MEASURABLE MULTIVALUED MAPPINGS AND LUSIN'S THEOREM(1)

BY

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1. **Introduction.** Let (X, ρ) be a metric space, and let 2^X denote the collection of nonempty closed subsets of X. If (T, d) is a compact metric space, and if μ is a positive Radon measure on T, then one may define measurable mappings $\Omega: T \to 2^X$ as in [5], [7], [20] and [15]. There are several ways of defining convergence in 2^X [17], [2, p. 118 ff.], and consequently there are several ways of defining continuity for mappings $\Omega: T \to 2^X$. The main purpose of this paper is to investigate the relationships between measurable multivalued mappings and continuous multivalued mappings. More specifically the central problem is to ascertain conditions under which the measurability of $\Omega: T \to 2^X$ will be equivalent to the property that for every $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that the restriction of Ω to $T \backslash E_\varepsilon$ is continuous (in some sense). Thus far the only results on this problem have been obtained by Plis [20] when X is compact, and by Castaing [7, Theorem 4.2] when X is separable and $\Omega(t)$ is compact for every $t \in T$. Our results are not restricted to the compact situation.

This study is motivated by the considerable interest recently manifested in the study of measurable multivalued mappings, e.g., see [5], [7], [15], [18], and [20] and their applications to optimal control problems, e.g., see [1], [6], [8], [13], [16], and [19].

2. **Measurable multivalued mappings.** Throughout this paper μ denotes a positive Radon measure [4] defined on a compact metric space, (T, d), and (X, ρ) denotes a metric space. If (Y, δ) is any metric space, then we use $\tau(\delta)$ to denote the topology on Y (collection of open sets) determined by δ. The symbol $\mathrm{cl}\,(A)$ is used to denote the closure of A, where A is a subset of Y. We shall use $\mathscr{A}(X)$ to denote the collection of nonempty subsets of X, 2^X to denote the collection of nonempty closed subsets of X, and $\mathscr{C}(X)$ to denote the collection of nonempty compact subsets of X. If $H: T \to \mathscr{A}(X)$ is a mapping, and if S is a subset of X, then we define $H^- S$ to be the set $\{t \in T \mid H(t) \cap S \neq \varnothing\}$. We say that the multivalued mapping H is *measurable* if $H^- F$ is measurable for every closed $F \subset X$.

In discussing topologies on 2^X we shall try to remain consistent with Michael's terminology [17]. The uniformity on X (see [10, p. 201]) determined by the metric ρ

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is denoted by $\mathscr{J}^\rho$, i.e., $\mathscr{J}^\rho=\{J_\varepsilon^\rho \mid \varepsilon>0\}$ where $J_\varepsilon^\rho=\{(x,y)\in X\times X \mid \rho(x,y)<\varepsilon\}$. If V is a subset of $X\times X$, and A is a subset of X, then

$$V[A] = \{y\in X \mid \exists x\in A\colon (x,y)\in V\}.$$

The uniformity $\mathscr{J}^\rho$ on X determines a uniformity $2^{[\rho]}$ on 2^X [17, p. 153]. Let $W(J_\varepsilon^\rho)$ denote the set

$$\{(A,B)\in 2^X\times 2^X \mid J_\varepsilon^\rho[A]\supset B \;\&\; J_\varepsilon^\rho[B]\supset A\}.$$

Then the uniformity $2^{[\rho]}$ is simply $\{W(J_\varepsilon^\rho) \mid \varepsilon>0\}$. The topology on 2^X determined by $2^{[\rho]}$ is called the *uniform topology on* 2^X *determined by* ρ. The uniform space $(2^X, 2^{[\rho]})$ is actually metrizable [17, Proposition 4.1] with the Hausdorff metric determined by $\rho^*=\rho/(1+\rho)$, but it will be more convenient to simply view the space $(2^X, 2^{[\rho]})$ as a uniform space.

If $A_1,\dots,A_n$ are subsets of X, then $\langle A_1,\dots,A_n\rangle$ is defined to be the collection

$$\left\{F\in 2^X \mid F\subset \bigcup_{i=1}^n A_i \;\&\; A_i\cap F\neq\varnothing,\ i=1,2,\dots,n\right\}.$$

The *finite topology* on 2^X has as an open base the open collections $\langle G_1,\dots,G_n\rangle$ where $G_1,\dots,G_n$ are open subsets of X [17, Proposition 2.1]. This topology is sometimes called the Vietoris or exponential topology on 2^X (cf. [14]).

