PSEUDOCOMPACT SPACES

BY

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1. **Introduction.** By definition, a pseudocompact space is a topological space on which every continuous real valued function is bounded. Pseudocompact spaces which are completely regular have often been studied [1], [4], [5], [7]–[10], [12], and [15], and some results have been obtained that apply to noncompletely regular pseudocompact spaces [1], [10], and [13]. In general, however, much more is known about completely regular pseudocompact spaces than is known about arbitrary pseudocompact spaces.

In this paper, we obtain results that apply to pseudocompact spaces which are not necessarily completely regular. We first obtain several characterizations of pseudocompactness. Then we investigate which subspaces of pseudocompact spaces are pseudocompact and consider some necessary and sufficient conditions that a collection of spaces have a pseudocompact product.

We use the same notation and terminology as that in [3]. For definitions of the terms "countably compact," "weakly normal," "normal," and "completely normal," see [5].

$R(N)$ will denote the set of real (natural) numbers. We shall denote the set of continuous mappings of a space $(X, \mathscr{V})$ into a space $(Y, \mathscr{W})$ by $C((X, \mathscr{V}), (Y, \mathscr{W}))$ or $C(X, Y)$ if no confusion is possible. $C(X, \mathscr{V})$ or $C(X)$ will denote the set of bounded functions in $C((X, \mathscr{V}), R)$, and $L(X, \mathscr{V})$ or $L(X)$ will denote the set of all functions in $C(X, \mathscr{V})$ which map $(X, \mathscr{V})$ into $[0, 1]$. Given a function $f \in C((X, \mathscr{V}), R)$, we shall denote the zero set $f^{-1}(0)$ by $Z(f)$ and the cozero set $X - Z(f)$ by $C(f)$. We shall write $\mathscr{Z}(X, \mathscr{V})$ for $\{Z(f) \mid f \in C((X, \mathscr{V}), R)\}$ and $\mathscr{C}(X, \mathscr{V})$ for $\{C(f) \mid f \in C((X, \mathscr{V}), R)\}$.

If $\mathscr{B}$ is a collection of sets, the set of all finite intersections of elements of $\mathscr{B}$ will be denoted by $\mathscr{B}^{\wedge}$. We shall say that $\dot{\mathscr{B}}$ is fixed (free) provided that $\bigcap \mathscr{B} \neq \varnothing$ ($\bigcap \mathscr{B} = \varnothing$).

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2. Characterizations.

DEFINITION 2.1. A filter base $\mathscr{B}$ on a space is said to be an open filter base if and only if all the sets belonging to $\mathscr{B}$ are open. An open filter base $\mathscr{B}$ on a space X is

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called completely regular provided that for each set $B \in \mathscr{B}$ there exist a set $B' \in \mathscr{B}$ and a function $f \in L(X)$ such that f vanishes on B' and equals 1 on $X - B$.

DEFINITION 2.2. An open cover $\mathscr{O}$ of a space X is said to be cocompletely regular provided that for each $O \in \mathscr{O}$ there exist a set $O' \in \mathscr{O}$ and a function $f \in L(X)$ such that f vanishes on O and equals 1 on $X - O'$.

THEOREM 2.3. *Let X be a topological space. The following are equivalent.*

(i) *X is pseudocompact.*

(ii) *For every space Y and function $f \in C(X, Y)$, $f(X)$ is pseudocompact.*

(iii) *For every completely normal space Y and function $f \in C(X, Y)$, $f(X)$ is countably compact.*

(iv) *For every metric space Y and function $f \in C(X, Y)$, $f(X)$ is compact.*

(v) *For every $f \in C(X, R)$, $f(X)$ is compact.*

(vi) *For every $f \in C(X)$, $f(X)$ is compact.*

(vii) *Every function in $C(X)$ assumes its greatest lower bound and its least upper bound for some point or points of X.*

(viii) *If $\mathscr{B}$ is a countable subset of $\mathscr{Z}(X)$ such that $\varnothing \notin \mathscr{B}^{\wedge}$, then $\mathscr{B}$ is fixed.*

(ix) *Every locally finite subset of $\mathscr{C}(X)$ is finite.*

(x) *Every countable completely regular filter base on X is fixed.*

(xi) *Every countable cocompletely regular cover of X has a finite subcover.*

Proof. We omit the proofs of the equivalence of (i), (vi), (vii), and (viii), for they are the same as Hewitt's proofs in [12, p. 67 and p. 70] that (i), (vi), (vii), and (viii) are equivalent on a completely regular space.

It is not difficult to prove that (i), (ii), (iii), (iv), and (v) are equivalent if one recalls the following facts: every subspace of a completely normal space is normal; a normal space is pseudocompact if and only if it is countably compact [12, p. 69]; compact and countably compact subsets of a metric space are identical.

(viii) implies (ix). Suppose that $\mathscr{L} = \{C_n \mid n \in N\}$ is an infinite locally finite system of nonempty elements of $\mathscr{C}(X)$. It follows from the normality of R that for each i there is a function $g_i \in L(X)$ which equals 1 on $X - C_i$ and vanishes at some point in C_i. For each n let h_n be given by $h_n(x) = \inf\{g_i(x) \mid i \geqq n\}$. Then each $Z(h_n)$ is nonempty and contains $Z(h_{n+1})$, and since $\mathscr{L}$ is locally finite, each $h_n \in L(X)$ and $\bigcap\{Z(h_n)\} = \varnothing$. This contradicts (viii).

