ON HAMILTONIAN LINE-GRAPHS(1)

BY
GARY CHARTRAND

Introduction. The *line-graph* $L(G)$ of a nonempty graph G is the graph whose point set can be put in one-to-one correspondence with the line set of G in such a way that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent. In this paper graphs whose line-graphs are eulerian or hamiltonian are investigated and characterizations of these graphs are given. Furthermore, necessary and sufficient conditions are presented for iterated line-graphs to be eulerian or hamiltonian. It is shown that for any connected graph G which is not a path, there exists an iterated line-graph of G which is hamiltonian.

Some elementary results on line-graphs. In the course of the article, it will be necessary to refer to several basic facts concerning line-graphs. In this section these results are presented. All the proofs are straightforward and are therefore omitted. In addition a few definitions are given.

If x is a line of a graph G joining the points u and v, written $x=uv$, then we define the *degree* of x by $\deg u+\deg v-2$. We note that if w is the point of $L(G)$ which corresponds to the line x, then the degree of w in $L(G)$ equals the degree of x in G. A point or line is called *odd* or *even* depending on whether it has odd or even degree.

If G is a connected graph having at least one line, then $L(G)$ is also a connected graph. For the most part then, we restrict ourselves to connected graphs for otherwise each connected component can be treated individually.

By $L^2(G)$ we shall mean $L(L(G))$ and, in general, $L^n(G)=L(L^{n-1}(G))$ for $n\geqq 1$, where $L^1(G)$ and $L^0(G)$ stand for $L(G)$ and G, respectively.

Two classes of graphs which have easily determined line-graphs are the cycles and simple paths. In particular, the line-graph of a cycle is a cycle of the same length, and the line-graph of a simple path of length n, $n\geqq 1$, is a simple path of length $n-1$. It therefore follows that if G is a path of length n, $n\geqq 1$, then $L^n(G)$ is the trivial path consisting of a single point while $L^m(G)$ does not exist for $m>n$. It is not difficult to see that if G is a connected graph which is not a path, then $L^n(G)$ exists for all positive integers n. Hence, if for some graph G, we wish to consider the infinite sequence $\{L^n(G)\}$ of graphs, then G must not be a path.

A *bridge* of a connected graph G is a line whose removal disconnects G, while a *cutpoint* of G is a point w of G such that the removal of w and all its incident lines

---

Received by the editors July 25, 1967.

(1) Definitions not presented in this article may be found in [3]. Work supported in part by a grant from the National Science Foundation (GN-2544).

results in a disconnected graph. In relation to this, we state the following three results.

PROPOSITION 1. *A necessary and sufficient condition that a point w of the line-graph $L(G)$ of a connected graph G be a cutpoint is that it corresponds to a bridge $x=uv$ of G in which neither of the points u and v has degree one.*

PROPOSITION 2. *A necessary and sufficient condition that a line $x=u_1u_2$ be a bridge of the line-graph $L(G)$ of a connected graph G is that the lines y_1 and y_2 in G which correspond to u_1 and u_2 be bridges which meet in a point of degree two.*

PROPOSITION 3. *A necessary and sufficient condition that the iterated line-graph $L^n(G)$ of a connected graph G contain a bridge x is that G contain a path of $n+1$ bridges, each consecutive two of which have a point of degree two in common.*

Eulerian line-graphs. A graph G is called *eulerian* if it has a closed path which contains every line of G exactly once and contains every point of G. Such a path is referred to as an *eulerian path.*

Eulerian graphs have been characterized by Euler [2] as those graphs which are connected and in which every point is even. It follows trivially that if G is an eulerian graph, then $L(G)$ too is eulerian; furthermore, if G is eulerian, then the sequence $\{L^n(G)\}$ contains only eulerian graphs. We now state necessary and sufficient conditions for a graph G in order that there exists a nonnegative integer n such that $L^n(G)$ is eulerian. Again the proof is routine and is omitted.