If t_0 is an element of T, then we use $\mathscr{S}(t_0)$ to denote the *neighborhood filterbase at* t_0 consisting of all $S_\varepsilon(t_0)$, $\varepsilon>0$, where $S_\varepsilon(t_0)=\{t\in T \mid d(t,t_0)<\varepsilon\}$ (cf. [10, p. 211]). The *grill* of $\mathscr{S}(t_0)$ [2, p. 11] is denoted by $\mathscr{S}''(t_0)$ and consists of all sets $S''(t_0)$ contained in T such that $S''(t_0)\cap S_\varepsilon(t_0)\neq\varnothing$ for every $\varepsilon>0$. If Ω is a mapping, $\Omega\colon T\to 2^X$, then the *pseudo-limit superior of* Ω *as* $t\to t_0$ (abbreviated: p-lim $\sup_{t\to t_0}\Omega(t)$) is defined to be

$$\bigcap_{S_\varepsilon(t_0)\in\mathscr{S}(t_0)} \operatorname{cl}\Big[\bigcup_{t\in S_\varepsilon(t_0)}\Omega(t)\Big],$$

and the *pseudo-limit inferior of* Ω *as* $t\to t_0$ (abbreviated: p-lim $\inf_{t\to t_0}\Omega(t)$) is defined to be

$$\bigcap_{S''(t_0)\in\mathscr{S}''(t_0)} \operatorname{cl}\Big[\bigcup_{t\in S''(t_0)}\Omega(t)\Big]$$

(cf. [2, Example 3, p. 120]).

We now state three different definitions of semicontinuity for mappings $\Omega\colon T\to 2^X$. Ω is *pseudo-upper semicontinuous at* $t_0\in T$ (abbreviated: p-usc at $t_0\in T$) if p-lim $\sup_{t\to t_0}\Omega(t)\subset\Omega(t_0)$. Dually Ω is *pseudo-lower semicontinuous at* $t_0\in T$ (abbreviated: p-lsc at $t_0\in T$) if $\Omega(t_0)\subset$ p-lim $\inf_{t\to t_0}\Omega(t)$. Ω is pseudo-continuous (abbreviated: p-continuous at $t_0\in T$) if Ω is p-usc and p-lsc at $t_0\in T$. For the justification of using the adjective "pseudo" see [2, p. 124 and p. 128]. Ω is *upper semicontinuous at* $t_0\in T$ (abbreviated: usc at $t_0\in T$) if for each open G containing $\Omega(t_0)$ there is an $S_\varepsilon(t_0)\in\mathscr{S}(t_0)$ such that

$$t\in S_\varepsilon(t_0) \qquad \text{implies} \qquad \Omega(t)\subset G.$$

Ω is *lower semicontinuous at* $t_0 \in T$ (abbreviated: lsc at $t_0 \in T$) if for every open G meeting $\Omega(t_0)$ there is an $S_\varepsilon(t_0) \in \mathscr{S}(t_0)$ such that $t \in S_\varepsilon(t_0)$ implies $\Omega(t) \cap G \neq \varnothing$. Let 2^X have the finite topology. Then Ω is continuous at $t_0 \in T$ if and only if Ω is usc at $t_0 \in T$ and lsc at $t_0 \in T$. Ω is *upper semicontinuous with respect to inclusion at* $t_0 \in T$ (abbreviated: usci at $t_0 \in T$) if for every $\varepsilon > 0$ there is an $S_\delta(t_0) \in \mathscr{S}(t_0)$ such that $t \in S_\delta(t_0)$ implies $J_\varepsilon^\rho[\Omega(t_0)] \supset \Omega(t)$. Ω is *lower semicontinuous with respect to inclusion at* $t_0 \in T$ (abbreviated: lsci at $t_0 \in T$) if for every $\varepsilon > 0$ there is an $S_\delta(t_0) \in \mathscr{S}(t_0)$ such that $t \in S_\delta(t_0)$ implies $J_\varepsilon^\rho[\Omega(t)] \supset \Omega(t_0)$. Let 2^X have the uniform topology determined by ρ. Then Ω is continuous if and only if Ω is usci at $t_0 \in T$ and lsci at $t_0 \in T$.

We enumerate three properties which will be used later.

(1) *Lusin C_p property.* For every $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that $\Omega | T \backslash E_\varepsilon$ is p-continuous.

(2) *Lusin C_u property.* Let 2^X have the uniform topology determined by ρ. For every $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that $\Omega | T \backslash E_\varepsilon$ is continuous.

(3) *Lusin C_f property.* The same as (2) only 2^X has the finite topology.

We shall say that a family of mappings $\{f_\alpha \mid \alpha \in A, f_\alpha : T \to X\}$ is *almost equicontinuous* if for every $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that $\{f_\alpha | T \backslash E_\varepsilon, \alpha \in A\}$ is an equicontinuous family on $T \backslash E_\varepsilon$.

Finally, if x is an element of X and A is a subset of X we define

$$\rho(x, A) = \inf \{\rho(x, y) \mid y \in A\}.$$

We state without proof the following lemma.

Lemma 2.1. *Let Ω be a mapping, $\Omega : T \to 2^X$. Then a necessary and sufficient condition that Ω be p-usc at each point of T is that if $\{x_n\}$ and $\{t_n\}$ are sequences in X and T respectively such that $x_n \in \Omega(t_n)$ for every n and such that $x_n \to x$ and $t_n \to t$ as $n \to \infty$, then $x \in \Omega(t)$.*

Lemma 2.2. *If A is a nonempty subset of X, then the mapping $f_A : X \to R$ defined by $f_A(x) = \rho(x, A)$ is Lipschitzian, i.e., $|f_A(x) - f_A(y)| \leqq \rho(x, y)$, $x, y \in X$.*

Proof. [10, p. 185].