(ix) implies (x). Let $\mathscr{F} = \{F(n) \mid n \in N\}$ be a completely regular filter base on X such that each $F(n) \supset F(n+1)$. For each $F \in \mathscr{F}$ choose a function $f_F \in L(X)$ which vanishes on $X - F$ and equals 1 on some set in $\mathscr{F}$. Then (ix) implies that there is a point x at which $\{C(f_F) \mid F \in \mathscr{F}\}$ is not locally finite. Evidently $x \in \bigcap\{\bar{F} \mid F \in \mathscr{F}\} = \bigcap \mathscr{F}$.

(x) implies (xi). Suppose that there is a countable cocompletely regular cover $\mathscr{O}$ of X which has no finite subcover. Then $\varnothing \notin \{X - O \mid O \in \mathscr{O}\}^{\wedge}$. Since for each set $O \in \mathscr{O}$ there exists a set $O' \in \mathscr{O}$ such that $O' \supset \bar{O}$, it is also true that $\varnothing \notin \mathscr{G} = \{X - \bar{O} \mid O \in \mathscr{O}\}^{\wedge}$. Thus $\mathscr{G}$ is a countable open filter base on X. Consider a set

$G = \bigcap \{X - \mathrm{Cl}\, O(i) \mid i = 1, \ldots, s\} \in \mathscr{G}$. For each i there exist $O(i)' \in \mathcal{O}$ and a function $f_i \in L(X)$ such that f_i vanishes on $O(i)$ and equals 1 on $X - O(i)'$. Define $G' = \bigcap \{X - \mathrm{Cl}\, O(i)' \mid i = 1, \ldots, s\}$ and $f = \min \{f_i \mid i = 1, \ldots, s\}$, and let g be the function given by $g(x) = 1 - f(x)$. Then $G' \in \mathscr{G}$, $g \in L(X)$, $g(G') = 0$, and $g(X - G) = 1$. Therefore, $\mathscr{G}$ is a countable completely regular filter base on X. Since $\mathcal{O}$ covers X, however, $\mathscr{G}$ is free.

(xi) implies (i). Consider an arbitrary function $f \in C(X, R)$. For each $n \in N$ let $U(n) = f^{-1}((-n, n))$, and define $\mathscr{U} = \{U(n) \mid n \in N\}$. Then $\mathscr{U}$ is a countable co-completely regular cover of X, so (xi) implies that there exists $k \in N$ such that $X \subset U(k)$. Therefore, $f \in C(X)$.

We consider next a number of modified pseudocompactness conditions on a space X.

DEFINITION 2.4. An open filter base $\mathscr{B}$ on a space is said to be *regular* provided that for each set $B \in \mathscr{B}$ there is a set $C \in \mathscr{C}$ such that $\bar{C} \subset B$.

DEFINITION 2.5. An open cover $\mathcal{O}$ of a space X is said to be *coregular* provided that for every set $O \in \mathcal{O}$ there is a set $P \in \mathcal{O}$ such that $\bar{O} \subset P$.

On any space X each of the following is a sufficient condition that X be pseudo-compact.

A(1) Every countable open cover of X has a finite subcover.

A(2) Every countable filter base on X has an adherent point.

B(1) Every locally finite system of open sets of X is finite.

B(2) Every countable, locally finite, disjoint system of open sets of X is finite.

B(3) If $\mathscr{U}$ is a countable open cover of X and A is an infinite subset of X, then the closure of some member of $\mathscr{U}$ contains infinitely many points of A.

B(4) If $\mathscr{U}$ is a countable open cover of X, then there is a finite subcollection of $\mathscr{U}$ whose closures cover X.

B(5) Every countable open filter base on X has an adherent point.

C(1) If $\mathscr{U} = \{U(n) \mid n \in N\}$ is a collection of nonempty open subsets of X such that $\mathrm{Cl}\, U(i) \cap \mathrm{Cl}\, U(j) = \varnothing$ whenever $i \neq j$, then $\mathscr{U}$ is not locally finite.

D(1) Every countable coregular cover of X has a finite subcover.

D(2) Every countable regular filter base on X is fixed.

THEOREM 2.6. *On a space X the following hold.*

(i) A(1), *and* A(2) *are equivalent.*

(ii) B(1), B(2), B(3), B(4), *and* B(5) *are equivalent.*

(iii) D(1) *and* D(2) *are equivalent.*

(iv) A(2) *implies* B(5).

(v) B(5) *implies* C(1).

(vi) C(1) *implies* D(2).

(vii) D(2) *implies that* X *is pseudocompact.*

(viii) *If each point of X has a fundamental system of closed neighborhoods, then* D(2) *implies* B(2).

(ix) *If X is a uniformizable pseudocompact space, then* B(1) *holds.*

(x) *If X is a weakly normal pseudocompact space, then* A(1) *holds.*

Proof. (i) A(1) and A(2) are obviously equivalent.

