 3 *are valid.*

THEOREM 5 (SELF-ADJOINT CASE). *Suppose $b_i=B_i=0$, $i=1, 2, \ldots, n$ in* (1) *and* (13) *and $G=0$. If there exists either* (i) *a nontrivial complex-valued solution $u \in \mathfrak{D}_V(R)$ of $lu=0$, or* (ii) *a positive l-supersolution $u \in \mathfrak{D}_V(R)$, such that $\{u, u\} \leqq 0$ and $V[u] \geqq 0$, then an L-subsolution* (*-supersolution*) *v satisfying $[|u|^2/v, v] \geqq 0$ cannot be everywhere positive* (*negative*) *in $R \cup P$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geqq 0$ must vanish at some point of $R \cup P$.*

Proof of Theorem 3. Since $u=0$ on P_a, it follows from Green's formula that

$$m_a[u] = -\int_{R_a} \operatorname{Re}(ul\bar{u})\, dx + \{u, u\}_a. \tag{17}$$

Since $lu=0$ and l has real-valued coefficients, also $l\bar{u}=0$. Since $\{u, u\} \leqq 0$, we obtain in the limit $a \to \infty$ that $m[u] \leqq 0$. The hypothesis $V[u]>0$ is equivalent to $M[u] < m[u]$. Hence $M[u]<0$ and Theorem 1 shows an L-subsolution (-supersolution) cannot be everywhere positive (negative) in $R \cup P$ under the hypothesis $[|u|^2/v, v] > 0$. The second statement of Theorem 3 also follows from Theorem 1. The last statement follows upon obvious modifications of the inequalities.

If u is a positive l-supersolution in R such that $\{u, u\} \leqq 0$, it follows again from [17] that $m[u] \leqq 0$. The proof of Theorem 4 is then completed in the same way as that of Theorem 3. The proof of Theorem 5 follows similarly from the statement in Theorem 1 relative to the self-adjoint case.

It follows from (14) by partial integration that

$$V[u; Q] = \int_Q \left[\sum_{i,j} (a_{ij}-A_{ij}) D_i u D_j \bar{u} + \delta |u|^2\right] dx + \Omega(Q),$$

where

$$\delta = \sum_{i=1}^{n} D_i(b_i - B_i) + C - c - G,$$

and

$$\Omega(Q) = \lim_{a \to \infty} \int_{Q \cap S_a} \sum_i (B_i - b_i)|u|^2 n_i\, ds,$$

whenever the limit exists.

L is called a *strict Sturmian majorant* of l in Q when the following conditions are fulfilled: (i) $(a_{ij}-A_{ij})$ is positive semidefinite and $\delta \geqq 0$ in Q; (ii) $\Omega(Q) \geqq 0$; and (iii) either $\delta > 0$ at some point in Q or $(a_{ij}-A_{ij})$ is positive definite and $c \neq 0$ at some point. A function defined in Q is said to be of class $C^{2,1}(Q)$ when all of its second partial derivatives exist and are Lipschitzian in Q.

THEOREM 6. *Suppose that L is a strict Sturmian majorant of l and that all the coefficients a_{ij} involved in l are of class $C^{2,1}(R)$. If there exists a nontrivial solution $u \in \mathfrak{D}_V(R)$ of $lu=0$ such that $\{u, u\} \leqq 0$, then no L-subsolution (-supersolution) v satisfying $[|u|^2/v, v] \geqq 0$ can be everywhere positive (negative) in $R \cup P$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geqq 0$ must vanish at some point of $R \cup P$.*

THEOREM 7 (SELF-ADJOINT CASE). *Suppose $b_i=B_i=0$, $i=1, 2, \ldots, n$ in* (1) *and* (13), *$G=0$, $C \geqq c$, and $(a_{ij}-A_{ij})$ is positive semidefinite in $R \cup P$. If there exists either* (i) *a nontrivial complex-valued solution $u \in \mathfrak{D}_V(R)$ of $lu=0$, or* (ii) *a positive l-supersolution $u \in \mathfrak{D}_V(R)$, such that $\{u, u\} \leqq 0$, then the conclusion of Theorem* 6 *is valid.*

Since the pointwise conditions $G=0$, $C \geqq c$, and $(a_{ij}-A_{ij})$ positive semidefinite obviously imply that $V[u] \geqq 0$, Theorem 7 is an immediate consequence of Theorem 5. The fact that the hypotheses of Theorem 6 imply $V[u] > 0$ was demonstrated in [12, p. 283], and consequently the conclusion of Theorem 6 follows from Theorems 3 and 4.

In the special case of the Schrödinger operator $-l = -\Delta + c(x)$ with $c(x)$ bounded from below in R, the hypothesis $\{u, u\} \leqq 0$ of Theorems 5 and 7 can be replaced by $u \in \mathfrak{H}$ and $lu \in \mathfrak{H}$ since these conditions imply that $\{u, u\} = 0$ [3, p. 56]. In the self-adjoint elliptic case, the same statement holds under quite general conditions on the coefficients, e.g. those stated prior to Theorem 2, as shown by Ikebe and Kato [5]. Also, the conclusion of Theorem 7 is valid even if (A_{ij}) is only positive *semidefinite* provided L is a strict Sturmian majorant of l and all the coefficients a_{ij} are of class $C^{2,1}(R)$ [12, p. 283].

5. **Oscillation theorems.** In [6] Kreith obtained oscillation theorems for self-adjoint elliptic equations of the form $Lv=0$ in the case that one variable x_n is separable. He considered the case of bounded domains for which part of the boundary is singular. Here we shall obtain oscillation theorems of a general nature on unbounded domains by appealing to the comparison Theorems 3–7.