In [12] we pointed out the validity and application of the following result.

Theorem 2.1. *Let 2^X have the uniform topology determined by ρ. Then a necessary and sufficient condition that $\Omega : T \to 2^X$ be continuous is that the family of mappings $\{t \to \rho(x, \Omega(t)) \mid x \in X\}$ be equicontinuous.*

Proof. Sufficiency. Given $\varepsilon > 0$ there is a $\delta > 0$ such that $d(t, t') < \delta$ implies $|\rho(x, \Omega(t)) - \rho(x, \Omega(t'))| < \varepsilon$, $x \in X$. Consequently $x \in \Omega(t)$, $d(t, t') < \delta$ imply $0 \leqq \rho(x, \Omega(t')) < \varepsilon$, and therefore there is a $y \in \Omega(t')$ such that $\rho(x, y) < \varepsilon$, i.e., $x \in J_\varepsilon^\rho[\Omega(t')]$. Whence $d(t, t') < \delta$ implies $J_\varepsilon^\rho[\Omega(t')] \supset \Omega(t)$, and by a symmetric argument $J_\varepsilon^\rho[\Omega(t)] \supset \Omega(t')$.

Necessity. Given $\varepsilon>0$ there is a $\delta>0$ such that $d(t,t')<\delta$ implies $J^{\rho}_{\varepsilon/2}[\Omega(t')] \supset \Omega(t)$ and $J^{\rho}_{\varepsilon/2}[\Omega(t)] \supset \Omega(t')$. Given $x\in X$ and $d(t,t')<\delta$ there is a $y\in\Omega(t)$ such that $\rho(x,\Omega(t))+\varepsilon/2>\rho(x,y)$, and there is a $y'\in\Omega(t')$ such that $\rho(x,\Omega(t'))+\varepsilon/2 >\rho(x,y')$. Moreover, there is a $b'\in\Omega(t')$ such that $\rho(b',y)<\varepsilon/2$, and there is a $b\in\Omega(t)$ such that $\rho(b,y')<\varepsilon/2$. Thus

$$\rho(x,\Omega(t)) \leqq \rho(x,b) \leqq \rho(x,y')+\rho(y',b) < \rho(x,\Omega(t'))+\varepsilon$$

and

$$\rho(x,\Omega(t')) \leqq \rho(x,b') \leqq \rho(x,y)+\rho(y,b') < \rho(x,\Omega(t))+\varepsilon,$$

thereby proving that the family $\{t\to\rho(x,\Omega(t)) \mid x\in X\}$ is equicontinuous.

REMARK (i). Let X be separable, and let $H\colon T\to\mathscr{A}(X)$ be measurable, then H^-G is measurable for every open $G\subset X$. This is evident since each G is an F_σ (cf. [15, p. 400] or [7]). Consequently if H is measurable, then each of the functions $t\to\rho(x,H(t))$, $x\in X$ is measurable (cf. [7, Theorem 3.2]). This follows from the fact that if $r>0$ and $x\in X$ are given, then the set $\{t\in T \mid \rho(x,H(t))<r\}$ is the same as the set $H^-J^{\rho}_r[x]$, where $J^{\rho}_r[x]=\{y \mid \rho(x,y)<r\}$.

LEMMA 2.3. *If H is a measurable mapping, $H\colon T\to\mathscr{A}(X)$, and if X is separable, then for every $\varepsilon>0$ there is an open $E_\varepsilon\subset T$ such that $\mu(E_\varepsilon)<\varepsilon$ and such that the restriction of the mapping $(t,x)\to\rho(x,H(t))$ to $(T\backslash E_\varepsilon)\times X$ is continuous.*

Proof. The conclusion is immediate from Lemma 2.2, Remark (i), and Theorem 2.1 of [13].

In the case where X is also locally compact this lemma can be obtained from [7, Theorem 3.1]. The theorem in [13] which we cited above is more general.

THEOREM 2.2. *Let X be separable. A necessary and sufficient condition that a mapping $\Omega\colon T\to 2^X$ have the Lusin C_u property is that the family of mappings $\{t\to\rho(x,\Omega(t)) \mid x\in X\}$ be almost equicontinuous.*

Proof. If Ω has the Lusin C_u property, then given $\varepsilon>0$ there is an open set $E_\varepsilon\subset T$ such that $\mu(E_\varepsilon)<\varepsilon$ and such that $\Omega|(T\backslash E_\varepsilon)$ is continuous when 2^X has the uniform topology determined by ρ. By Theorem 2.1 the family of mappings $\{t\to\rho(x,\Omega(t))(t\in T\backslash E_\varepsilon) \mid x\in X\}$ is equicontinuous. Therefore, the family of mappings $\{t\to\rho(x,\Omega(t))\ (t\in T) \mid x\in X\}$ is almost equicontinuous.