(ii). It was proved in [1] that B(1), B(2), B(3), and B(4) are equivalent.

B(4) implies B(5). If $\mathscr{F}$ is a countable open filter base on X which has empty adherence, then $\mathscr{O} = \{X - \bar{F} \mid F \in \mathscr{F}\}$ is a countable open cover of X such that for every finite subcollection $\mathscr{P}$ of $\mathscr{O}$, $\bigcup \mathscr{P}$ is not dense in X.

B(5) implies B(4). If $\mathscr{U}$ is a countable open cover of X, and if for every finite subcollection $\mathscr{M}$ of $\mathscr{U}$, $X \not\subset \bigcup \{\bar{M} \mid M \in \mathscr{M}\}$, then $\{X - \bar{U} \mid U \in \mathscr{U}\}^{\wedge}$ is a countable open filter base on X which has empty adherence.

The proof of (iii) is similar to the proof of the equivalence of B(4) and B(5). (iv) is trivial.

(v). Suppose that B(5) holds, and let $\{U(n) \mid n \in N\}$ be a collection of nonempty open subsets of X such that Cl $U(i) \cap$ Cl $U(j) = \varnothing$ whenever $i \neq j$. For each $n \in N$ let $V(n) = \bigcup \{U(j) \mid j \geqq n\}$. Then $\{V(n)\}$ is a countable open filter base on X and hence has an adherent point x. Clearly, for any neighborhood V of x, $V \cap U(n) \neq \varnothing$ for infinitely many n.

(vi). Suppose that there exists a free, countable, regular filter base on X. Then an inductive argument shows that there exists a free open filter base $\mathscr{G} = \{G(n) \mid n \in N\}$ such that for each $n \in N$, $G(n) \supset$ Cl $G(n+1)$ and $G(n) -$ Cl $G(n+1) \neq \varnothing$. Define $U(n) = G(2n) -$ Cl $G(2n+1)$ for each $n \in N$, and consider $\mathscr{U} = \{U(n) \mid n \in N\}$. Since $\mathscr{G}$ is free, no point of X has the property that each of its neighborhoods intersects infinitely many of the sets in $\mathscr{U}$. On the other hand, $\mathscr{U}$ is a collection of nonempty open subsets of X, and whenever $i < j$, Cl $U(j) \subset G(2i+1)$, but Cl $U(i) \subset G(2i-1) - G(2i+1)$. This contradicts C(1).

(vii). Every countable completely regular filter base is a countable regular filter base, so it follows from D(2) and (x) of Theorem 2.3 that X is pseudocompact.

(viii). In [2] Banaschewski proved that if X is a regular space on which every regular filter base is fixed, then X does not contain a countably infinite family of open sets whose closures are mutually disjoint and have a closed union. A corollary to his method of proof is that if X is a space in which every point has a fundamental system of closed neighborhoods, then D(2) implies C(1).

In order to see that C(1) implies B(2), suppose that $\mathscr{L} = \{L(n) \mid n \in N\}$ is a locally finite system of nonempty open sets of X such that $L(i) \cap L(j) = \varnothing$ whenever $i \neq j$. For each $n \in N$ choose a nonempty open set $M(n)$ such that Cl $M(n) \subset L(n)$, and set $\mathscr{M} = \{M(n) \mid n \in N\}$. Then $\mathscr{M}$ is locally finite, and Cl $M(i) \cap$ Cl $M(j) \subset L(i) \cap L(j) = \varnothing$ whenever $i \neq j$.

(ix). In [1] it was proved that on a uniformizable space B(1) is equivalent to pseudocompactness.

(x) is a consequence of the fact that a weakly normal pseudocompact space is countably compact [5].

In §3 we shall construct several examples of pseudocompact spaces in order to show that the converses of (v), (vi), and (vii) of Theorem 2.6 are false. Of course, the converse of (iv) is also false, since, for example, the Tychonoff plank is pseudocompact and completely regular, but not countably compact [5].

We conclude this section with a slight generalization of Theorem 17 in [12].

THEOREM 2.7. *Let X be a completely regular space. The following are equivalent.*
(i) *X is pseudocompact.*
(ii) *Whenever X is embedded in a Hausdorff (regular, completely regular) space Y and $Y-X$ is first countable, then X is closed in Y.*

Proof. (i) implies (ii). Let Y be a Hausdorff space in which X is embedded, and suppose that $\mathcal{O}$ is a countable fundamental system of open neighborhoods of a point $y \in \bar{X}$. Then $\mathcal{P} = \mathcal{O} | X$ is a countable open filter base on X, so there is a point $x \in X$ such that $x \in \bar{O}$ for all $O \in \mathcal{O}$. Therefore, $y = x \in X$.

(ii) implies (i). Suppose that X is not pseudocompact. Then by (x) of Theorem 2.3 there is a countable completely regular filter base $\mathcal{F}$ on X which is free. Let $Y = X \cup \{\mathcal{F}\}$, and define a subset F of Y to be open if and only if (a) $F \cap X$ is open in X, and (b) if $\mathcal{F} \in F$, then $F \cap X \supset G$ for some $G \in \mathcal{F}$. Then $Y - X$ is first countable, and Y is a completely regular space containing X as a dense proper subspace. This contradicts (ii).

