ANALYTIC CANONICAL FORMS FOR NONLINEAR DIFFERENCE EQUATIONS[1]

BY
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1. **Introduction.** We consider a system of nonlinear difference equations of the form

$$y(x+1) = F(x, y(x)) \tag{1.1}$$

where x is a complex variable, y is an n-dimensional vector, and F is an n-dimensional vector. In general, the components of $F(x, y)$ will be holomorphic in the region $|x| > R$, $\|y\| < \delta$. In the expansion

$$F(x, y) = F_0(x) + \hat{B}(x)y + \hat{F}(x, y),$$

where

$$\hat{F}(x, y) = \sum_{|\mathfrak{p}| \geq 2} F_{\mathfrak{p}}(x)[y]^{\mathfrak{p}},$$

the vectors $F_0(x)$, $F_{\mathfrak{p}}(x)$, and the matrix $\hat{B}(x)$ will be holomorphic for $|x| > R$. B_0 will be the constant matrix in the power series expansion of $\hat{B}(x)$ and λ_j, $j = 1, \ldots, n$, will denote the eigenvalues of B_0.

Throughout this paper we will use the following notation:

(i) If v is an m-dimensional vector with components v_j, $j = 1, \ldots, m$, then

$$\|v\| = \sum_{j=1}^{m} |v_j| \quad \text{and} \quad [v]^{\mathfrak{p}} = v_1^{\mathfrak{p}_1} v_2^{\mathfrak{p}_2} \cdots v_m^{\mathfrak{p}_m},$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are nonnegative integers and $|\mathfrak{p}| = \sum_{j=1}^{m} \mathfrak{p}_j$.

(ii) If A is an $m \times m$ matrix, then $\|A\| = \sup_{\|v\|=1} \|Av\|$.

Let $y(x)$ be a solution of the difference equation (1.1) which approaches a limit, y_0, as x tends to infinity in some direction in the complex plane. Then

$$y_0 = F(\infty, y_0). \tag{1.2}$$

On the other hand, if y_0 is a solution of equation (1.2), it is natural to ask if a solution of equation (1.1) in a neighborhood of y_0 will approach y_0 as x tends to infinity. L. J. Grimm and W. A. Harris, Jr. [1] have answered this question in the affirmative under the assumption

$$0 < |\lambda_j| < 1, \qquad j = 1, \ldots, n, \tag{1.3}$$

by constructing the general solution of (1.1) in a sector of the form

$$-\frac{\pi}{2} < l_1 < \arg(x-a) < l_2 < \frac{\pi}{2}.$$

Presented to the Society, January 25, 1967; received by the editors July 24, 1967.
(1) The work was partially supported by ONR grant 5151.

However if not all of the eigenvalues of B_0 have modulus less than one it is expected that some, but not all, of the solutions of equation (1.1) in a neighborhood of y_0 will approach y_0 as x tends to infinity. We expect that if B_0 has ν, $1 \leqq \nu \leqq n$, eigenvalues satisfying (1.3) there will be a ν-dimensional manifold, containing y_0, such that any solution of (1.1) in a neighborhood of y_0 on the manifold will approach y_0 as x approaches infinity through an appropriate sector.

The purpose of this paper is to construct an analytic change of variables which will transform the difference equation (1.1) into a canonical form for which the stable manifold is clearly exhibited and such that on this stable manifold the equation has elementary form. Then the equation of the stable manifold in y-space can be displayed.

Let S be any sector of the form

$$S\colon |\arg (xe^{-i\theta}-a)| < \pi/2+\rho$$

where a^{-1} and ρ are sufficiently small positive constants. Assume that B_0 and B_0-I are nonsingular and that θ satisfies

$$\begin{aligned} \theta &\neq \arg(-\log \lambda_j), && j = 1,\dots,n, \\ \theta &\neq \arg(-\log \lambda_j/\lambda_k), && \lambda_j \neq \lambda_k. \end{aligned}$$

Then from the results of W. A. Harris, Jr. and Y. Sibuya [4], [6], and L. J. Grimm and W. A. Harris, Jr. [1] there exists a linear transformation, holomorphic in S, of the form $z(x)=T(x)(y(x)-\phi(x))$ which reduces the difference equation (1.1) into the form

$$z(x+1) = B(x)z(x)+f(x, z(x)) \tag{1.4}$$

where

$$f(x, z) = \sum_{|\mathfrak{p}|\geqq 2} f_{\mathfrak{p}}(x)[z]^{\mathfrak{p}}$$

is holomorphic for $x \in S$, $\|z\| < \delta$. The vector coefficients $f_{\mathfrak{p}}(x)$ and the matrix $B(x)$ will be holomorphic in S and will have asymptotic expansions

$$f_{\mathfrak{p}}(x) \cong \sum_{k=0}^{\infty} f_{\mathfrak{p}k}x^{-k}, \qquad B(x) \cong \sum_{k=0}^{\infty} B_k x^{-k}$$

which are valid as x tends to infinity in S. Moreover $B(x)$ will have the form

$$B(x) = \operatorname{diag}[B_1(x),\dots, B_s(x)]$$

where the blocks correspond to the distinct eigenvalues of B_0. The matrix B_0 will be in Jordan canonical form with 0's or 1's on the subdiagonal and eigenvalues satisfying

$$0 < |\lambda_\nu| \leqq |\lambda_{\nu-1}| \leqq \cdots \leqq |\lambda_1| < 1 \leqq |\lambda_{\nu+1}| \leqq \cdots \leqq |\lambda_n|. \tag{1.5}$$

Corresponding to the division of eigenvalues of B_0, for any n-dimensional vector v, we will use the notation

$$v^1 = \begin{pmatrix} v_1 \\ \vdots \\ v_\nu \end{pmatrix}, \qquad v^2 = \begin{pmatrix} v_{\nu+1} \\ \vdots \\ v_n \end{pmatrix}.$$

Consequently, we can write $f(x, z) = f(x, z^1, z^2)$. $B^1(x)$ and $B^2(x)$ will denote the $\nu \times \nu$ and $(n-\nu)\times(n-\nu)$ matrices for which $B(x) = \operatorname{diag}(B^1(x), B^2(x))$. Finally, we will let Λ represent the ν-dimensional vector

$$\Lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\nu \end{pmatrix}.$$

We now state our first theorem.

THEOREM 1. *Consider the system* (1.4) *where* $B(x)$ *and* $f(x, z)$ *have the properties described above. Assume* θ *satisfies*

(i) $|\theta| < \pi/2$,

(ii) $\theta \neq \arg(-\log \lambda_j/[\Lambda]^{\mathfrak{p}}), j = 1, \ldots, n, |\mathfrak{p}| \geqq 2$.

Let S_0 *and* U_0 *be the domains*

$$S_0\colon |\arg(xe^{-i\theta} - a_0)| < \pi/2 + \rho_0, \qquad U_0\colon \|z^1\| < \delta_0.$$

Then for a_0^{-1}, ρ_0, *and* δ_0 *sufficiently small, there exists an n-dimensional vector,* $P(x, z^1)$, *holomorphic in* $S_0 \times U_0$, *with power series expansion*

$$P(x, z^1) = \sum_{|\mathfrak{p}| \geqq 2} P_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}$$

such that the transformation

$$u(x) = z(x) - P(x, z^1(x)) \tag{1.6}$$

reduces the difference equation (1.4) *to the canonical form*

$$u(x+1) = B(x)u(x) + g(x, u(x)) \tag{1.7}$$

where

$$g(x, u) = \sum_{|\mathfrak{p}| \geqq 2} g_{\mathfrak{p}}(x)[u]^{\mathfrak{p}}.$$

If we write $g(x, u) = g(x, u^1, u^2)$ *and set* $g(x, u^1, 0) = h(x, u^1)$, *then the components of* $h(x, u^1)$ *will be polynomials in the components of* u^1 *of the form*

$$h_j(x, u^1) = \sum_{\lambda_j = [\Lambda]^{\mathfrak{p}}} h_{j\mathfrak{p}}(x)[u^1]^{\mathfrak{p}} \tag{1.8}$$

for all $j = 1, \ldots, n$, $|\mathfrak{p}| \geqq 2$. *Finally the coefficients* $P_{\mathfrak{p}}(x)$, $g_{\mathfrak{p}}(x)$, *and* $h_{\mathfrak{p}}(x)$ *will be holomorphic and have asymptotic expansions as x approaches infinity through* S_0.