PROPOSITION 4. *Let G be a connected graph which is not a simple path. Then exactly one of the following must occur:*

(1) *G is eulerian,*
(2) *$L(G)$ is eulerian but G is not,*
(3) *$L^2(G)$ is eulerian but $L(G)$ is not,*
(4) *there exists no $n \geqq 0$ such that $L^n(G)$ is eulerian,*

where

(1) *occurs if and only if every point of G is even,*
(2) *occurs if and only if every point of G is odd,*
(3) *occurs if and only if every line of G is odd, and*
(4) *occurs otherwise.*

COROLLARY 4a. *Let G be a connected graph which is other than a simple path. If the sequence $\{L^n(G)\}$ of iterated line-graphs of G contains an eulerian graph, then the degrees of the lines of G are of the same parity and $L^n(G)$ is eulerian for $n \geqq 2$.*

Hamiltonian line-graphs. A graph G is called *hamiltonian* if G has a cycle containing all the points of G; such a cycle is also called *hamiltonian.* If C is a hamiltonian cycle of hamiltonian graph G, then any line of G which does not lie on C is referred to as a *diagonal* of C. Clearly, every hamiltonian graph is connected and has at least three points.

The following concept will be of considerable use to us. A graph G with q lines, where $q \geqq 3$, is called *sequential* if the lines of G can be ordered as $x_0, x_1, \ldots, x_{q-1}$, $x_q = x_0$ so that x_i and x_{i+1}, $i = 0, 1, \ldots, q-1$, are adjacent. Two types of graphs in which we are interested are sequential, as we now see.

PROPOSITION 5. *Every eulerian graph is sequential.*

Proof. If G is an eulerian graph, then G contains a closed path P containing each line of G exactly once, say P: $x_0, x_1, \ldots, x_{q-1}, x_q = x_0$, where x_i and x_{i+1} are adjacent for $i = 0, 1, \ldots, q-1$. This ordering of the lines of G serves to show that G is sequential.

PROPOSITION 6. *Every hamiltonian graph is sequential.*

Proof. Let C be a hamiltonian cycle of a hamiltonian graph G whose points are arranged cyclically as, say, $v_0, v_1, \ldots, v_{p-1}, v_p = v_0$. To show that G is sequential, we exhibit an appropriate ordering of the lines of G. We begin the sequence of lines by selecting all those diagonals incident with v_0 (there may be none). These lines may be taken in any order, and, clearly, each two are adjacent with each other. We follow these with the line v_0v_1. The next lines in the sequence are those diagonals incident with v_1 (again, there may be none). As before, these lines may be taken in any order. The next line in the sequence is v_1v_2, followed by all those diagonals incident with v_2 which are not in the part of the sequence already formed. We continue this until we finally arrive at the line $v_{p-1}v_p = v_{p-1}v_0$, which is adjacent with the first line in the sequence. From the way the sequence was produced, it is now clear that every line of G appears exactly once and that any two consecutive lines in the sequence are adjacent as are the first and last lines. Thus G is sequential.

The primary purpose for introducing sequential graphs lies in the following theorem.

THEOREM 1. *A necessary and sufficient condition that the line-graph $L(G)$ of a graph G be hamiltonian is that G is sequential.*

Proof. The result follows by simply observing that the points of $L(G)$ can be ordered $v_0, v_1, \ldots, v_{p-1}, v_p = v_0$, where v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots, p-1$, if and only if $L(G)$ is hamiltonian, and such an ordering is possible if and only if the lines of G can be ordered $x_0, x_1, \ldots, x_{p-1}, x_p = x_0$, where x_i and x_{i+1} are adjacent for $i = 0, 1, \ldots, p-1$. This latter condition states that G is sequential.

Propositions 5 and 6 and Theorem 1 yield the following corollaries.

COROLLARY 1A. *If G is an eulerian graph, then $L(G)$ is both eulerian and hamiltonian. Furthermore, $L^n(G)$ is both eulerian and hamiltonian for all $n \geqq 1$.*

COROLLARY 1B. *If G is a hamiltonian graph; then $L(G)$ is hamiltonian. Furthermore, $L^n(G)$ is hamiltonian for all $n \geqq 1$.*

As with eulerian graphs, we now determine for what connected graphs G which are not simple paths does the sequence $\{L^n(G)\}$ contain a hamiltonian graph. Unlike the situation for eulerian graphs, however, we find that for all connected graphs G which are not simple paths, the sequence $\{L^n(G)\}$ contains a (in fact, infinitely many) hamiltonian graph. A proof of this was outlined in [1]. Before proving it in detail, we present two lemmas.