COROLLARY 2.8. *Let X be a topological space. The following are equivalent.*
(i) *X is pseudocompact.*
(ii) *For every first countable completely regular space Y and function $f \in C(X, Y)$, $f(X)$ is a closed subspace of Y.*

Proof. (i) implies (ii). If X is pseudocompact, then $f(X)$ is pseudocompact, so Theorem 2.7 implies that $f(X)$ is a closed subset of Y.

(ii) implies (i). If there is a function $g \in C(X, R) - C(X)$, then the function $h = 1/(1 + |g|) \in L(X)$ has the property that $0 \in \text{Cl } h(X) - h(X)$.

3. **Examples of pseudocompact spaces.** As noted earlier, the Tychonoff plank shows that B(5) does not imply A(1). A more simple example of the same thing is the following:

EXAMPLE 3.1. Let $X = [0, 1]$, denote the usual topology on X by $\mathcal{V}$, and let $\mathcal{M}$ be the topology on X generated by $\mathcal{V} \cup \{\{X - 1/n \mid n \in N\}\}$.

For every set $W \in \mathcal{M}$, $\text{Cl}_{\mathcal{M}} W$ is a closed subset of $(X, \mathcal{V})$. Thus every open filter base on $(X, \mathcal{M})$ has an adherent point.

Since $\{1/n \mid n \in N\}$ has no limit point in $(X, \mathcal{M})$, $(X, \mathcal{M})$ is not countably compact.

In [1] and [14] examples were constructed in order to show that on a T_1-space B(1) is not a necessary condition for pseudocompactness. This result can be sharpened. The following example is a Hausdorff space on which C(1) does not imply B(1).

EXAMPLE 3.2.   The example given here is the countable connected Hausdorff space constructed by Bing.

Denote by Y the set of all points (p, q) in the plane such that p and q are rational, and $q \geqq 0$. If $a = (p, 0) \in Y$ and $e > 0$, let $W(a, e) = \{(s, 0) \in Y \mid |s - p| < e\}$. If $a = (p, q) \in Y$, $q > 0$, and $e > 0$, choose $c, d \in R$ so that (p, q), $(c, 0)$, and $(d, 0)$ are vertices of an equilateral triangle, and define $W(a, e)$ to be

$$\{(p, q)\} \cup \{(s, O) \in Y \mid |s - c| < e \text{ or } |s - d| < e\}.$$

Let $\mathscr{W}$ be the topology on Y generated by $\{W(a, e) \mid a \in Y \text{ and } e > 0\}$.

$(Y, \mathscr{W})$ satisfies C(1) vacuously, for no two nonempty open sets in $(Y, \mathscr{W})$ have disjoint closures.

$\{\{(p, 0) \in Y \mid p > n\} \mid n \in N\}$ is an infinite locally finite system of open sets.

DEFINITION 3.3.   A space X is said to be a Stone space (or a completely Hausdorff space) provided that $C(X, R)$ separates the points of X.

According to Theorem 2.6, B(1), C(1), and D(2) are equivalent on a regular space X. The next example shows that even on a Stone space D(2) may not imply C(1).

EXAMPLE 3.4.   The example given here was given by Herrlich in [11] in order to show that there exists a space X with the following properties: (a) not every open filter base on X has an adherent point; (b) if $\mathscr{J}$ and $\mathscr{L}$ are any open filter bases on X such that each $J \in \mathscr{J}$ contains the closure of some $L \in \mathscr{L}$, then $\mathscr{J}$ has nonempty adherence.

Let $(X, \mathscr{V})$ be as in 3.1. Choose disjoint dense subsets $X(1)$, $X(2)$, and $X(3)$ of $(X, \mathscr{V})$ such that $X = X(1) \cup X(2) \cup X(3)$, and let $\mathscr{H}$ be the topology on X generated by $\mathscr{V} \cup \{X(1), X(2)\}$.

Since $\mathscr{V} \subset \mathscr{H}$, $(X, \mathscr{H})$ is a Stone space. D(2) is an immediate consequence of (b).

Fix a point $x \in [0, 1) \cap X(1)$. For each $n \in N$ define $U(n) = (x + 1/(2n + 1), x + 1/2n) \cap X(2)$. Then $\{U(n) \mid n \geqq 1/(2 - 2x)\}$ is an infinite locally finite collection of nonempty open subsets of $(X, \mathscr{H})$ whose closures are disjoint.

The following example is a pseudocompact Stone space which does not have property D(2).

EXAMPLE 3.5.   Let $(X, \mathscr{V})$ be as above, and choose a collection of sets $X(n) \subset X$, $n = 1, 2, \ldots$, which have the following properties: for each $n \in N$, $X(n)$ is a dense subset of $(X, \mathscr{V})$; $X(i) \cap X(j) = \varnothing$ whenever $i \neq j$; $X = \bigcup \{X(n) \mid n \in N\}$. Denote by $\mathscr{W}$ the topology on X generated by $\mathscr{V} \cup \{X(2n - 1) \cup X(2n) \cup X(2n + 1) \mid n \in N\} \cup \{X(2n - 1) \mid n \in N\}$.