From the ordering of the eigenvalues, (1.5), it is seen that $h_j(x, u^1)=0$ for $j \geqq \nu+1$ and $h_j(x, u^1)$ is a polynomial in $u_1, \ldots, u_{j-1}$ for $j \leqq \nu$. Hence on the manifold, $u^2=0$, the difference equation (1.7) becomes

$$u^1(x+1) = B^1(x)u^1(x)+h^1(x, u^1(x))$$

which can be solved recursively as linear difference equations to yield solutions which tend exponentially to zero as x approaches infinity through a subsector of S_0. Hence $u^2=0$ is the stable manifold of the system (1.7). Setting $u^2=0$ in the transformation (1.6), we obtain $z^2=P^2(x, z^1)$ which is the explicit representation of the stable manifold in z-space.

Condition (i) in Theorem 1 is a natural restriction since we are concerned with the stable manifold associated with those eigenvalues having modulus less than one.

In the power series expansion of $P(x, z^1)$, the coefficients $P_{\mathfrak{p}}(x)$ have asymptotic expansions in powers of x^{-1}. This does not imply that $P(x, z^1)$ has a uniform asymptotic expansion in terms of x^{-1}; see, for example, Y. Sibuya [7], or W. Wasow [8]. However we may obtain such a representation by strengthening the hypotheses of Theorem 1.

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that* $f(x, z)$ *has a uniform asymptotic expansion*

$$f(x, z) \cong \sum_{k=0}^{\infty} f_k(z)x^{-k}$$

for $\|z\| < \delta$ *as* x *tends to infinity in* S. *Then the functions* $P(x, z^1)$, $g(x, u)$, *and* $h(x, u^1)$ *will have uniform asymptotic expansions*

$$P(x, z^1) \cong \sum_{k=0}^{\infty} P_k(z^1)x^{-k}, \qquad g(x, u) \cong \sum_{k=0}^{\infty} g_k(u)x^{-k}, \qquad h(x, u^1) \cong \sum_{k=0}^{\infty} h_k(u^1)x^{-k},$$

as x *tends to infinity in* S_0. *Moreover, the holomorphic coefficients* $P_k(z^1)$, $g_k(u)$, *and* $h_k(u^1)$ *will have power series expansions beginning with quadratic terms and, in particular, the components of the* $h_k(u^1)$ *will be polynomials in the components of* u^1 *of the form*

$$h_{jk}(u^1) = \sum_{\lambda_j=[\Lambda]^{\mathfrak{p}}} h_{jk\mathfrak{p}}[u^1]^{\mathfrak{p}}, \qquad j = 1, \ldots, n, \tag{1.9}$$

for all k.

When $\nu=n$, results of the same general nature as Theorems 1 and 2 have been obtained by Harris and Sibuya [5] in half-planes of the form $|\mathrm{Im}\, x| > \mathcal{N}$ under the assumptions

(i) $0<|\lambda_j|<1, \quad j=1, \ldots, n,$

(ii) $\prod_{j=1}^{n} |\lambda_j|^{\mathfrak{p}_j} \neq |\lambda_i|, \quad i=1, \ldots, n, \ |\mathfrak{p}| \geqq 2.$

W. A. Harris, Jr. [2] has given a proof of Theorem 1 when the matrix $B^1(x)$ is diagonal and $[\Lambda]^{\mathfrak{p}} \neq \lambda_j, j=1, \ldots, n, |\mathfrak{p}| \geqq 2$. In this case $h(x, u^1)=0$. L. J. Grimm and W. A. Harris, Jr., [1], have treated the problem considered in Theorem 1 using a transformation of the form $z(x)=u(x)+Q(x, u(x))$ under assumption (i) above. This transformation reduces (1.4) to canonical form and is the inverse of (1.6) when $\nu=n$.

2. **Lemmas on nonhomogeneous difference equations.** The following lemmas will be instrumental in the proof of Theorems 1 and 2. Lemma 1 is a specialization of the results of Harris and Sibuya [6]. For a proof of Lemma 2, see Harris and Sibuya [5] or Grimm and Harris [1].

Consider the system of linear nonhomogeneous difference equations

$$E(x)y(x+1) = D(x)y(x)+b(x) \tag{2.1}$$

where $y(x)$ and $b(x)$ are m-dimensional vectors and $E(x)$ and $D(x)$ are $m \times m$ matrices. Assume $E(x)$, $D(x)$, and $b(x)$ are holomorphic in the sector R,

$$R\colon |\arg (xe^{-i\theta}-c)| < \pi/2+\gamma,$$

and that

$$E(x) \cong \sum_{k=0}^{\infty} E_k x^{-k}, \qquad D(x) \cong \sum_{k=0}^{\infty} D_k x^{-k}, \qquad b(x) \cong \sum_{k=0}^{\infty} b_k x^{-k},$$

as x tends to infinity in R. Then we have

LEMMA 1. *Assume $E^{-1}(x)$ exists in R and let the eigenvalues of $E_0^{-1}D_0$, u_j, satisfy*

(i) $u_j \neq 0, 1, \quad j=1, \ldots, n,$

(ii) $\theta \neq \arg(-\log u_j), \quad j=1, \ldots, n.$

Then there exists a unique formal solution of (2.1) *in powers of x^{-1}, and in the sector R_1,*

$$R_1\colon |\arg (xe^{-i\theta}-c_1)| < \pi/2+\gamma_1$$

for c_1^{-1}, γ_1, sufficiently small, there exists a solution, $y(x)$, of (2.1) *which is holomorphic in R_1 and asymptotic to the formal solution as x tends to infinity in R_1.*

LEMMA 2. *Assume that $D^{-1}(x)$ exists and that*

$$\|D^{-1}(x)E(x)\| \leqq r < 1$$

for $x \in R$. Further assume that $x \in R$ implies $x+1 \in R$, i.e.

$$0 < \gamma < \pi/2, \qquad -\pi/2 < \theta-\gamma < \theta+\gamma < \pi/2.$$

Then there exists a unique bounded solution, $y(x)$, of the system (2.1) *which is holomorphic in R and admits an asymptotic expansion in powers of x^{-1} as x tends to infinity through R. Moreover, for any sector $R_d \subseteq R$,*

$$R_d\colon |\arg (xe^{i\theta}-d)| < \pi/2+\gamma, \qquad d \geqq c,$$

there is a constant C, independent of d, such that

$$\|y(x)\| \leqq C \sup_{x\in R_d} \|b(x)\|$$

for all $x \in R_d$.

3. **Formal transformation.** If a transformation of the form (1.6) exists and satisfies the conditions of Theorem 1, then $P(x, z^1)$ must satisfy

$$P(x+1, z^1(x+1)) = B(x)P(x, z^1(x))+f(x, z^1(x), z^2(x))-g(x, u^1(x), u^2(x)). \tag{3.1}$$

Upon setting $u^2(x)\equiv 0$ in the transformation (1.6), we obtain

$$u^1(x) = z^1(x)-P^1(x, z^1(x))$$

and

$$0 = z^2(x)-P^2(x, z^1(x)).$$

Further,

$$z^1(x+1) = B^1(x)z^1(x)+f^1(x, z^1(x), P^2(x, z^1(x))).$$

Hence, when $u^2(x)\equiv 0$, equation (3.1) can be written in the form

$$\begin{aligned} P(x+1, B^1(x)z^1+f^1(x, z^1, P^2(x, z^1))) = {} & B(x)P(x, z^1)+f(x, z^1, P^2(x, z^1)) \\ & -h(x, z^1-P^1(x, z^1)) \end{aligned} \tag{3.2}$$

where the argument of z^1 has been suppressed. It is sufficient to construct $P(x, z^1)$ to satisfy the functional equation (3.2) provided $h(x, u^1)$ is chosen properly.