LEMMA 1. *If a graph G has a cycle C with the property that every line of G is incident with at least one point of C, then $L(G)$ is hamiltonian.*

Proof. The graph G stated in the lemma is sequential so that, by Theorem 1, $L(G)$ is hamiltonian. To see that G is sequential, one need only observe that an appropriate ordering of the lines of G can be accomplished by using the same procedure as that in Proposition 6 except that after considering all the diagonals at a given point, we next insert in the sequence all the lines of G which are incident with that point but with no other point of C. After this, we proceed as before. The graph G is therefore easily seen to be sequential.

LEMMA 2. *Let G be a graph consisting of a cycle C, its diagonals, and m paths $P_1, P_2, \ldots, P_m$ where* (i) *each path has precisely one endpoint in common with C and* (ii) *for $i \neq j$, P_i and P_j are disjoint except possibly having an endpoint in common if this point is also common to C. Then, if the maximum of the lengths of the P_i is M, $L^n(G)$ is hamiltonian for all $n \geqq M$.*

Proof. The line-graph $L(G)$ has the same properties as G except that the length of each of the m paths is one less than in G so that the maximum length of the paths is $M-1$. Thus, we can apply Lemma 1 to $L^{M-1}(G)$ thereby showing that $L^n(G)$ is hamiltonian for all $n \geqq M$.

THEOREM 2. *Let G be a connected graph which is not a simple path. If G has p points, then $L^n(G)$ is hamiltonian for all $n \geqq p-3$.*

Proof. The proof is by induction on p. Later developments in the proof make it convenient to investigate individually all the graphs under consideration for which $p=3$, 4, or 5. The only connected graph with three points which is not a path is a triangle, but this graph is already hamiltonian so that the result follows.

For $p=4$, there are two connected graphs which are not simple paths and which are not already hamiltonian. We denote these graphs by $G_{4,1}$ and $G_{4,2}$; they are shown in Figure 1. One readily sees that the line-graph of each of these two graphs is hamiltonian, and the result is established for $p=4$.

There are twelve connected graphs with five points which are not paths and which do not contain hamiltonian cycles. These are also shown in Figure 1. It is a routine matter to verify that $L^2(G_{5,1})$ and $L^2(G_{5,2})$ are hamiltonian and that $L(G_{5,i})$ is hamiltonian for $i=3, 4, \ldots, 12$. This proves the theorem for $p=5$.

Let us assume then for all connected graphs G' which are not simple paths and which have s points, where $s<p$ and $p \geqq 6$, that $L^n(G')$ is hamiltonian for all