The space $(X, \mathscr{W})$ has the following properties.

(i) The sets $F(n) = \bigcup \{X(j) \mid j \geqq 2n - 1\}$, $n = 1, 2, \ldots$, form a free regular filter base on $(X, \mathscr{W})$.

(ii) If $a, b \in R$, O and P are open subsets of R, $O \supset \bar{P}$, $f \in C((X, \mathscr{W}), R)$, and $f((a, b) \cap X(j)) \subset P$, then $f((a, b) \cap X) \subset O$.

Proof. (i). Each $F(n)$ is clearly an open set of $(X, \mathscr{W})$, and for any $n \in N$,

$\mathrm{Cl}_{\mathscr{W}} F(n+1) = \bigcup \{X(j) \mid j \geq 2n\} \subset F(n)$. Thus $\{F(n)\}$ is a regular filter base. It follows from the properties of the $X(n)$ that $\{F(n)\}$ is free.

(ii). We first observe that the following equations hold for any $n \in N$ and $W \in \mathscr{W}$.

$$\mathrm{Cl}_{\mathscr{W}} ((a, b) \cap X(1)) = [a, b] \cap (X(1) \cup X(2)).$$

$$\mathrm{Cl}_{\mathscr{W}} ((a, b) \cap X(2n+1)) = [a, b] \cap (X(2n) \cup X(2n+1) \cup X(2n+2)).$$

$$\mathrm{Cl}_{\mathscr{W}} ((a, b) \cap X(2n)) = [a, b] \cap X(2n).$$

If $(a, b) \cap X(1) \subset W$, then $[a, b] \cap (X(1) \cup X(2)) \subset \mathrm{Cl}_{\mathscr{W}} W$.

If $(a, b) \cap X(2n+1) \subset W$, then $[a, b] \cap (X(2n) \cup X(2n+1) \cup X(2n+2)) \subset \mathrm{Cl}_{\mathscr{W}} W$.

If $(a, b) \cap X(2) \subset W$, then $[a, b] \cap (X(1) \cup X(2) \cup X(3) \cup X(4)) \subset \mathrm{Cl}_{\mathscr{W}} W$.

If $(a, b) \cap X(2n+2) \subset W$, then $[a, b] \cap (X(2n) \cup X(2n+1) \cup X(2n+2)$
$\cup X(2n+3) \cup X(2n+4)) \subset \mathrm{Cl}_{\mathscr{W}} W$.

Since R is normal, there exist open sets $Z(n) \subset R$ such that for each $n \in N$, $\bar{P} \subset Z(n) \subset \mathrm{Cl}\, Z(n) \subset Z(n+1) \subset O$. Define $Y(n) = f^{-1}(Z(n))$, $n = 1, 2, \ldots$. Then $(a, b) \cap X(j) \subset Y(1)$, $\{Y(n) \mid n \in N\} \subset \mathscr{W}$, and for each $n \in N$, $\mathrm{Cl}_{\mathscr{W}} Y(n) \subset Y(n+1)$. An inductive argument based on the equations above shows that if j is odd, then for every integer $n \geq (j-2)/2$, $(a, b) \cap \bigcup \{X(k) \mid k \leq j+1+2n\} \subset Y(2+n)$, and if j is even, then for every integer $n \geq (j+2)/2$, $(a, b) \cap \bigcup \{X(k) \mid k \leq j+2n-2\} \subset Y(n)$. Thus

$$(a, b) \cap X = \bigcup \{(a, b) \cap \bigcup \{X(k) \mid k \leq j+2n-2\} \mid n \in N, n \geq (j+2)/2\}$$
$$\subset \bigcup \{Y(2+n) \mid n \geq (j+2)/2\} = \bigcup \{Y(n) \mid n \in N\}.$$

Therefore, $f((a, b) \cap X) \subset f(\bigcup \{Y(n)\}) = \bigcup \{f(Y(n))\} \subset \bigcup \{Z(n)\} \subset O$.

4. Pseudocompact product spaces.

Terasaka's example (see [8, p. 135]) shows that if X is a countably compact completely regular space, then $X \times X$ need not be pseudocompact. Comfort's example in [4] shows that if $\{X(n) \mid n \in N\}$ are completely regular spaces, then $\prod \{X(n) \mid n \in N\}$ is not necessarily pseudocompact even if $\prod \{X(n) \mid n \in B\}$ is pseudocompact for every finite (nonempty) subset B of N. On the other hand, Glicksberg [9] and Bagley, Connell, and McKnight [1] have proved that under certain supplementary hypotheses the product of a collection of pseudocompact completely regular spaces is pseudocompact.

Our aim in this section is to obtain several product theorems which apply to noncompletely regular pseudocompact spaces.

THEOREM 4.1. *Let $\{X(a) \mid a \in A\}$ be a collection of topological spaces. If $X = \prod \{X(a) \mid a \in A\}$ is pseudocompact, then $\prod \{X(b) \mid b \in B\}$ is pseudocompact for every nonempty subset B of A. If A is infinite and X is not pseudocompact, then there is a countably infinite subset J of A such that $Y = \prod \{X(j) \mid j \in J\}$ is not pseudocompact.*

The observation that Theorem 4.1 holds for completely regular spaces is due to Glicksberg [9].