Our first step is to construct formal series of the form

$$P(x, z^1) = \sum_{|\mathfrak{p}|\geqq 2} P_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}, \qquad h(x, z^1) = \sum_{|\mathfrak{p}|\geqq 2} h_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}},$$

which formally satisfy equation (3.2). Assuming that $P(x, z^1)$ and $h(x, z^1)$ have the above form, we may construct formal series

$$\sum_{|\mathfrak{p}|\geqq 2} \hat{f}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = f(x, z^1, P^2(x, z^1)),$$

$$\sum_{|\mathfrak{p}|\geqq 2} H_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = h(x, z^1-P^1(x, z^1))-h(x, z^1),$$

$$\sum_{|\mathfrak{p}|\geqq 3} q_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = P(x+1, B^1(x)z^1+f(x, z^1, P^2(x, z^1)))-P(x+1, B^1(x)z^1),$$

where $\hat{f}_{\mathfrak{p}}(x)$, $H_{\mathfrak{p}}(x)$, and $q_{\mathfrak{p}}(x)$ are n-dimensional vectors whose components are polynomials in the components of the coefficients $P_{\mathfrak{q}}(x)$ and $h_{\mathfrak{q}}(x)$ for $|\mathfrak{q}|<|\mathfrak{p}|$.

As previously indicated, $\mathfrak{p}$ represents a ν-tuple of positive integers, $(\mathfrak{p}_1, \ldots, \mathfrak{p}_\nu)$, with $|\mathfrak{p}|=\mathfrak{p}_1+\cdots+\mathfrak{p}_\nu$. For each positive integer m there are

$$\nu_m = (\nu+m-1)!/m!(\nu-1)!$$

distinct ν-tuples, $\mathfrak{p}$, with $|\mathfrak{p}|=m$. In order to distinguish between them we order them in the following way. Let $\mathfrak{p}=(\mathfrak{p}_1, \ldots, \mathfrak{p}_\nu)$ and $\mathfrak{q}=(\mathfrak{q}_1, \ldots, \mathfrak{q}_\nu)$ with $|\mathfrak{p}|=|\mathfrak{q}|$.

Then we say $\mathfrak{p}=\mathfrak{q}$ if all of the components of $\mathfrak{p}-\mathfrak{q}=(\mathfrak{p}_1-\mathfrak{q}_1,\ldots,\mathfrak{p}_\nu-\mathfrak{q}_\nu)$ are zero and $\mathfrak{p}>\mathfrak{q}$ if the first nonzero component of $\mathfrak{p}-\mathfrak{q}$ is positive. We denote those $\mathfrak{p}$ with $|\mathfrak{p}|=m$ in increasing order by $\mathfrak{p}^j, j=1,\ldots,\nu_m$.

We can now formally write equation (3.2) in the form:

$$\begin{aligned}\sum_{m=2}^{\infty}\sum_{j=1}^{\nu_m} P_{\mathfrak{p}^j}(x+1)[B^1(x)z^1]^{\mathfrak{p}^j} &= B(x)\sum_{m=2}^{\infty}\sum_{j=1}^{\nu_m} P_{\mathfrak{p}^j}(x)[z^1]^{\mathfrak{p}^j}\\ &\quad -\sum_{m=2}^{\infty}\sum_{j=1}^{\nu_m} h_{\mathfrak{p}^j}(x)[z^1]^{\mathfrak{p}^j}+\sum_{m=2}^{\infty}\sum_{j=1}^{\nu_m} R_{\mathfrak{p}^j}(x)[z^1]^{\mathfrak{p}^j},\end{aligned} \tag{3.3}$$

where $R_{\mathfrak{p}^j}(x)=\hat{f}_{\mathfrak{p}^j}(x)-H_{\mathfrak{p}^j}(x)-q_{\mathfrak{p}^j}(x)$.

It follows from the properties of $B^1(x)$ that

$$[B^1(x)z^1]^{\mathfrak{p}^j} = \sum_{l=1}^{\nu_m}(a_{lj}^m+c_{lj}^m(x))[z^1]^{\mathfrak{p}^l}$$

where $|\mathfrak{p}^j|=|\mathfrak{p}^l|=m$, $c_{lj}^m(x)=O(|x|^{-1})$ for $x\in S$, $a_{lj}^m=0$ for $l<j$, $a_{jj}^m=[\Lambda]^{\mathfrak{p}^j}$, and $a_{lj}^m+c_{lj}^m(x)\equiv 0$ for $[\Lambda]^{\mathfrak{p}^l}\neq[\Lambda]^{\mathfrak{p}^j}$. Substituting this form into equation (3.3) and identifying terms gives the system of linear nonhomogeneous difference equations

$$\sum_{j=1}^{\nu_m}(a_{lj}^m+c_{lj}^m(x))P_{\mathfrak{p}^j}(x+1) = B(x)P_{\mathfrak{p}^l}(x)-h_{\mathfrak{p}^l}(x)+R_{\mathfrak{p}^l}(x) \tag{3.4}$$

for each $l=1,\ldots,\nu_m$ and each $m\geqq 2$.

We define the following $(n\cdot\nu_m)$-dimensional vectors

$$\mathfrak{P}_m(x)=\begin{pmatrix}P_{\mathfrak{p}^1}(x)\\ \vdots\\ P_{\mathfrak{p}^{\nu_m}}(x)\end{pmatrix},\qquad \mathfrak{H}_m(x)=\begin{pmatrix}h_{\mathfrak{p}^1}(x)\\ \vdots\\ h_{\mathfrak{p}^{\nu_m}}(x)\end{pmatrix},\qquad \mathfrak{R}_m(x)=\begin{pmatrix}R_{\mathfrak{p}^1}(x)\\ \vdots\\ R_{\mathfrak{p}^{\nu_m}}(x)\end{pmatrix}$$

where $|\mathfrak{p}^l|=m$, $l=1,\ldots,\nu_m$. We also define $\mathfrak{B}_m(x)$ to be the $(n\cdot\nu_m)\times(n\cdot\nu_m)$ block diagonal matrix with diagonal blocks $B(x)$ and $\mathfrak{A}_m(x)$ to be the $(n\cdot\nu_m)\times(n\cdot\nu_m)$ matrix composed of $n\times n$ component blocks $A_{lj}^m(x)$, $1\leqq l,j\leqq\nu_m$, where

$$A_{lj}^m(x) = (a_{lj}^m+c_{lj}^m(x))I_n.$$

With this notation the equations (3.4) with $|\mathfrak{p}^l|=m$ can be combined into the $(n\cdot\nu_m)$-dimensional system

$$\mathfrak{A}_m(x)\mathfrak{P}_m(x+1) = \mathfrak{B}_m(x)\mathfrak{P}_m(x)-\mathfrak{H}_m(x)+\mathfrak{R}_m(x) \tag{3.5}$$

for each $m\geqq 2$. We will solve the systems (3.5) recursively to determine functions $\mathfrak{P}_m(x)$ and $\mathfrak{H}_m(x)$ holomorphic in a sector independent of m.

We begin by supposing that for all $t<m$, $\mathfrak{P}_t(x)$ and $\mathfrak{H}_t(x)$ have been determined as holomorphic solutions to (3.5) for $x\in\hat{S}_m\subset S$, where

$$\hat{S}_m\colon |\arg(xe^{-i\theta}-\hat{a}_m)| < \pi/2+\hat{\rho}_m$$

with $\hat{a}_m \geqq a$ and $\hat{\rho}_m \leqq \rho$. The constant $\hat{a}_m$ is assumed to be sufficiently large to insure that $\mathfrak{A}_m^{-1}(x)$ exists for $x \in \hat{S}_m$. Now we apply Lemma 1 to the system (3.5). The eigenvalues are $\lambda_j/[\Lambda]^{\mathfrak{p}^l}$, where $|\mathfrak{p}^l| = m, j = 1, \ldots, n, l = 1, \ldots, \nu_m$. The solution is separated into two cases.