Conversely if the family of mappings $\{t\to\rho(x,\Omega(t))\ (t\in T) \mid x\in X\}$ is an almost equicontinuous family, then Ω has the Lusin C_u property by Theorem 2.1.

The next theorem gives various criteria for the measurability of multivalued functions. The equivalence (ii) $\Leftrightarrow$ (v) shows the intimate relationship between measurable functions and p-continuous functions. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) were obtained by Castaing [7, Theorem 3.2] in the special case where X is locally compact and $H(t)\in 2^X$ for every $t\in T$ (cf. also [15, §1 Theorem and Corollary 1]).

THEOREM 2.3. *Let X be a Polish space* [3, Part 2, p. 195]. *Let a mapping $H: T \to \mathscr{A}(X)$ be given. Then statements* (ii) *through* (v) *are equivalent. Moreover,* (i) *implies any of the remaining four statements.*

(i) *H is measurable*;

(ii) *$t \to \mathrm{cl}\,(H(t))$ is measurable*;

(iii) *Each of the mappings $t \to \rho(x, H(t))$, $x \in X$ is measurable*;

(iv) *$H^{-}G$ is measurable for every open set $G \subset X$*;

(v) *$t \to \mathrm{cl}\,(H(t))$ has the Lusin C_p property.*

Proof. The validity of [(i) $\Rightarrow$ (iv) $\Rightarrow$ (iii)] was discerned in Remark (i). We observe that $\rho(x, H(t)) = \rho(x, \mathrm{cl}\,(H(t)))$ for every $(t, x) \in T \times X$. Thus (iii) implies that for each $x \in X$ the mapping $t \to \rho(x, \mathrm{cl}\,(H(t)))$ $(t \in T)$ is measurable. Thus given $\varepsilon > 0$ Lemma 2.3 guarantees that there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that the mapping $(t, x) \to \rho(x, \mathrm{cl}\,(H(t)))$ $((t, x) \in (T \backslash E_\varepsilon) \times X)$ is continuous. Let $\mathscr{S}_\varepsilon(t)$ denote the neighborhood filter base relative to $T \backslash E_\varepsilon$ at $t \in T \backslash E_\varepsilon$, and as usual $\mathscr{S}_\varepsilon''(t)$ denotes the grill of $\mathscr{S}_\varepsilon(t)$. Let $\{x_n\}$ and $\{t_n\}$ be sequences in X and $T \backslash E_\varepsilon$ respectively such that $x_n \in \mathrm{cl}\,(H(t_n))$ for every n and such that $x_n \to x$ and $t_n \to t$. Then we have that $\rho(x_n, \mathrm{cl}\,(H(t_n))) = 0 \to \rho(x, \mathrm{cl}\,(H(t)))$ (by continuity), and this implies that $x \in \mathrm{cl}\,(H(t))$ [10, p. 185]. Consequently $t \to \mathrm{cl}\,(H(t))$ is p-usc at each $t \in T \backslash E_\varepsilon$ by Lemma 2.1. Now let $t_0 \in T \backslash E_\varepsilon$ be given, and select $x_0 \in \mathrm{cl}\,(H(t_0))$. If $S''(t_0)$ is an element of $\mathscr{S}_\varepsilon''(t_0)$, then

$$\rho(x, \mathrm{cl}\,(H(t))) \geqq \rho\Big(x, \mathrm{cl}\Big[\bigcup_{s \in S''(t_0)} \mathrm{cl}\,(H(s))\Big]\Big)$$

for every $(t, x) \in S''(t_0) \times X$. If we select $S_{1/n}(t_0) \in \mathscr{S}_\varepsilon(t_0)$ for $n = 1, 2, 3, \ldots$, then there exist elements $t_n \in S''(t_0) \cap S_{1/n}(t_0)$ for $n = 1, 2, 3, \ldots$. Such a sequence $\{t_n\}$ converges to t_0. If $\{x_n\}$ is any sequence in X which converges to x_0, then we have

$$\begin{aligned} 0 = \rho(x_0, \mathrm{cl}\,(H(t_0))) &= \lim \rho(x_n, \mathrm{cl}\,(H(t_n))) \\ &\geqq \lim \rho\Big(x_n, \mathrm{cl}\Big[\bigcup_{s \in S''(t_0)} \mathrm{cl}\,(H(s))\Big]\Big) \\ &= \rho\Big(x_0, \mathrm{cl}\Big[\bigcup_{s \in S''(t_0)} \mathrm{cl}\,(H(s))\Big]\Big) \geqq 0 \end{aligned}$$