Let T'_a denote the complement of T_a in E^n. A function u is said to be *oscillatory in R at ∞*, or simply *oscillatory in R*, whenever u has a zero in $R \cap T'_a$ for all $a > 0$.

A domain (not necessarily bounded) $Q \subset R$ is called a *nodal domain* of a function u iff $u=0$ on ∂Q and $\{u, u; Q\} \leqq 0$. If Q is bounded, the latter condition is understood to be void, and the definition reduces to the standard definition of a nodal domain. If $-l$ is the Schrödinger operator with potential $c(x)$ bounded from below, sufficient conditions for Q to be a nodal domain of $u \in D_l(Q)$ are $u=0$ on ∂Q, $u \in \mathfrak{H}$, and $lu \in \mathfrak{H}$ [3, p. 56]. A function u is said to have the *nodal property* in R whenever u has a nodal domain $Q \subset R \cap T'_a$ for all $a>0$.

The following results are immediate consequences of Theorems 3–7.

THEOREM 8. *Suppose G is a continuous function in R satisfying* (4). *Suppose there exists either* (i) *a nontrivial complex-valued solution u of $lu=0$, or* (ii) *a positive l-supersolution u, with the nodal property in R such that $V[u;\, Q]>0$ for every nodal domain Q. Then every real solution of $Lv=0$ is oscillatory in R provided $[|u|^2/v,\, v;\, Q] \geqq 0$ for every Q. If the nodal domains are all bounded, every solution of $Lv=0$ is oscillatory in R. In the self-adjoint case $b_i=B_i=0$, $i = 1, 2, \ldots, n$, the same conclusions hold under the weaker condition $V[u;\, Q] \geqq 0$ for every nodal domain Q.*

THEOREM 9. *Suppose that L is a strict Sturmian majorant of l and that all the coefficients involved in l are of class $C^{2,1}(R)$. If there exists a nontrivial complex-valued solution of $lu=0$ with the nodal property in R, then every real solution of $Lv=0$ is oscillatory in R provided $[|u|^2/v,\, v;\, Q] \geqq 0$ for every nodal domain Q. If the nodal domains are all bounded, every solution of $Lv=0$ is oscillatory in R. In the self-adjoint case $b_i=B_i=0$, $i=1, 2, \ldots, n$, the same conclusions hold under the weaker hypotheses $G=0$, $C \geqq c$, and $(a_{ij}-A_{ij})$ positive semidefinite in $R \cup P$.*

Kreith has shown [6] that equations of the form

$$(18) \quad D_n[a(x_n)D_n u] + \sum_{i,j=1}^{n-1} D_i[a_{ij}(\bar{x})D_j u] + c(x_n)u = 0, \qquad \bar{x} = (x_1, x_2, \ldots, x_{n-1}),$$

have bounded nodal domains in the form of cylinders, under suitable hypotheses, when R is a *bounded* domain with an $(n-1)$-dimensional singular boundary. We shall show that the analogous construction for unbounded domains is valid provided R is *limit cylindrical*, i.e. contains an infinitely long cylinder. Without loss of generality we can assume that R contains a cylinder of the form

$$G \times \{x_n : 0 \leqq x_n < \infty\},$$

where G is a bounded $(n-1)$-dimensional domain.

Let μ be the smallest eigenvalue of the boundary problem

$$(19) \quad \begin{aligned} -\sum_{i,j=1}^{n-1} D_i[a_{ij}(\bar{x})D_j\phi] &= \mu\phi \quad \text{in } G, \\ \phi &= 0 \quad \text{on } \partial G. \end{aligned}$$

THEOREM 10. *If there exists a positive number b such that*

$$(20) \qquad \int_b^\infty \frac{dt}{a(t)} = \infty \quad \textit{and} \quad \int_b^\infty [c(t)-\mu]\, dt = \infty,$$

then equation (18) *has a solution u with the nodal property in R. If $V[u;\, Q] \geqq 0$ for every nodal domain Q, every solution of $Lv=0$ is oscillatory in R. In particular,* *every solution of the self-adjoint equation* $Lv=0$ *is oscillatory provided* $C \geqq c$ *and* $(a_{ij}-A_{ij})$ *is positive semidefinite in* $R \cup P$.

Proof. The hypotheses (20) imply that the ordinary differential equation

$$D_n[a(x_n)D_nw]+[c(x_n)-\mu]w = 0$$

is oscillatory at $x_n=\infty$ on account of well-known theorems of Leighton [7] and Wintner [13]. Let w be a solution with zeros at $x_n=\delta_1, \delta_2, \ldots, \delta_m, \ldots,$ where $\delta_m \uparrow \infty$. If ϕ is an eigenfunction of (19) corresponding to the eigenvalue μ, then the function u defined by $u(x)=w(x_n)\phi(\bar{x})$ is a solution of (18) by direct calculation, with nodal domains in the form of cylinders

$$G_m = G\times\{x_n : \delta_m < x_n < \delta_{m+1}\}, \qquad m = 1, 2, \ldots.$$

Thus u has a nodal domain $G_m \subset R \cap T'_a$ for all $a>0$. In fact, given $a>0$, choose m large enough so that $\delta_m \geqq a$. Then $x \in G_m$ implies $|x| \geqq |x_n| > a$ so $x \in T'_a$. Hence (18) has a solution u with the nodal property. The second statement of Theorem 10 follows from Theorem 8 and the last statement follows from Theorem 9.

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University of British Columbia,
Vancouver, B. C., Canada