Case I. $\lambda_j \neq [\Lambda]^{\mathfrak{p}^l}$ for $j = 1, \ldots, n, l = 1, \ldots, \nu_m$. Since $\lambda_j/[\Lambda]^{\mathfrak{p}^l} \neq 1$, we can set $\mathfrak{H}_m(x) \equiv 0$ and apply Lemma 1 directly to obtain a solution, $\mathfrak{P}_m(x)$.

Case II. $\lambda_j = [\Lambda]^{\mathfrak{p}^l}$ for some $j = 1, \ldots, n, l = 1, \ldots, \nu_m$. In this case at least one of the eigenvalues of the system is equal to one and Lemma 1 cannot be applied directly. We must make a nontrivial choice of the components of $\mathfrak{H}_m(x)$ and reduce the system to one of the form solved by Case I. Each component of equation (3.5) has the form

$$\sum_{k=1}^{\nu_m} (a_{lk}^m + c_{lk}^m(x)) P_{\mathfrak{p}^k j}(x+1) = \sum_{i=1}^{n} b_{ji}(x) P_{\mathfrak{p}^l i}(x) - h_{\mathfrak{p}^l j}(x) + R_{\mathfrak{p}^l j}(x), \tag{3.6}$$

where the $b_{ji}(x)$ are the elements of $B(x)$ and $1 \leqq j \leqq n$. We choose the components of $\mathfrak{H}_m(x)$ in the following manner

$$\begin{aligned} h_{\mathfrak{p}^l j}(x) &= R_{\mathfrak{p}^l j}(x) \qquad && \text{if } \lambda_j = [\Lambda]^{\mathfrak{p}^l}, \\ &= 0 && \text{if } \lambda_j \neq [\Lambda]^{\mathfrak{p}^l}, \end{aligned}$$

and partially determine the $\mathfrak{P}_m(x)$ by

$$P_{\mathfrak{p}^l j}(x) \equiv 0, \qquad \text{if } \lambda_j = [\Lambda]^{\mathfrak{p}^l}.$$

From the properties of $B(x)$ and $a_{lk}^m + c_{lk}^m(x)$ it is seen that when $\lambda_j = [\Lambda]^{\mathfrak{p}^l}$ both sides of equation (3.6) vanish identically. Hence we may remove these equations from the system (3.5) to yield a smaller system

$$\hat{\mathfrak{A}}_m(x)\hat{\mathfrak{P}}_m(x+1) = \hat{\mathfrak{B}}_m(x)\hat{\mathfrak{P}}_m(x) + \hat{\mathfrak{R}}_m(x)$$

which can be solved by direct application of Lemma 1 as in Case I.

In either Case I or Case II, we obtain a solution, $\mathfrak{P}_m(x)$, which is holomorphic and has an asymptotic expansion in a sector $S_m \subset \hat{S}$ where

$$S_m\colon |\arg(xe^{-i\theta} - a_m)| \leqq \pi/2 + \rho_m$$

with $a_m \geqq \hat{a}_m$ and $\rho_m \leqq \hat{\rho}_m$. Moreover, since $\mathfrak{P}_m(x)$ and $\mathfrak{R}_m(x)$ are bounded in S_m, there exists a positive constant C_m such that

$$\|\mathfrak{P}_m(x)\| \leqq C_m \sup_{x \in S_m} \|\mathfrak{R}_m(x) - \mathfrak{H}_m(x)\|$$

for all $x \in S_m$.

From the structure of $\mathfrak{B}_m(x)$ we have that $\|\mathfrak{B}_m^{-1}(x)\| = \|B^{-1}(x)\|$ for all m and $|x|$ sufficiently large. Also for $|x|$ large,

$$\|B(x)\| \leqq \sigma < 1. \tag{3.7}$$

Let $S_1\colon |\arg(xe^{-i\theta} - a_1)| \leqq \pi/2 + \rho_1$ be the sector where $B^{-1}(x)$ exists and inequality

(3.7) holds. Grimm and Harris [1] have shown for all $x \in S_1$, $\|\mathfrak{A}_m(x)\| \leqq n(m+1)^n \sigma^m$. Let $K = \sup_{x \in S_1} \|B^{-1}(x)\|$ and M_0 be a positive integer such that $Kn(m+1)^n\sigma^m < 1$ for all $m \geqq M_0$. Then

$$(3.8) \qquad \|\mathfrak{B}_m^{-1}(x)\mathfrak{A}_m(x))\| \leqq r < 1$$

for all $m \geqq M_0$ and all $x \in S_1$. Now let S_0 be the sector

$$S_0 \colon |\arg(xe^{-i\theta} - a_0)| < \pi/2 + \rho_0$$

where a_0^{-1} and ρ_0 are chosen so small that $S_0 \subset S_1$, $S_0 \subset S_{M_0}$, and condition (2.2) in Lemma 2 is satisfied. Inequality (3.8) allows us to solve equations (3.5) recursively for all $m \geqq M_0$ by means of Lemma 2. For $m \geqq M_0$, we choose $\mathfrak{H}_m(x) \equiv 0$. The solutions will be holomorphic and have asymptotic expansions in S_0 and also will satisfy the inequalities

$$\|\mathfrak{P}_m(x)\| \leqq C_0 \sup_{x \in S_0} \|\mathfrak{R}_m(x) - \mathfrak{H}_m(x)\|$$

for $x \in S_0$, $m \geqq M_0$. The constant C_0 is independent of m.

We have constructed a formal series

$$(3.9) \qquad \sum_{|\mathfrak{p}| \geqq 2} P_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}$$

and a polynomial

$$(3.10) \qquad \sum_{|\mathfrak{p}| \geqq 2}^{M_0} h_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}$$

which formally satisfy equation (3.2). The coefficients $P_{\mathfrak{p}}(x)$ and $h_{\mathfrak{p}}(x)$ are holomorphic in the sector S_0 and have asymptotic expansions

$$P_{\mathfrak{p}}(x) \cong \sum_{k=0}^{\infty} P_{\mathfrak{p}k} x^{-k} \qquad h_{\mathfrak{p}}(x) \cong \sum_{h=0}^{\infty} h_{\mathfrak{p}k} x^{-k}$$

as x tends to infinity through S_0. By our construction the components of $h_{\mathfrak{p}}(x)$ have the form (1.8). Finally we have the fundamental inequality

$$(3.11) \qquad \|\mathfrak{P}_m(x)\| \leqq C \sup_{x \in S_0} \|\mathfrak{R}_m(x) - \mathfrak{H}_m(x)\|$$

for all $x \in S_0$ and each m with C independent of m. This inequality will be used to prove that the formal series $\sum_{|\mathfrak{p}| \geqq 2} P_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}$ actually converges for $\|z^1\|$ sufficiently small and hence represents a holomorphic function in x and z^1.

4. **Construction of a majorant series.** We denote the components of the series (3.9) and (3.10) by

$$P_j(x, z^1) = \sum_{|\mathfrak{p}| \geqq 2} P_{\mathfrak{p}j}(x)[z^1]^{\mathfrak{p}}, \qquad h_j(x, z^1) = \sum_{|\mathfrak{p}| \geqq 2} h_{\mathfrak{p}j}(x)[z^1]^{\mathfrak{p}}$$

for $j = 1, \ldots, n$. We construct the formal scalar series

$$h_j(x, z^1 - P^1(x, z^1)) = \sum_{|\mathfrak{p}| \geqq 2} \hat{h}_{\mathfrak{p}j}(x)[z^1]^{\mathfrak{p}}, \qquad j = 1, \ldots, n.$$

Letting $\hat{f}_{\mathfrak{p}j}(x)$ and $q_{\mathfrak{p}j}(x)$ be the components of $\hat{f}_{\mathfrak{p}}(x)$ and $q_{\mathfrak{p}}(x)$ respectively for $j=1,\ldots,n$, the inequality (3.11) can be written in the form

$$\sum_{|\mathfrak{p}|=m}\left\{\sum_{j=1}^{n}|P_{\mathfrak{p}j}(x)|\right\} \leqq C \sup_{x\in S_0}\sum_{|\mathfrak{p}|=m}\left\{\sum_{j=1}^{n}(|\hat{f}_{\mathfrak{p}j}(x)|+|\hat{h}_{\mathfrak{p}j}(x)|+|q_{\mathfrak{p}j}(x)|)\right\}. \tag{4.1}$$