DEFINITION 4.2. A space X is said to be feebly compact [13], or lightly compact [1], provided that X has property B(1).

In [13] A. H. Stone proved that every product of feebly compact spaces, of which all but one (at most) are sequentially compact, is feebly compact. Since on a uniformizable space feeble compactness and pseudocompactness are equivalent, it follows from Stone's theorem that every product of pseudocompact uniformizable spaces, of which all but one (at most) are sequentially compact, is pseudocompact. One can also obtain a similar result for spaces which are not necessarily uniformizable.

LEMMA 4.3. *The product of a sequentially compact space and a pseudocompact space is pseudocompact.*

THEOREM 4.4. *Every product of pseudocompact spaces, of which all but one (at most) are sequentially compact, is pseudocompact.*

DEFINITION 4.5. A mapping f of a space X into a space Y is said to be Z-closed provided that for every $Z \in \mathscr{Z}(X)$, $f(Z)$ is a closed subset of Y.

In [15] Tamano proved that for pseudocompact completely regular spaces X and Y, $X \times Y$ is pseudocompact if and only if $\prod_x$ is Z-closed. The next theorem is a partial generalization of this result.

LEMMA 4.6. *Let $\mathscr{T}$ and $\mathscr{U}$ be completely regular topologies on a set X, and suppose that $(X, \mathscr{U})$ is pseudocompact and $(X, \mathscr{T})$ is first countable. Then $\mathscr{T} \subset \mathscr{U}$ if and only if $\mathscr{T} = \mathscr{U}$.*

Proof. Let F be a closed subset of $(X, \mathscr{U})$. Then there is a subset $\mathscr{V}$ of $\mathscr{U}$ such that $F = \bigcap \{\mathrm{Cl}_{\mathscr{U}} V \mid V \in \mathscr{V}\}$. In [9] Glicksberg proved that the closure of every open subset of a pseudocompact completely regular space is pseudocompact. Thus each $(\mathrm{Cl}_{\mathscr{U}} V, \mathscr{U}|\mathrm{Cl}_{\mathscr{U}} V)$ and, consequently, each $(\mathrm{Cl}_{\mathscr{U}} V, \mathscr{T}|\mathrm{Cl}_{\mathscr{U}} V)$ is pseudocompact. Since every pseudocompact subset of a first countable completely regular space is closed, each $\mathrm{Cl}_{\mathscr{U}} V$ is a closed subset of $(X, \mathscr{T})$. Therefore, F is a closed subset of $(X, \mathscr{T})$.

DEFINITION 4.7. If X is a topological space, we shall denote by wX the uniformizable space which has the same points and the same continuous real valued functions as those of X.

THEOREM 4.8. *Let X and Y be pseudocompact spaces.*

(i) *If $\prod_x$ is Z-closed, then $X \times Y$ is pseudocompact.*

(ii) *If $X \times Y$ is a Stone space, $wX \times wY$ is first countable, and $X \times Y$ is pseudocompact, then $\prod_x$ is Z-closed.*

Proof. (i). Suppose that Y is pseudocompact, $\prod_x$ is Z-closed, and $f \in C(X \times Y, R)$. Then the function $g(x) \equiv \sup \{f(x, y) \mid y \in Y\}$ is continuous, for consider a point $x' \in X$. If $\varepsilon > 0$, $\prod_x(\{(x, y) \mid |f(x, y) - f(x', y)| \geqq \varepsilon\})$ is a closed set not containing x',

so its complement is a neighborhood V of x'. V has the property: if $x \in V$, then $|f(x, y) - f(x', y)| < \varepsilon$ for all $y \in Y$. Now $g(x') = f(x', y')$ and $g(x) = f(x, y'')$ for some $y', y'' \in Y$, by pseudocompactness. Thus $f(x, y'') \geqq f(x, y') > f(x', y') - \varepsilon$ and $f(x, y'') < f(x', y'') + \varepsilon \leqq f(x', y') + \varepsilon$, so $|g(x) - g(x')| < \varepsilon$.

If $X \times Y$ is not pseudocompact, then there exists $f \in C(X \times Y, (-\infty, 0))$ such that $\sup \{f(x, y)\} = 0$. Since Y is pseudocompact, $g(x) < 0$ for each $x \in X$, but

$$\sup \{g(x) \mid x \in X\} = \sup \{f(x, y)\} = 0.$$

Thus X cannot be pseudocompact.

(ii) Lemma 4.6 implies that $wX \times wY = w(X \times Y)$. Thus $\mathscr{Z}(X \times Y) = \mathscr{Z}(w(X \times Y)) = \mathscr{Z}(wX \times wY)$. Since $\prod_x: wX \times wY \to wX$ is Z-closed, and the closed subsets of wX are closed in X, the mapping $\prod_x: X \times Y \to X$ is Z-closed.