which shows that $x_0 \in \mathrm{cl}\,[\bigcup_{s \in S''(t_0)} \mathrm{cl}\,(H(s))]$, thereby proving the relation $\mathrm{cl}\,(H(t_0)) \subset$ p-lim $\inf_{t \to t_0} (\mathrm{cl}\, H(t))$. Thus we have shown $t \to \mathrm{cl}\,(H(t))$ restricted to $T \backslash E_\varepsilon$ is p-continuous at every $t \in T \backslash E_\varepsilon$, and we conclude that [(iii) $\Rightarrow$ (v)]. Statement (v) implies that there is a sequence $\{T_n\}$ of compact subsets of T and a set $\mathscr{N}$ of measure zero contained in T such that $\mathrm{cl}\,(H)|T_n$ is p-continuous for every n and $\bigcup_{n=1}^{\infty} T_n = T \backslash \mathscr{N}$ ($\mathrm{cl}\,(H)|T_n$ denotes the restriction of the mapping $t \to \mathrm{cl}\,(H(t))$ to T_n). By Lemma 2.1 each mapping, $\mathrm{cl}\,(H)|T_n$, has a closed graph, i.e., $\bigcup_{t \in T_n} \{t\} \times \mathrm{cl}\,(H(t))$ is a closed subset of $T_n \times X$. Therefore $\mathrm{cl}\,(H)|T_n$ is measurable for every

n [7, Lemma 3.2]. Define a sequence of mappings $H_n^*: T \to 2^X \cup \{\varnothing\}$ by the relation

$$H_n^*(t) = \mathrm{cl}\,(H(t)), \quad \text{if } t \in T_n$$
$$= \varnothing, \qquad \text{if } t \notin T_n.$$

Then each H_n^* has the property that $H_n^{*-}F$ is measurable for every closed $F \subset X$. The measurability of $\mathrm{cl}\,(H)$ then follows from the relation $\mathrm{cl}\,(H(t)) = \bigcup_{n=1}^{\infty} H_n^*(t)$, if $t \notin N$. This shows that [(v) $\Rightarrow$ (ii)]. Since [(i) $\Rightarrow$ (iv)] has already been demonstrated it follows that (ii) $\Rightarrow$ (iv). Whence statements (ii) through (v) are all equivalent.

REMARK (ii). If the mapping, $t \to \mathrm{cl}\,(H(t))$ $(t \in T)$, in Theorem 2.3 is denoted by Φ, then we observe that $H^-G = \Phi^-G$ for every open $G \subset X$.

LEMMA 2.4. *Let X be a Polish space. If $\Omega: T \to 2^X$ has the Lusin C_u property, then Ω is measurable.*

Proof. There is a set N of measure zero contained in T and a sequence of compact sets, $\{T_n\}$, contained in T such that $\bigcup_{n=1}^{\infty} T_n = T \backslash N$ and such that the mappings $\Omega | T_n: T_n \to 2^X$ are continuous when 2^X has the uniform topology determined by ρ. Thus the mappings $\Omega | T_n$ are p-usc. The measurability of Ω follows as in the above proof of (v) $\Rightarrow$ (ii) in Theorem 2.3.

It is interesting to note in passing that Plis' theorem [20] is an immediate consequence of Lemmas 2.3, 2.4 and Theorem 2.2.

COROLLARY 2.1 (PLIS). *Let X be compact. Then a necessary and sufficient condition that $\Omega: T \to 2^X$ be measurable is that Ω have the Lusin C_u property.*

THEOREM 2.4. *Let X be separable and locally compact. Let a mapping $\Omega: T \to 2^X$ be given. Then there is a metric ρ_∞ on X such that the two topologies $\tau(\rho)$ and $\tau(\rho_\infty)$ coincide, and such that the following two statements are equivalent.*

(i) *Ω is measurable;*

(ii) *Ω has the Lusin C_u property when 2^X has the uniform topology determined by ρ_∞.*

Proof. X is a Polish space [3, Part 2, Corollary, p. 196]. Whence (ii) implies (i) by Lemma 2.4. In the special case where X is compact the theorem follows from Corollary 2.1. We may therefore assume that X is not compact. Let $X_\infty = X \cup \{\infty\}$ denote the one-point compactification of X. Then X_∞ is metrizable [10, p. 247]. Since X_∞ is compact, it follows that X_∞ is complete with respect to any metric defining its topology. Let ρ_∞ be a metric on X_∞ defining the topology of X_∞ and therefore also defining the topology of X. Hereafter both X and X_∞ will be assumed to be metrized with ρ_∞, and no further use will be made of the metric ρ. Let Ω be measurable, and let Ω_∞ denote the mapping, $\Omega_\infty: T \to \mathscr{A}(X_\infty)$, where $\Omega_\infty(t)$ is the image of $\Omega(t)$ under the inclusion mapping $i_\infty: X \subset X_\infty$. Let G_∞ be an open subset of X_∞. Then we have $\Omega_\infty^- G_\infty = \Omega^-(G_\infty \backslash \{\infty\})$. Since $G_\infty \backslash \{\infty\}$ is open in X, and $\Omega: T \to 2^X$ is measurable, it follows that $\Omega_\infty^- G_\infty$ is measurable. Therefore by