Because the coefficients of $f(x, z^1, z^2)$ and $h(x, z^1)$ and the elements of $B^1(x)$ are holomorphic and bounded for $x \in S_0$, we can construct holomorphic functions $f^*(z^1, z^2)$ and $h^*(z^1)$ and constants b^*_{ij}, $i, j=1,\ldots,n$, such that for all $x \in S_0$

$$\begin{aligned} f_j(x, z^1, z^2) &\ll f^*(z^1, z^2), \qquad j = 1,\ldots,n,\\ h_j(x, z^1) &\ll h^*(z^1), \qquad j = 1,\ldots,n, \end{aligned}$$

and $|b_{ij}(x)| \leqq b^*_{ij}$ for each element, $b_{ij}(x)$, of $B^1(x)$. We let B^* be the matrix with elements b^*_{ij} and $P^*(x, z^1)$ be the formal scalar series

$$P^*(x, z^1) = \sum_{|\mathfrak{p}|\geqq 2}\left(\sum_{j=1}^{n}|P_{\mathfrak{p}j}(x)|\right)[z^1]^{\mathfrak{p}}.$$

Clearly, for each j, we have

$$P_j(x, z^1) \ll P^*(x, z^1). \tag{4.2}$$

Let $\tilde{\mu}=(\mu,\ldots,\mu)$ be the n-dimensional vector with all components identical and $\tilde{\mu}^1$ and $\tilde{\mu}^2$ the corresponding ν- and $(n-\nu)$-dimensional vectors. The following formal scalar series can now be defined:

$$\sum_{|\mathfrak{p}|\geqq 2}\hat{f}^*_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = f^*(z^1, \tilde{P}^{*2}(x, z^1)),$$

$$\sum_{|\mathfrak{p}|\geqq 2}\hat{h}^*_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = h^*(z^1+\tilde{P}^{*1}(x, z^1)),$$

$$\hat{C}\sum_{|\mathfrak{p}|\geqq 3}q^*_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} = P^*(x+1, B^*z^1+\hat{C}\tilde{f}^{*1}(z^1, \tilde{P}^{*2}(x, z^1)))-P^*(x+1, B^*z^1).$$

Assuming that $\hat{C}\geqq 1$ it can be shown by induction that for all $x \in S_0$

$$\sum_{j=1}^{n}|f_{\mathfrak{p}j}(x)| \leqq n\hat{f}^*_{\mathfrak{p}}(x), \quad \sum_{j=1}^{n}|\hat{h}_{\mathfrak{p}j}(x)| \leqq n\hat{h}^*_{\mathfrak{p}}(x), \quad \sum_{j=1}^{n}|q_{\mathfrak{p}j}(x)| \leqq q^*_{\mathfrak{p}}(x)$$

for each p. Since the constant C in inequality (3.11) can be chosen larger than one, we have, on setting $C=\hat{C}$,

$$\sum_{|\mathfrak{p}|=m}\left(\sum_{j=1}^{n}|P_{\mathfrak{p}j}(x)|\right) \leqq C \sup_{x\in S_0}\sum_{|\mathfrak{p}|=m}(n\hat{f}^*_{\mathfrak{p}}(x)+n\hat{h}^*_{\mathfrak{p}}(x)+q^*_{\mathfrak{p}}(x)) \tag{4.3}$$

for every m.

Next we assume that all of the components of z^1 are identical, i.e. $z_1 = z_2 = \cdots = z_\nu = s$, and we consider the following scalar series:

$$\hat{P}(x, s) = \sum_{k=2}^{\infty} \hat{P}_k(x)s^k = P^*(x, \tilde{s}^1),$$

$$F^*(x, s) = \sum_{k=2}^{\infty} F_k^*(x)s^k = f^*(\tilde{s}^1, \tilde{P}^{*2}(x, \tilde{s}^1)),$$

$$H^*(x, s) = \sum_{k=2}^{\infty} H_k^*(x)s^k = h^*(\tilde{s}^1 + \tilde{P}^{*1}(x, \tilde{s}^1)),$$

$$Q(x, s) = C\sum_{k=3}^{\infty} Q_k(x)s^k = \hat{P}(x+1, \sigma s + CF^*(x, s)) - \hat{P}(x+1, \sigma s),$$

where $\sigma = \max_{1 \leqq i \leqq \nu} (\sum_{j=1}^{\nu} b_{ij}^*)$. Formally, we have

$$\hat{P}_k(x) = \sum_{|\mathfrak{p}|=k} \left(\sum_{j=1}^{n} |P_{\mathfrak{p}j}(x)| \right), \quad F_k^*(x) = \sum_{|\mathfrak{p}|=k} \hat{f}_{\mathfrak{p}}^*(x), \quad H_k^*(x) = \sum_{|\mathfrak{p}|=k} \hat{h}_{\mathfrak{p}}^*(x).$$

Further, it follows that $\sum_{|\mathfrak{p}|=k} q_{\mathfrak{p}}^*(x) \leqq Q_k(x)$ for all k. Hence, using inequality (4.3), we have

(4.4) $$\hat{P}_k(x) \leqq C \sup_{x \in S_0} [nF_k^*(x) + nH_k^*(x) + Q_k(x)]$$

for every k and all $x \in S_0$.

We consider the scalar functional equation

(4.5) $$T(s) = T(\sigma s + Cf^*(\tilde{s}^1, \tilde{T}^2(s))) + Cnf^*(\tilde{s}^1, \tilde{T}^2(s)) + Cnh^*(\tilde{s}^1 + \tilde{T}^1(s))$$

where s is a complex variable. This equation has a formal power series solution

(4.6) $$T(s) = \sum_{k=2}^{\infty} T_k s^k$$

satisfying

(4.7) $$\sum_{k=2}^{\infty} T_k s^k = \sum_{k=2}^{\infty} T_k(\sigma s)^k + C \sum_{k=2}^{\infty} R_k^* s^k$$

where $R_k^* \geqq nF_k^*(x) + nH_k^*(x) + Q_k(x)$ for $x \in S_0$ if $T_{k'} \geqq \hat{P}_{k'}(x)$ for all $k' < k$. Because of the structure of $B^1(x)$ we may assume without loss of generality that $\sigma < 1$. Using this assumption and inequality (4.4), it is easily shown that

(4.8) $$T_k \geqq CR_k^* \geqq \hat{P}_k(x) = \sum_{|\mathfrak{p}|=k} \left(\sum_{j=1}^{n} |P_{\mathfrak{p}j}(x)| \right)$$

for all $x \in S_0$.

If it can be shown that the formal series (4.6) is convergent for $|s| < \delta_0$, then from inequalities (4.8) and (4.2) it follows that the series $\sum_{|\mathfrak{p}|\geqq 2} P_{\mathfrak{p}j}(x)[z^1]^{\mathfrak{p}}$ will converge absolutely and uniformly for $x \in S_0$ and $\|z^1\| < \delta_0$.

5. **Convergence of the majorant series.** The coefficients of the formal series (4.6) are uniquely determined positive constants, so it is sufficient to prove that equation (4.5) has an analytic solution such that $T(0)=0$. To prove that this is the case we will apply the Schauder Fixed Point Theorem.

In the following we write $f_*(u, v)=f^*(\tilde{u}^1, \tilde{v}^2)$ and $h_*(u)=h^*(\tilde{u}^1)$. By definition $f_*(u, v)$ and $h_*(u)$ are analytic for $|u| < \delta'$, $|v| < \delta''$ for δ', δ'' sufficiently small. Moreover, there exist constants K and L such that

$$(5.1)\qquad |f_*(u, v)| \leqq K(|u|+|v|)^2,$$

$$(5.2)\qquad |h_*(u)| \leqq L|u|^2$$

for $|u| < \delta'$, $|v| < \delta''$.