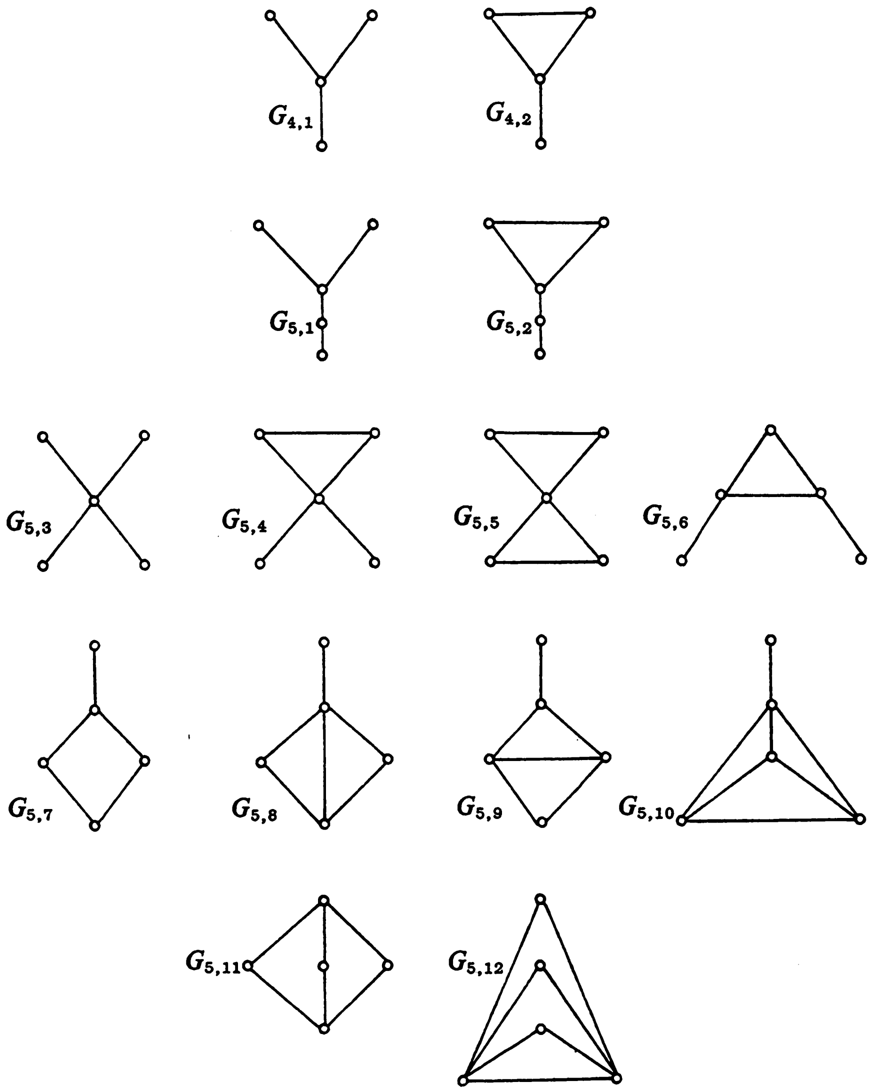

FIGURE 1

$n \geqq s-3$. Let G be a connected graph with p points which is not a simple path. We show that $L^{p-3}(G)$ is hamiltonian which, with the aid of Corollary 1B, establishes the result.

The theorem is clearly evident if G itself is a cycle, so, without loss of generality, we assume G is not a cycle implying the existence of a point v having degree three or more. By H we shall mean the connected star subgraph whose lines are all those incident with v, and we let Q denote the subgraph whose point set consists of all the points of G different from v and whose lines are all those in G which are not in H. The subgraphs H and Q have $\deg v$ points in common but are line-disjoint. We adopt the notation $G=H \oplus Q$ to mean that the line set of G is partitioned by H and Q. Also, we denote the connected components of Q by $G_1, G_2, \ldots, G_k$, where G_i has p_i points for $i=1, 2, \ldots, k$. Clearly, $\sum p_i = p-1$.

If G_i is a path, then $L^{p_i}(G_i)$ does not exist whereas if G_i is not a path, then $L^n(G_i)$ is hamiltonian for $n \geqq p_i - 3$, by the inductive hypothesis.

The line-graph $H_1 = L(H)$ is a complete subgraph of $L(G)$, which, therefore, has a cycle containing all the points of H_1. Let J_1 denote the connected subgraph of $L(G)$ consisting of H_1 and all those lines incident with a point of H_1. Thus, $L(G)$ can be expressed as $J_1 \oplus L(G_1) \oplus L(G_2) \oplus \cdots \oplus L(G_k)$, where $L(G_i)$ and $L(G_j)$ are disjoint for $i \neq j$.

Now let $H_2 = L(J_1)$ and let J_2 denote the connected subgraph of $L^2(G)$ consisting of H_2 and all lines incident with a point of H_2. By Lemma 1, H_2 has a cycle containing all the points of H_2. Thus, $L^2(G) = J_2 \oplus L^2(G_1) \oplus L^2(G_2) \oplus \cdots \oplus L^2(G_k)$. In general, we let J_m denote the connected subgraph of $L^m(G)$ consisting of H_m and all those lines incident with a point of H_m and let $H_{m+1} = L(J_m)$, where H_{m+1} has a cycle containing all the points of H_{m+1} by Lemma 1. The graph $L^m(G)$ can therefore be expressed as $J_m \oplus L^m(G_1) \oplus L^m(G_2) \oplus \cdots \oplus L^m(G_k)$.