Theorem 2.3 [(iii) $\Leftrightarrow$ (v)], the mappings $t \to \rho_\infty(x, \Omega_\infty(t)) = \rho_\infty(x, \text{cl}(\Omega_\infty(t)))$ $(t \in T)$, $x \in X$, are measurable. By Lemma 2.2, the compactness of $T \times X_\infty$, and [13, Theorem 2.1], the aforementioned mappings form an almost equicontinuous family. Whence given $\varepsilon > 0$ there is an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that the mappings, $t \to \rho_\infty(x, \Omega_\infty(t)) = \rho_\infty(x, \text{cl}(\Omega_\infty(t)))$, $x \in X$, when restricted to $T \backslash E_\varepsilon$ are equicontinuous. Let(2)

$$\mathscr{J}^\infty = \{J_\delta^\infty \mid \delta > 0\}, \qquad (J_\delta^\infty = \{(x, y) \in X_\infty \times X_\infty \mid \rho_\infty(x, y) < \delta\})$$

denote the uniformity which determines the topology of X_∞. Then $\mathscr{J}^\infty \cap (X \times X) = \{J_\delta^\infty \cap (X \times X) \mid \delta > 0\}$ determines the topology of X. Given $\delta > 0$ there is a $\beta > 0$ such that $t, t' \in T \backslash E_\varepsilon$, $d(t, t') < \beta$ imply

$$|\rho_\infty(x, \Omega_\infty(t)) - \rho_\infty(x, \Omega_\infty(t'))| < \delta, \qquad x \in X_\infty.$$

Hence $t, t' \in T \backslash E_\varepsilon$, $d(t, t') < \beta$ imply

$$(J_\delta^\infty \cap (X \times X))[\Omega(t')] \supset \Omega(t) \;\&\; (J_\delta^\infty \cap X \times X))[\Omega(t)] \supset \Omega(t')$$

(cf. the sufficiency part of the proof of Theorem 2.1). This proves (ii).

The following corollary is evident.

COROLLARY 2.2. *Let X be separable and locally compact. A necessary and sufficient condition that a mapping, $\Omega: T \to 2^X$, be measurable is that for every $\varepsilon > 0$ there exists an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that $\Omega | T \backslash E_\varepsilon$ is* lsc.

Now $2^{[\rho]}$ and $2^{[\rho_\infty]}$ define equivalent topologies on $\mathscr{C}(X)$ [17, Theorem 3.3], but since ρ and ρ_∞ are not uniformly equivalent we cannot expect that $2^{[\rho]}$ and $2^{[\rho_\infty]}$ will define equivalent topologies on 2^X. For example, let R denote the set of real numbers with the usual topology determined by the metric ρ where $\rho(t, t') = |t - t'|$, and let ρ_∞ denote a metric defining the topology of the one-point compactification of R. Metrics ρ and ρ_∞ define equivalent topologies on R. Let $\{K_n\}$ denote the sequence of compact intervals $\{[-n, n]\}$. Then $\{K_n\}$ does not converge in the uniform topology determined by ρ, but the sequence converges to R in the uniform topology determined by ρ_∞.

It is noted that in the special case where X is locally compact, Michael's Theorem 3.3 [17], and Theorem 2.4 may be applied to obtain a result of Castaing's [7, Theorem 2.4].

COROLLARY 2.3. *Let X be separable and locally compact. Let a mapping $\Omega: T \to \mathscr{C}(X)$ be given. Then the following are equivalent*:

(i) Ω *is measurable*;

(ii) Ω *has the Lusin C_f property*;

(iii) Ω *has the Lusin C_u property*.

The following corollary is embodied in the proof of Theorem 2.4.

(2) According to our agreement in §2 we should write $\mathscr{J}^{\rho_\infty} = \{J_\delta^{\rho_\infty} \mid \delta > 0\}$, but in order to avoid too many typographical levels we are suppressing ρ and simply writing $\mathscr{J}^\infty = \{J_\delta^\infty \mid \delta > 0\}$.

COROLLARY 2.4. *Let X be separable and locally compact. Let X_∞ denote the one-point compactification of X. Let $\Omega: T \to 2^X$ be a given mapping, and let $\Omega_\infty: T \to \mathscr{A}(X_\infty)$ be the mapping defined by the condition that $\Omega_\infty(t)$ is the image of $\Omega(t)$ under the inclusion mapping $i_\infty: X \subset X_\infty$. Then Ω is measurable if and only if the function* $\mathrm{cl}\,(\Omega_\infty)$ *from T into 2^{X_∞}, defined by $t \to \mathrm{cl}\,(\Omega_\infty(t))$, is measurable.*

A mapping $S: T \to \mathscr{A}(X)$ is *simple* if there is a finite partition of T, say $\{T_1, \ldots, T_n\}$, such that S has a constant value on each of the T_i. If the sets T_i are measurable, then S is a measurable simple function, i.e. $S^- F$ is measurable for every closed $F \subset X$.