Let $\mathfrak{F}$ be the family of functions $\{\Phi(s)\}$ which are analytic and satisfy $|\Phi(s)| \leqq M|s|$ for $|s| < \delta_0$. Here M is an arbitrary but fixed constant and δ_0 is chosen to satisfy the inequalities

$$(5.3)\qquad \begin{gathered} \delta_0 < \delta'; \delta_0 < \delta''/M; \qquad \sigma + CK(1+M)^2\,\delta_0 < 1; \\ \sigma + C(1+M)^2(K+MK+ML)\,\delta_0 \leqq 1. \end{gathered}$$

$\mathfrak{F}$ is convex and compact with respect to the topology of uniform convergence on each compact subset of the region $|s| < \delta_0$. We define the continuous mapping $\mathfrak{G}$ on $\mathfrak{F}$ by

$$\mathfrak{G}[\Phi](s) = \Phi(\sigma s + Cf_*(s, \Phi(s))) + Cnf_*(s, \Phi(s)) + Cnh_*(s+\Phi(s)).$$

The inequalities (5.1), (5.2), and (5.3) show that $\mathfrak{G}[\Phi](s)$ is analytic for $|s| < \delta_0$ and satisfies $|\mathfrak{G}[\Phi](s)| \leqq M|s|$. Hence $\mathfrak{G}$ maps $\mathfrak{F}$ into itself. An application of the Schauder Fixed Point Theorem then guarantees the existence of a fixed point of the mapping $\mathfrak{G}$ which is the desired analytic solution of equation (4.5).

6. **Formal solution, Theorem 2.** Our first step in proving Theorem 2 is to construct formal series

$$(6.1)\qquad P(x, z^1) = \sum_{k=0}^{\infty} P_k(z^1)x^{-k}, \qquad h(x, z^1) = \sum_{k=0}^{\infty} h_k(z^1)x^{-k}$$

such that if these series, along with the asymptotic representations of $B(x)$ and $f(x, z^1, z^2)$, are substituted into equation (3.2), then the equations obtained by equating coefficients of like powers of x^{-1} will be satisfied. In addition, we shall show that there exists a constant $\delta_0 > 0$, independent of k, such that for $\|z^1\| < \delta_0$, the coefficients $P_k(z^1)$ and $h_k(z^1)$ will be analytic and $O(\|z^1\|^2)$. Moreover, the components of $h_k(z^1)$ will be shown to be polynomials of the form (1.9).

The coefficients $P_k(z^1)$ and $h_k(z^1)$ will be determined recursively. Letting x tend to infinity formally in equation (3.2) gives the nonlinear functional equation

$$P_0(B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))) = B_0 P_0(z^1) + f_0(z^1, P_0^2(z^1)) - h_0(z^1 - P_0^1(z^1)). \tag{6.2}$$

Equation (6.2) is of the same general form as equation (3.2), but it is an algebraic equation rather than a difference equation. Techniques similar to those used to solve equation (3.2) can be used to determine an analytic function $P_0(z_1)$ and a polynomial $h_0(z^1)$ which satisfy equation (6.1) in a region $\|z^1\| < \delta_0$. Further, $P_0(z^1)$ and $h_0(z^1)$ will be $O(\|z^1\|^2)$ and $h_0(z^1)$ will have the desired form. The details are omitted.

In determining the remaining coefficients we first assume that δ_0 is so small that for $\|z^1\| < \delta_0$ the coefficients $f_k(z^1, P_0^2(z^1))$ are analytic and

$$\|B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))\| < \varepsilon \|z^1\| \tag{6.3}$$

where $\varepsilon < 1$. Putting the formal expansions in equation (3.2), yields

$$\begin{aligned} \sum_{k=0}^{\infty} & P_k(B^1(x) z^1 + f^1(x, z^1, P^2(x, z^1)))(x+1)^{-k} \\ &= \left(\sum_{k=0}^{\infty} B_k x^{-k}\right)\left(\sum_{k=0}^{\infty} P_k(z^1) x^{-k}\right) \\ &\quad + \sum_{h=0}^{\infty} f_k(z^1, P^2(x, z^1)) x^{-k} - \sum_{k=0}^{\infty} h_k(z^1 - P^1(x, z^1)) x^{-k}. \end{aligned} \tag{6.4}$$

Assuming that the $P_k(z^1)$ and $h_k(z^1)$ are analytic for $\|z^1\| < \delta_0$, expanding and identifying coefficients in equation (6.4) gives equations of the form

$$P_k(B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))) = C(z^1) P_k(z^1) + R_k(z^1) - h_k(z^1) \tag{6.5}$$

for $k \geqq 2$. The matrix $C(z^1) = B_0 + O(\|z^1\|)$ is analytic for $\|z^1\| < \delta_0$ and is independent of k. The components of $R_k(z^1)$ are polynomials in the components of the coefficients $P_s(z^1)$ and $h_s(z^1)$ and their mth order derivatives for $m \leqq s < k$. Moreover, if $P_s(z^1)$ and $h_s(z^1)$ are $O(\|z^1\|^2)$ then $R_k(z^1)$ will be $O(\|z^1\|^2)$.

If the formal series (6.1) are to be formal solutions of equation (3.2) with coefficients having the desired properties, then the coefficients must satisfy the linear functional equations (6.5). On the other hand if analytic functions $P_k(z^1)$ and $h_k(z^1)$ are found to satisfy equation (6.5) for each k, the corresponding formal series will be a formal solution to equation (3.2).

We proceed in a recursive manner to prove the existence of solutions to equation (6.5) which have the required properties for $\|z^1\| < \delta_0$. Assume that for all $k' < k$, solutions to equation (6.5) have been constructed which are holomorphic and $O(\|z^1\|^2)$ for $\|z^1\| < \delta_0$ and such that the $h_{k'}(z^1)$ are polynomials of the form (1.9). In the usual manner, formal series solutions

$$\sum_{|\mathfrak{p}| \geqq 2} P_{k\mathfrak{p}}[z^1]^{\mathfrak{p}}; \quad \sum_{|\mathfrak{p}| \geqq 2} h_{k\mathfrak{p}}[z^1]^{\mathfrak{p}}$$

to equation (6.5) can be constructed, where $h_k(z^1)$ is actually a polynomial in the components of z^1. We set

$$P_k^N(z^1) = \sum_{|\mathfrak{p}|\geqq 2}^{N-1} P_{k\mathfrak{p}}[z^1]^{\mathfrak{p}}$$

and

$$Q_k^N(z^1) = P_k(z^1) - P_k^N(z^1). \tag{6.6}$$

Then $Q_k^N(z^1)$ satisfies the equation

$$C(z^1)Q_k^N(z^1) = Q_k^N(B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))) + I_k^N(z^1), \tag{6.7}$$

where

$$I_k^N(z^1) = P_k^N(B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))) - C(z^1)P_k^N(z^1) - R_k(z^1) + h_k(z^1)$$

is analytic for $\|z^1\| < \delta_0$ and satisfies

$$\|I_k^N(z^1)\| \leqq L_N\|z^1\|^N \tag{6.8}$$

for some positive constant L_N. Since B_0 is nonsingular and $C(z^1)$ is independent of k, we may assume that $C^{-1}(z^1)$ exists for $\|z^1\| < \delta_0$. We choose N sufficiently large so that

$$\varepsilon^N\|C^{-1}(z^1)\| < \tfrac{1}{2} \tag{6.9}$$

for $\|z^1\| < \delta_0$, where ε is defined by inequality (6.3). Let K be any constant satisfying

$$K \geqq 2\|C^{-1}(z^1)\|L_N. \tag{6.10}$$

Let $\mathfrak{F}$ be the family of functions, $\{Q(z^1)\}$, which are analytic and satisfy $\|Q(z^1)\| \leqq K\|z^1\|^N$ for $\|z^1\| < \delta_0$ and define the continuous mapping, $\mathfrak{G}$, on $\mathfrak{F}$ by

$$\mathfrak{G}[Q](z^1) = C^{-1}(z^1)Q(B_0^1 z^1 + f_0^1(z^1, P_0^2(z^1))) + C^{-1}(z^1)I_k^N(z^1).$$

From the inequalities (6.3), (6.8), (6.9), and (6.10) it follows that $\mathfrak{G}$ maps $\mathfrak{F}$ into itself. Hence equation (6.7) has an analytic solution of order $O(\|z^1\|^N)$. Defining $P_k(z^1)$ by equation (6.6) yields the desired solution to equation (6.5).