We now consider two cases.

Case 1. Suppose each of the components $G_1, G_2, \ldots, G_k$ of Q is a path. (This includes the possibility that some of these components may be the trivial path consisting of a single point.)

If $k \geqq 3$, then $p_i \leqq p-3$ for all i. Hence, $L^{p-3}(G) = H_{p-3}$, which contains a hamiltonian cycle. If $k=2$ and neither p_1 nor p_2 exceeds $p-3$, then, as before, $L^{p-3}(G) = H_{p-3}$. If, on the other hand, $k=2$ and one component, say G_1, has $p-2$ points while G_2 is a single point, then H and G_1 have at least two points in common. Thus G contains a cycle plus possibly diagonals and j pairwise disjoint paths, $1 \leqq j \leqq 3$, each path having precisely one endpoint in common with the cycle. Since none of these paths has length exceeding $p-4$, it follows, by Lemma 2, that $L^{p-4}(G)$ (and so also $L^{p-3}(G)$) contains a hamiltonian cycle.

If $k=1$, then Q is a path having at least three points in common with H so that G consists of a cycle (with some diagonals) and j pairwise disjoint paths, $0 \leqq j \leqq 2$, each path having exactly one endpoint in common with the cycle. If $j=0$, G is hamiltonian while if $j>0$, no path extending from the aforementioned cycle can have length exceeding $p-4$, and by Lemma 2, $L^{p-4}(G)$ is hamiltonian as is $L^{p-3}(G)$.

Case 2. Assume the first t subgraphs, $1 \leqq t \leqq k$, of $G_1, G_2, \ldots, G_k$ are not paths. Clearly, then, each of $G_1, G_2, \ldots, G_t$ has at least three points.

If $t<k$, then $G_{t+1}, G_{t+2}, \ldots, G_k$ are paths, each having at most $p-4$ points so that $L^{p-4}(G) = J_{p-4} \oplus L^{p-4}(G_1) \oplus L^{p-4}(G_2) \oplus \cdots \oplus L^{p-4}(G_t)$. Since each G_i, $1 \leqq i \leqq t$, has at most $p-1$ points, each subgraph $L^{p-4}(G_i)$ of $L^{p-4}(G)$ has a cycle containing all points of $L^{p-4}(G_i)$ by the inductive hypothesis.

For each $i = 1, 2, \ldots, t$, there is clearly at least one line joining a point of H_{p-5}

to a point of $L^{p-5}(G_i)$. We now show that for each i such a line exists with the added property that it is adjacent with at least two lines of $L^{p-5}(G_i)$.

Suppose $t=1$ so that G_1 is the only component of Q which is not a path. If $k>1$, then G_1 has at most $p-2$ points so that $L^{p-5}(G_1)$ contains a hamiltonian cycle and clearly such a line exists. If $k=1$, then $Q=G_1$ and all lines of H are incident with points of G_1. Since each line which joins H_m to $L^m(G_1)$ results in one or more lines joining H_{m+1} with $L^{m+1}(G_1)$, there are at least three lines joining H_{p-5} and $L^{p-5}(G_1)$. If no such line is adjacent with at least two lines of $L^{p-5}(G_1)$, then each of the three or more lines joining H_{p-5} and $L^{p-5}(G_1)$ is adjacent with precisely one line of $L^{p-5}(G_1)$. Hence, $L^{p-5}(G_1)$ contains at least three lines which are incident with points of degree one, i.e., $L^{p-5}(G_1)$ contains at least three bridges. By Proposition 3, G_1 must contain a path of $p-4$ bridges for each bridge of $L^{p-5}(G_1)$. Since the bridges of $L^{p-5}(G_1)$ are incident with points of degree one and since $L^{p-5}(G_1)$ is not itself a path, the three or more paths of G_1 are line-disjoint. This implies that G_1 contains at least $3(p-4)+1$ points but since $p\geqq 6$, $3(p-4)+1>p-1$, which contradicts the number of points in G_1.