COROLLARY 2.5. *Let X be separable and locally compact. Let 2^X have the uniform topology determined by ρ_∞. A necessary and sufficient condition that a mapping $\Omega: T \to 2^X$ be measurable is that there exists a sequence $\{S_n\}$ of measurable simple functions $S_n: T \to 2^X$ such that $S_n(t) \to \Omega(t)$ a.e. on T.*

Proof. By Theorem 2.4 the mapping Ω is measurable if and only if Ω is measurable in the Bourbaki sense [4, p. 169], when considered as a mapping from T into the metrizable space 2^X with the uniform topology determined by ρ_∞. Thus Corollary 2.5 follows immediately as in Bourbaki [4, Theorem 3, p. 178].

The next two theorems are improved statements of Castaing's Corollaries 5.2 and 5.2′ [7].

THEOREM 2.5. *Let X be a Polish space. Let a measurable mapping $\Omega: T \to 2^X$ be given. Let f be a continuous mapping from $T \times X$ into a Hausdorff space Y, and let y be a measurable mapping, $y: T \to Y$ such that $y(t) \in f(t, \Omega(t))$ for every $t \in T$. Then there is a measurable mapping $x: T \to X$ such that $x(t) \in \Omega(t)$ and $y(t) = f(t, x(t))$ for every $t \in T$.*

Proof. The function $\Gamma: T \to 2^X$ defined by $\Gamma(t) = \{u \in \Omega(t) \mid f(t, u) = y(t)\}$ is measurable (the $\Gamma(t)$ are closed by the continuity of f). For if $\varepsilon > 0$ is given, then there exists an open set $E_\varepsilon \subset T$ such that $\mu(E_\varepsilon) < \varepsilon$ and such that (1) $\Omega|(T \backslash E_\varepsilon)$ is p-continuous (by Theorem 2.3) and (2) $y|T \backslash E_\varepsilon$ is continuous [4, Proposition 1]. It follows from Lemma 2.1 that Γ is p-usc on $T \backslash E_\varepsilon$. Therefore $\Gamma|(T \backslash E_\varepsilon)$ is measurable [7, Lemma 3.2] and $\mu(E_\varepsilon) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, Γ is measurable. The conclusion of the theorem follows from either of the three results [15, §1, Theorem], [5, Theorem 3], or [7, Theorem 5.2].

Let (Y, δ) be a metric space. We shall say that a mapping $h: X \to Y$ is *locally uniformly continuous* if for each $x \in X$ there is a neighborhood of x (i.e., a set which contains an open set containing x) on which h is uniformly continuous (X is assumed to carry the metric ρ).

THEOREM 2.5′ *Let (X, ρ) be a Polish space, and let (Y, δ) be a metric space. Let $\Omega: T \to 2^X$ be a measurable mapping, and let $f: T \times X \to Y$ be a mapping such that the mappings $t \to f(t, x)$, $x \in X$ are measurable, and such that the mappings*

$x \to f(t, x)$, $t \in T$ are locally uniformly continuous. Let y be a measurable mapping, $y: T \to Y$, such that $y(t) \in f(t, \Omega(t))$ for every $t \in T$. Then there is a measurable mapping $x: T \to X$ such that $x(t) \in \Omega(t)$ and $y(t) = f(t, x(t))$ for every $t \in T$.

Proof. Using a result we obtained in [13, Corollary 2.1] the proof follows the same line of reasoning as the proof of Theorem 2.5. We omit the details.

The euclidean space R^q is ordered lexicographically [10, p. 57] by a relation, $\geqq$, defined by $x_1 = (x_1^1, \ldots, x_1^q) \geqq x_2 = (x_2^1, \ldots, x_2^q)$ if in the first coordinate k where x_1 and x_2 differ we have $x_1^k > x_2^k$. Let F be a mapping, $F: T \to \mathscr{C}(R^q)$. Then there is a unique lexicographic minimum of $F(t)$ (denoted by lex. min $F(t)$) for each $t \in T$.

LEMMA 2.5 (FILIPPOV [11]). *Let $F: T \to \mathscr{C}(R^q)$ be a measurable mapping. Then the function $x_L: T \to R^q$ defined by $x_L(t) =$ lex. min $F(t)$ is measurable and $x_L(t) \in F(t)$ for every $t \in T$.*

Proof. The conclusion is immediate from Filippov's proof [11, pp. 78–79] and Corollary 2.3.

Our main purpose in mentioning Lemma 2.5 is that we wish to use it in discussing the following example.