DEFINITION 4.9. If K is an infinite cardinal number, a space X will be called K-pseudocompact provided that the following holds: whenever $\mathscr{F}$ is a filter base on X such that $\mathscr{F}$ contains K or fewer sets and each set belonging to $\mathscr{F}$ is a union of cozero sets of X, then $\mathscr{F}$ has an adherent point.

THEOREM 4.10. *Let Y be a K-pseudocompact space, and suppose that X is a pseudocompact space in which every point has a fundamental system of neighborhoods containing K or fewer sets. Then $X \times Y$ is pseudocompact.*

Proof. Suppose that $X \times Y$ is not pseudocompact. Then there exist $Z = Z(f)$ in $X \times Y$ (say $f \geqq 0$) and $x_0 \in X$ with $x_0 \in \text{Cl} (\prod_x (Z)) - \prod_x (Z)$. Let $\mathscr{V}$ be a fundamental system of neighborhoods of x_0 containing $\leqq K$ sets, and for each $V \in \mathscr{V}$ choose a point $x_V \in V \cap \prod_x (Z)$. Since Y is pseudocompact and $f(x_0, y) > 0$ for all y, there is an $a > 0$ with $f(x_0, y) \geqq a$ for all y. For each $V \in \mathscr{V}$ let $F(V)$ be the cozero set $\{y \in Y \mid f(x_V, y) < a/3\}$, and set $G(V) = \bigcup \{F(S) \mid S \in \mathscr{V} \text{ and } S \subset V\}$. Let y_0 be an adherent point of $\{G(V)\}$. Choose a neighborhood $U \times W$ of (x_0, y_0) such that $f \mid U \times W > 2a/3$. Since $U \supset \{x_V \mid V \subset U\}$, and $W \cap F(V) \neq \varnothing$ for some $V \subset U$, a contradiction is obtained.

COROLLARY 4.11. *Let X and Y be pseudocompact spaces such that every point of X has a fundamental system of neighborhoods containing K or fewer sets, Y is uniformizable, and every point of Y has a K-pseudocompact neighborhood. Then $X \times Y$ is pseudocompact.*

Proof. Let $\mathscr{F}$ be a countable completely regular filter base on $X \times Y$. Because Y is feebly compact, $\prod_y (\mathscr{F})$ has an adherent point y. Let W be a K-pseudocompact neighborhood of y. Then $X \times W$ is pseudocompact by Theorem 4.10, and since $\mathscr{F} \mid (X \times W)$ is a countable completely regular filter base on $X \times W$, $\mathscr{F} \mid (X \times W)$ is fixed.

In [13] C. T. Scarborough studied properties of spaces on which every open filter base has an adherent point. He called a space of this type an $H(i)$ space.

COROLLARY 4.12. *If X and Y are pseudocompact spaces, one of which is an $H(\mathrm{i})$ space, then $X \times Y$ is pseudocompact.*

COROLLARY 4.13. *Let $\{X(i) \mid i = 1, \ldots, n\}$ be a collection of pseudocompact spaces such that for each $j \geqq 2$, $X(j)$ is uniformizable, and each point of $X(j)$ has an $H(\mathrm{i})$ neighborhood. Then $\prod \{X(i) \mid i = 1, \ldots, n\}$ is pseudocompact.*

In [6] S. P. Franklin calls a space X sequential if every sequentially closed subset of X is closed.

COROLLARY 4.14. *If X and Y are pseudocompact spaces, one of which is sequential, then $X \times Y$ is pseudocompact.*

Proof. The proof of Theorem 4.10 for the case $K = \aleph_0$ shows that $X \times Y$ is pseudocompact if Y is K-pseudocompact and X is a pseudocompact sequential space. One can use (ix) of Theorem 2.3 to show that $\aleph_0$-pseudocompactness and pseudocompactness are equivalent conditions.

THEOREM 4.15. *If X is a pseudocompact k-space [5] and Y is a pseudocompact completely regular space, then $X \times Y$ is pseudocompact.*

The proof is the same as Tamano's proof in [15] of Proposition 2.

In [7] Frolik obtained a useful necessary and sufficient condition that a space belong to the class $\mathscr{P}$ of all completely regular spaces X such that for every pseudocompact completely regular space Y the product $X \times Y$ is pseudocompact. He also constructed a space $X \in \mathscr{P}$ containing an infinite disjoint family $\mathscr{U}$ of nonempty open subsets of X such that for every compact subset K of X, $K \cap U = \varnothing$ for all but finitely many $U \in \mathscr{U}$. The following result is also of interest.

THEOREM 4.16. *There exists a Stone space $(X, \mathscr{W})$ which has the following properties.*

(i) There is an infinite disjoint family $\mathscr{U}$ of nonempty open subsets of X such that for every subset K of X which has the property $D(2)$, $K \cap U = \varnothing$ for all but finitely many $U \in \mathscr{U}$.

(ii) For every pseudocompact space Y, $(X, \mathscr{W}) \times Y$ is pseudocompact.

(iii) If A is any nonempty set and if $Y_a = (X, \mathscr{W})$ for all $a \in A$, then $Y = \prod \{Y_a \mid a \in A\}$ is pseudocompact.