7. **Application of the Ritt theorem.** We have constructed formal series solutions to equation (3.2) of the form (6.1) in a region $S \times U_0$ where

$$S\colon |\arg(xe^{-i\theta} - a)| < \pi/2 + \rho, \qquad U_0\colon \|z^1\| < \delta_0.$$

In general these series will not converge. However we will show that these formal series are uniform asymptotic representations of holomorphic solutions of equations (3.2) in a subregion of $S \times U_0$.

Let U_1 be the region $\|z^1\| \leqq \delta_1 < \delta_0$. By the Ritt theorem [8], we can construct functions $\bar{P}(x, z^1)$ and $\bar{h}(x, z^1)$ which are holomorphic in $S \times U_1$, $O(\|z^1\|^2)$, and such that

$$\bar{P}(x, z^1) \cong \sum_{k=0}^{\infty} P_k(z^1)x^{-k}, \qquad \bar{h}(x, z^1) \cong \sum_{k=0}^{\infty} h_k(z^1)x^{-k}$$

uniformly for $z^1 \in U_1$ as x tends to infinity through S. Because of the form of the $h_k(z^1)$, $\bar{h}(x, z^1)$ can be assumed to be a polynomial in the components of z^1 of the form (1.8).

We set

$$\hat{P}(x, z^1) = P(x, z^1) - \bar{P}(x, z^1), \qquad \hat{H}(x, z^1) = h(x, z^1) - \bar{h}(x, z^1). \tag{7.1}$$

Then equation (3.2) becomes

$$\begin{aligned} \hat{P}(x+1, B^1(x)z^1 + f^1(x, z^1, \hat{P}^2 + \bar{P}^2)) &= B(x)\hat{P}(x, z^1) \\ &\quad + f(x, z^1, \hat{P} + \bar{P}^2) - \hat{H}(x, z^1 - \hat{P}^1 - \bar{P}^1) + B(x)\bar{P}(x, z^1) \\ &\quad - \bar{P}(x+1, B^1(x)z^1 + f^1(x, z^1, \hat{P}^2 + \bar{P}^2)) - \bar{h}(x, z^1 - \hat{P}^1 - \bar{P}^1). \end{aligned} \tag{7.2}$$

We will show that there are functions $\hat{P}(x, z^1)$ and $\hat{H}(x, z^1)$ satisfying equation (7.2) in a region $S_* \times U_*$ where

$$S_*: |\arg(xe^{-i\theta} - a_*)| < \pi/2 + \rho_*, \qquad U_*: \|z^1\| < \delta_*$$

with $a_* > a$, $\rho > \rho_*$, and $\delta_1 > \delta_*$. Moreover, in $S_* \times U_*$, $\hat{P}(x, z^1)$ and $\hat{H}(x, z^1)$ will have the following properties:

I. $\hat{P}(x, z^1)$ and $\hat{H}(x, z^1)$ will be holomorphic and $O(\|z^1\|^2)$.

II. $\hat{P}(x, z^1)$ and $\hat{H}(x, z^1)$ will be uniformly asymptotic to zero for $z^1 \in U_*$ as x tends to infinity in S_*.

III. The components of $\hat{H}(x, z^1)$ will be polynomials in the components of z^1 of the form (1.8).

Then $P(x, z^1)$ and $h(x, z^1)$ as defined by equations (7.1) are the desired solutions of equation (3.2).

8. **Solutions to equation (8.2).** The construction is similar to that of Theorem 1. We let

$$\begin{aligned} Q(x, z^1, \hat{P}) = f(x, z^1, \hat{P}^2 + \bar{P}^2) + B(x)\bar{P}(x, z^1) - \bar{h}(x, z^1 - \hat{P}^1 - \bar{P}^1) \\ - \bar{P}(x+1, B^1(x)z^1 + f^1(x, z^1, \hat{P}^2 + \bar{P}^2)). \end{aligned}$$

From the properties of $\bar{P}(x, z^1)$, $\bar{h}(x, z^1)$, and $f(x, z^1, z^2)$ it is seen that $Q(x, z^1, \hat{P})$ will be holomorphic in a region $S \times U_2 \times V$ where

$$U_2: \|z^1\| \leqq \delta_2, \qquad V: \|\hat{P}\| \leqq \eta$$

for δ_2 and η sufficiently small. Hence, we may write

$$Q(x, z^1, \hat{P}) = Q_0(x, z^1) + Q_1(x, z^1)\hat{P} + \sum_{|\mathfrak{p}| \geqq 2} Q_{\mathfrak{p}}(x, z^1)[\hat{P}]^{\mathfrak{p}}$$

where the coefficients will be holomorphic in $S \times U_2$. Because $\bar{P}(x, z^1)$ is uniformly asymptotic to a formal solution of equation (3.2), we have

$$Q_0(x, z^1) = Q(x, z^1, 0) \cong 0$$

uniformly for $z^1 \in U_2$ as x tends to infinity in S. Moreover, it follows that

$$Q_0(x, z^1) = O(\|z^1\|^2), \qquad Q_1(x, z^1) = O(\|z^1\|)$$

in $S \times U_2$.

Assuming that $\hat{P}(x, z^1)$ and $\hat{H}(x, z^1)$ have the form

$$\hat{P}(x, z^1) = \sum_{|\mathfrak{p}|\geqq 2} \hat{P}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}, \qquad \hat{H}(x, z^1) = \sum_{|\mathfrak{p}|\geqq 2} \hat{H}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}, \tag{8.1}$$

we can write equation (7.2) as follows:

$$\begin{aligned}\sum_{|\mathfrak{p}|\geqq 2} \hat{P}_{\mathfrak{p}}(x+1)[B^1(x)z^1]^{\mathfrak{p}} = B(x) &\sum_{|\mathfrak{p}|\geqq 2} \hat{P}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} \\ &+ \sum_{|\mathfrak{p}|\geqq 2} Q_{0\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} + \sum_{|\mathfrak{p}|\geqq 2} R_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} \\ &- \sum_{|\mathfrak{p}|\geqq 2} \hat{H}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}},\end{aligned} \tag{8.2}$$

where $\sum_{|\mathfrak{p}|\geqq 2} Q_{0\mathfrak{p}}(x)[z^1]^{\mathfrak{p}}$ is the power series expansion of $Q_0(x, z^1)$ and the components of $R_{\mathfrak{p}}(x)$ are polynomials in the components of $\hat{P}_{\mathfrak{q}}(x)$ and $\hat{H}_{\mathfrak{q}}(x)$ for $|\mathfrak{q}| < |\mathfrak{p}|$ with no constant terms.

Equation (8.2) can now be written in exactly the same form as equation (3.3) and a formal solution, (8.1), can be constructed as before. The coefficients $\hat{P}_{\mathfrak{p}}(x)$ and $\hat{H}_{\mathfrak{p}}(x)$ will be holomorphic in a sector $S_1 \subset S$,

$$S_1: |\arg (xe^{-i\theta} - a_1)| < \pi/2 + \rho_1$$

which is independent of $\mathfrak{p}$. The components of $\hat{H}(x, z^1)$ will have the form (1.8) since their form depends only upon the structure of $B(x)$. Further for each $k \geqq 2$ and all $x \in S_1$, we have the inequality

$$\sum_{|\mathfrak{p}|=k} \|\hat{P}_{\mathfrak{p}}(x)\| \leqq C \sup_{x\in S_1} \sum_{|\mathfrak{p}|=k} \|Q_{0\mathfrak{p}}(x) + R_{\mathfrak{p}}(x) - \hat{H}_{\mathfrak{p}}(x)\|$$

where the constant C is independent of k.