EXAMPLE. The constructions of the function x in Theorem 2.5 and 2.5′ given by Kuratowski [15] or Castaing [7], [5] are somewhat complicated. In the finite dimensional situation, however, there is another interesting construction (cf. [8, p. 384]). Let $\Omega: T \to 2^{R^q}$ be a measurable mapping, let $y: T \to R^p$ be a measurable mapping, and let $f: T \times R^q \to R^p$ be a mapping such that the mappings $t \to f(t, x)$, $x \in R^q$ are measurable and such that the mappings $x \to f(t, x)$, $t \in T$ are continuous. We suppose $y(t) \in f(t, \Omega(t))$ for every $t \in T$. Then the mapping $t \to \Gamma(t) \in 2^{R^q}$ defined by $\Gamma(t) = \{u \in \Omega(t) \mid f(t, u) = y(t)\}$ is measurable [see the proof of Theorem 2.5′]. We are content to construct a measurable function $x: T \to R^q$ such that $x(t) \in \Gamma(t)$ for every $t \in T$. Let R^q_∞ denote the one-point compactification of R^q. Let $\Gamma_\infty: T \to \mathscr{A}(R^q_\infty)$ denote the mapping defined by requiring $\Gamma_\infty(t)$ to be the image of $\Gamma(t)$ under the inclusion mapping, $i_\infty: R^q \subset R_q^\infty$. By Corollary 2.4 $t \to \mathrm{cl}\,(\Gamma_\infty(t)) \in 2^{R^q_\infty}$ is a measurable mapping. There is a homeomorphism h of R^q_∞ onto the unit sphere $S^q \subset R^{q+1}$, $S^q = \{(\xi^1, \ldots, \xi^q, \xi^{q+1}) \mid \sum_{i=1}^{q+1} (\xi^i)^2 = 1\}$, such that $h(\infty) = (1, 0, \ldots, 0)$ [10, p. 246]. Now since

$$\{t \mid \mathrm{cl}\,(\Gamma_\infty(t)) \cap h^{-1}(F) \neq \varnothing\} = \{t \mid h(\mathrm{cl}\,(\Gamma_\infty(t))) \cap F \neq \varnothing\}$$

for every closed $F \subset S^q$, it follows that $t \to h(\mathrm{cl}\,(\Gamma_\infty(t))) \in \mathscr{C}(S^q)$ is measurable. We define $x_\infty: T \to S^q$ by the relation $x_\infty(t) =$ lex. min $h(\mathrm{cl}\,(\Gamma_\infty(t)))$, $t \in T$. We observe that $x_\infty(t) \neq (1, 0, \ldots, 0) = h(\infty)$ for every $t \in T$, since if $x_\infty(t) = (1, 0, \ldots, 0)$, then $(\xi^1, \ldots, \xi^q, \xi^{q+1}) \in h(\mathrm{cl}\,(\Gamma_\infty(t)))$ would imply that $\xi^1 \geqq 1$, and thus $\xi^1 = 1$, $\xi^i = 0$, $i > 1$, i.e., $h(\mathrm{cl}\,(\Gamma_\infty(t))) = \{h(\infty)\}$ a contradiction. The mapping x_∞ is measurable by Filippov's lemma and thus so also is the mapping $i_\infty^{-1} \circ h^{-1} \circ x_\infty: T \to R^q$ which is the required mapping, i.e. $i_\infty^{-1} \circ h^{-1} \circ x_\infty$ is measurable and $(i_\infty^{-1} \circ h^{-1} \circ x_\infty)(t) \in \Gamma(t)$, for every $t \in T$.

REMARK (iii). The idea of using the one-point compactification of X, which we described in Theorem 2.4, is particularly well suited for applications in euclidean space R^q. Using the euclidean distance $\rho(x, y) = \|x-y\|$, $x, y \in R^q$ as the metric ρ, then the metric ρ_∞ on R^q determined by ρ and stereographic projection on the Riemann sphere is

$$\rho_\infty(x, y) = \frac{\rho(x, y)}{[1+\|x\|^2]^{1/2}[1+\|y\|^2]^{1/2}}, \qquad x, y \in R^q.$$

Evidently the two metrics satisfy the relation $\rho_\infty(x, y) \leqq \rho(x, y)$, $x, y \in R^q$. Thus the topology on 2^{R^q} determined by $2^{[\rho_\infty]}$ is weaker than the topology on 2^{R^q} determined by $2^{[\rho]}$, i.e., $2^{[\rho_\infty]} \subset 2^{[\rho]}$. If E_1 and E_2 are two topological spaces, then $C(E_1, E_2)$ denotes the collection of continuous mappings of E_1 into E_2. Thus we have $C(E_1, (2^{R^q}, 2^{[\rho]})) \subset C(E_1, (2^{R^q}, 2^{[\rho_\infty]}))$ for any topological space E_1. In general the inclusion is strict, as is shown by an example given by Cesari [8, pp. 374–375], viz., pick $q=2$, $E_1=[0, 1]$, and define $t \to \Omega(t) \in 2^{R^2}$ by the relation

$$\Omega(t) = \{(x, y) \in R^2 \mid x \geqq 0 \ \&\ 0 \leqq y \leqq tx\}.$$

As Cesari points out, the mapping Ω is not in the class $C([0, 1], (2^{R^2}, 2^{[\rho]}))$. Also it is not difficult to show from Cesari's remarks that the mapping Ω does *not* enjoy the Lusin C_u property when 2^{R^2} has the uniform topology determined by ρ. Geometrically, however, it is fairly clear that Ω belongs to the class $C([0, 1], (2^{R^2}, 2^{[\rho_\infty]}))$.

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