Proof. Let $(X, \mathscr{W})$ be the space constructed in Example 3.5 and let

$$\mathscr{U} = \{X(2n-1) \mid n \in N\}.$$

(i) If K is any subset of X such that $K \cap U \neq \varnothing$ for infinitely many $U \in \mathscr{U}$, then $\{K \cap F(n) \mid n \in N\}$, where the $F(n)$ are as in 3.5, is a free, countable, regular filter base on K.

Since $(X, \mathscr{W})$ is a first countable pseudocompact space, Corollary 4.14 implies that (ii) holds.

(iii) Consider a function $f \in C(Y, R)$. If $y \in Y$, and O and P are open subsets of R such that $f(y) \in P$ and $\bar{P} \subset O$, then there is a basic open set $B \subset Y$ such that $y \in B$ and $f(B) \subset P$. Repeated application of (ii) of 3.5 to the restrictions of f to appropriate subspaces of Y shows that there is a basic open set B' of the compact space $M = \prod \{ w Y_a \mid a \in A \}$ such that $y \in B'$ and $f(B') \subset 0$. Thus $f \in C(M, R) = C(M)$.

5. Pseudocompact subspaces of pseudocompact spaces.
In [9] Glicksberg observed that in a pseudocompact completely regular space the closure of every open subset is pseudocompact. In [1] it was proved more generally that a space X is feebly compact if and only if the closure of each open subset of X is feebly compact. Thus every open subset of a feebly compact space has pseudocompact closure. Similar results can also be obtained for spaces with property C(1).

THEOREM 5.1. *A space X has property* C(1) *if and only if the closure of every open subset of X has property* C(1).

The proof is immediate. As the following example shows, the closure of an open subset of a pseudocompact space X need not be pseudocompact even if X has property D(1).

EXAMPLE 5.2.   Let $(X, \mathscr{H})$ be as in 3.4. Choose $a \in X(1) \cap [0, 1)$, let $A = X(2) \cap (a, 1]$, and define $f(x) = 1/(x - a)$ if $x \in \mathrm{Cl}_{\mathscr{H}} A$. Then $A \in \mathscr{H}$ and $a \notin \mathrm{Cl}_{\mathscr{H}} A$, so f is an unbounded continuous mapping of $(\mathrm{Cl}_{\mathscr{H}} A, \mathscr{H} \mid \mathrm{Cl}_{\mathscr{H}} A)$ into R.

We can prove the following result.

THEOREM 5.3. *Let X be a space with property* D(1), *and suppose that A is an open subset of X such that $\bar{A} - A$ has property* D(1). *Then $\bar{A}$ has property* D(1).

Proof. Let $\mathscr{F}$ be a countable regular filter base on $\bar{A}$. If $\varnothing \notin \mathscr{F} \mid (\bar{A} - A)$, then $\mathscr{F} \mid (\bar{A} - A)$ is a regular filter base on $\bar{A} - A$ and hence must be fixed.

Suppose that there exists a set $F \in \mathscr{F}$ such that $F \cap (\bar{A} - A) = \varnothing$. Since the closure of a subset B of $\bar{A}$ with respect to $\bar{A}$ is the same as $\bar{B}$, we can choose a set $H \in \mathscr{F}$ such that $\bar{H} \subset F$. Let $\mathscr{G} = \{ G \in \mathscr{F} \mid G \subset H \}$. Then $\mathscr{G}$ is easily seen to be a countable regular filter base on X. Because X has property D(1), $\mathscr{G}$ and, consequently, $\mathscr{F}$ must be fixed.

An analogous theorem holds for pseudocompact spaces.

THEOREM 5.4. *Let X be a pseudocompact space, and suppose that A is an open subset of X such that $\bar{A} - A$ is pseudocompact. Then $\bar{A}$ is pseudocompact.*

Proof. Let $f \in C(\bar{A}, R)$. Since $f \mid (\bar{A} - A) \in C(\bar{A} - A, R) = C(\bar{A} - A)$, there is an $n \in N$ such that $|f(x)| < n$ for all $x \in \bar{A} - A$. Define g by $g(x) = n$ if $x \in X - \bar{A}$ and $g(x) = \max \{ |f(x)|, n \}$ if $x \in \bar{A}$. Then $g \in C(X, R) = C(X)$, so there is a $k \in N$ such that $\max \{ |f(x)|, n \} \leqq k$ for all $x \in \bar{A}$, i.e., $f \in C(\bar{A})$.

REMARK 5.5.   If X is the Tychonoff plank, then the closure of every open subset of X is pseudocompact, but there is an open dense subset O of X such that $X - O$ is an infinite discrete space.

COROLLARY 5.6. *Let X be a pseudocompact space, and suppose that a closed subset C of X is quasi-compact. Then* Cl $(X-C)$ *is pseudocompact.*

COROLLARY 5.7. *Let X and Y be pseudocompact spaces, and suppose that an open subset W of Y has quasi-compact closure. The following are equivalent.*

(i) $X \times Y$ *is pseudocompact.*

(ii) $X \times (Y-W)$ *is pseudocompact.*

This is an extension of a result in [9, p. 377]. Corollary 4.12 and Theorem 5.4 show that Glicksberg's proof can be applied here.

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