Suppose next that $t>1$, i.e., suppose Q contains two or more components which are not paths. Therefore, G_1 and G_2 are not paths, and each contains at most $p-4$ points. If there is a line joining a point of H_{p-5} to a point of $L^{p-5}(G_1)$, say, which is adjacent with only one line of $L^{p-5}(G_1)$, then $L^{p-5}(G_1)$ contains a bridge implying that G_1 contains a path of $p-4$ bridges, but this contradicts the number of points of G_1.

We therefore conclude that for each $i=1, 2, \dots, t$, there exists a line joining H_{p-5} and $L^{p-5}(G_i)$ which is adjacent to two lines of $L^{p-5}(G_i)$. This implies that for each $i=1, 2, \dots, t$, there is a point u_i in H_{p-4} adjacent to both endpoints of a line in $L^{p-4}(G_i)$. It is not difficult to see that $u_i \neq u_j$ for $i\neq j$. Let x_{i1} and x_{i2} be lines of $L^{p-4}(G)$ which join u_i to the distinct endpoints of a line y_i of $L^{p-4}(G_i)$.

We now claim that $L^{p-4}(G)$ is a sequential graph so that $L^{p-3}(G)$ is hamiltonian. Recall first that $L^{p-4}(G_i)$ for $1\leqq i\leqq t$ has a cycle containing all the points of $L^{p-4}(G_i)$ and so is sequential by Proposition 6. Thus for $1\leqq i\leqq t$, the lines of $L^{p-4}(G_i)$ can be arranged in a sequence s_i such that each pair of successive lines in s_i are adjacent and the first and last lines in s_i are adjacent. Let z_i be the term following y_i in s_i (or the first term of s_i if y_i is the last term). Now y_i is adjacent to both x_{i1} and x_{i2}, and z_i, being adjacent to y_i, is adjacent to one of x_{i1} and x_{i2}. Therefore, by cyclically permuting the terms of s_i if necessary and reversing their order if necessary, we can convert s_i into a sequence s_i' whose first and last terms are adjacent to x_{i1} and x_{i2}, respectively. Now H_{p-4} has a cycle C containing all the points of H_{p-4} and every line of J_{p-4} is incident with at least one point of C. Therefore, the procedure of the proof of Lemma 1 enables us to order the lines of J_{p-4} in a sequence (s, say) such that each pair of successive lines in s are adjacent as are the first and last lines. Moreover, since x_{i1} and x_{i2} are lines incident with the point u_i of C and with no other point of C, it is evident that, in applying the procedure of the proof of Lemma 1, we can arrange the lines incident with u_i so that x_{i2} will immediately follow x_{i1} in s for $i=1, 2, \ldots, t$. If we now insert the sequence s_i' between the terms x_{i1} and x_{i2} of s for $i=1, 2, \ldots, t$, the resulting sequence has the properties required for $L^{p-4}(G)$ to be a sequential graph. This completes the proof.

The preceding theorem now permits us to make the following definition. Let G be a connected graph which is not a simple path. The *hamiltonian index* of G, denoted $h(G)$, is the smallest nonnegative integer n such that $L^n(G)$ is hamiltonian. According to Theorem 2 then, if G is a connected graph with p points which is not a simple path, then $h(G)$ exists and $h(G) \leqq p-3$. This bound cannot, in general, be improved since for each $p \geqq 3$ the graph whose point set is $\{v_i \mid 1 \leqq i \leqq p\}$ and whose line set is $\{v_1v_3\} \cup \{v_iv_{i+1} \mid 1 \leqq i \leqq p-1\}$ has a hamiltonian index of $p-3$.

REFERENCES

1. G. Chartrand, *The existence of complete cycles in repeated line-graphs*, Bull. Amer. Math. Soc. **71** (1965), 668–670.

2. L. Euler, *Solutio problematis ad geometriam situs pertinentis*, Comment. Acad. Petropolitanae **8** (1736), 128–140.

3. F. Harary, *A seminar on graph theory*, Holt, Rinehart and Winston, New York, 1967, 1–41.

WESTERN MICHIGAN UNIVERSITY,
KALAMAZOO, MICHIGAN