The difference equations which arise from identifying terms in equation (8.2) are solved by means of Lemmas 1 and 2. In particular, the first $M_0 - 1$ equations are solved by application of Lemma 1 and the remainder by Lemma 2. M_0 is a positive integer defined as in §3. According to Lemma 1, the solutions to the nonhomogeneous linear systems are asymptotic to the formal solution determined in powers of x^{-1}. Thus if the nonhomogeneous term is asymptotically zero, then the particular solution will be asymptotic to zero. Since $Q_0(x, z^1) \cong 0$ uniformly and $R_{\mathfrak{p}}(x)$ is a polynomial in $\hat{P}_{\mathfrak{p}'}(x)$ and $\hat{H}_{\mathfrak{p}'}(x)$ for $|\mathfrak{p}'| < |\mathfrak{p}|$ with no constant terms, it follows from the way in which the $\hat{H}_{\mathfrak{p}'}(x)$ are chosen that

$$\hat{P}_{\mathfrak{p}}(x) \cong 0, \quad \hat{H}_{\mathfrak{p}}(x) \cong 0$$

for $|\mathfrak{p}| < M_0$. Therefore, since $\hat{H}_{\mathfrak{p}}(x) \equiv 0$ for $|\mathfrak{p}| \geqq M_0$, we have

$$\hat{H}(x, z^1) \cong 0, \qquad \sum_{|\mathfrak{p}|\geqq 2}^{M_0 - 1} \hat{P}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} \cong 0 \tag{8.4}$$

uniformly in U_2 as x tends to infinity in S_1.

It remains to establish the convergence of the formal solution (8.1) and to show that $\hat{P}(x, z^1) \cong 0$ uniformly. We prove convergence of the formal solution by use of a majorant equation as in the proof of Theorem 1. However, we must be more careful in our choice of majorants in order to prove that $\hat{P}(x, z^1) \cong 0$.

Let majorants for the components of $\bar{P}(x, z^1)$, $Q_1(x, z^1)$, $Q_{\mathfrak{p}}(x, z^1)$, $|\mathfrak{p}| \geqq 2$, and $f(x, z^1, z^2)$ be given. Let ξ be a constant satisfying

$$\|Q_0(x, z^1)\| \leqq \xi \tag{8.5}$$

in $S_1 \times U_2$. Then a majorant equation can be constructed as in §4. The majorant equation will have a formal solution $\sum_{k=2}^{\infty} T_k(\xi)s^k$ satisfying

$$\sum_{k=2}^{\infty} T_k(\xi)s^k = \sum_{k=2}^{\infty} T_k(\xi)(\sigma s)^k + C\xi \sum_{k=2}^{\infty} \gamma_k s^k + C \sum_{k=3}^{\infty} R_k^* s^k + C \sum_{k=2}^{\infty} \hat{H}_k^* s^k$$

where $\gamma_k = n \sum_{|\mathfrak{p}|=k} \delta_2^{-|\mathfrak{p}|}$, R_k^* is defined in a manner similar to the way it was defined in equation (4.7), and $\hat{H}_k^* = \sum_{|\mathfrak{p}|=k} \hat{H}_{\mathfrak{p}}^*$, where

$$\begin{aligned} H_{\mathfrak{p}}^* &= n\xi\gamma_2 && \text{for } |\mathfrak{p}| = 2, \\ &= n\xi\gamma_k + R_k^* && \text{for } |\mathfrak{p}| = k, 2 < k < M_0, \\ &= 0 && \text{for } |\mathfrak{p}| \geqq M_0. \end{aligned}$$

With this choice of H_k^*, it follows from the definition of R_k^*, that $R_k^* = O(\xi)$ and hence $T_k(\xi) = O(\xi)$ for all k. By induction, it can be shown that

$$\sum_{|\mathfrak{p}|=k} \|\hat{P}_{\mathfrak{p}}(x)\| \leqq T_k(\xi), \qquad k = 2, 3, \ldots,$$

for all $x \in S_1$ and each ξ such that inequality (8.5) holds in $S_1 \times U_2$. The formal series $T(\xi, s) = \sum_{k=2}^{\infty} T_k(\xi)s^k$ can be shown to converge uniformly for $|s| < \delta_3 < \delta_2$ and $\xi < \xi_0$. Furthermore, the inequality $|T(\xi, s)| \leqq L\xi$ holds for $|s| < \delta$, $\xi < \xi_0$. We conclude that the formal solution (8.1) converges absolutely and uniformly for $x \in S_1$ and $\|z^1\| < \delta_3$ and satisfies $\|\hat{P}(x, z^1)\| \leqq \hat{L}\xi$ for some positive constant $\hat{L}$ depending on δ_3 and ξ_0.

We now show that $\hat{P}(x, z^1) \cong 0$. Let $a > a_1$ and define the sector $\bar{S}_a$ by

$$\bar{S}_a \colon |\arg(xe^{-i\theta} - a)| \leqq \pi/2 + \rho_1.$$

Then from inequality (8.3) we have, by means of Lemma 2,

$$\sum_{|\mathfrak{p}|=k} \|\hat{P}_{\mathfrak{p}}(x)\| \leqq C \sup_{x \in S_a} \sum_{|\mathfrak{p}|=k} (\|Q_{0\mathfrak{p}}(x) + R_{\mathfrak{p}}(x) - \hat{H}_{\mathfrak{p}}(x)\|)$$

for all $x \in S_a$ and all $k \geqq M_0$. Our choice of majorants will hold in the smaller sector $\bar{S}_a$, so we will have

$$\left\| \hat{P}(x, z^1) - \sum_{|\mathfrak{p}|=2}^{M_0-1} \hat{P}_{\mathfrak{p}}(x)[z^1]^{\mathfrak{p}} \right\| \leqq K\xi$$

for all $x \in \bar{S}_a$.

Now fix m. Since $Q_0(x, z^1) \cong 0$ uniformly we have

$$\|Q_0(x, z^1)\| \leqq L_m|x|^{-m}$$

for $x \in S_1$. Choose

$$a_m = (L_m \xi_0^{-1})^{1/m} (\cos \rho_1)^{-1}$$

and let $\bar{x} \in \bigcup_{a > a_m} S_a$. Then there exists an $\bar{a} > a_m$ such that

$$|\arg (\bar{x}e^{-i\theta} - \bar{a})| = \pi/2 + \rho_1.$$

Setting $\xi = L_m(\cos \rho_1)^{-m}(\bar{a})^{-m}$ yields $\|Q_0(x, z^1)\| \leqq \xi < \xi_0$ for all $x \in \bar{S}_{\bar{a}}$, $\|z^1\| < \delta_3$. Hence

$$\left\| \hat{P}(\bar{x}, z^1) - \sum_{|\mathfrak{p}|=2}^{M_0-1} \hat{P}_{\mathfrak{p}}(\bar{x})[z^1]^{\mathfrak{p}} \right\| \leqq KL_m (\cos \rho_1)^{-m}(\bar{a})^{-m}$$

for each $\bar{x} \in \bigcup_{a > a_m} S_a$.

Define the sector S_* by

$$S_*\colon |\arg (xe^{-i\theta})| \leqq \pi/2 + \rho_1 - \gamma$$

where $\gamma > 0$ is an arbitrarily small constant. For $\bar{x} \in S_1 \cap S_*$, we have

$$|\bar{x}| \leqq \bar{a} \cos \rho_1 (\sin \gamma)^{-1}.$$

Hence for $\bar{x} \in \{\bigcup_{a > a_m} S_a\} \cap S_*$,

$$\left\| \hat{P}(\bar{x}, z^1) - \sum_{|\mathfrak{p}|=2}^{M_0-1} \hat{P}_{\mathfrak{p}}(\bar{x})[z^1]^{\mathfrak{p}} \right\| \leqq KL_m(\sin \gamma)^{-m}|\bar{x}|^{-m} = L_m'|\bar{x}|^{-m}$$

when $\|z^1\| < \delta_3$. This inequality along with statement (6.4) implies $\hat{P}(x, z^1) \cong 0$ uniformly for $\|z^1\| < \delta_3$ as x tends to infinity in the sector S_2: $|\arg (xe^{-i\theta} - a_1)| < \pi/2 + \rho_1 - \gamma$. This completes the proof of Theorem 2.

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