THE UNIQUENESS OF HERMITE SERIES UNDER POISSON-ABEL SUMMABILITY⁽¹⁾

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Introduction. The differential equation

$$(1) \quad y''(x) - 2xy'(x) + 2nyx = 0 \quad (n = 0, 1, 2, \dots),$$

known as Hermite's equation arises in quantum mechanics in connection with the one-dimensional simple harmonic oscillator. It can be solved by the method of undetermined coefficients and yields the Hermite polynomials

$$(2) \quad H_n(x) = (-1)^n \exp(x^2) [d^n \exp(-x^2)/dx^n] \quad (n = 0, 1, 2, \dots).$$

These polynomials are orthogonal with respect to the weight function $\exp(-x^2)$. We will work with the normalized Hermite functions

$$(3) \quad \Phi_n(x) = \exp(-x^2/2) H_n(x) / \pi^{1/4} (n!)^{1/2} 2^{n/2} \quad (n = 0, 1, 2, \dots),$$

which are orthonormal on $(-\infty, \infty)$ and complete in $\mathcal{L}_2$ [1, p. 288].

Notations and definitions will be as in the main reference of this paper [2]. For any real number p , we say that $f \in \mathcal{H}_p$ if the function $f \in \mathcal{L}$ on every finite interval, and if

$$(4) \quad \int_{-\infty}^{\infty} |x^p f(x)| \exp(-x^2/2) dx < +\infty.$$

If $f \in \mathcal{H}_p$, for every $p \geq 0$, we say that $f \in \mathcal{H}$. If $f \in \mathcal{H}$, and if

$$a_n = \int_{-\infty}^{\infty} f(x) \Phi_n(x) dx \quad (n = 0, 1, 2, \dots),$$

the generalized Fourier series coefficients, then we say that the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ is the Hermite series of $f(x)$, and write

$$(5) \quad f(x) \sim \sum_{n=0}^{\infty} a_n \Phi_n(x).$$

If the series $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ converges, for $0 \leq r < 1$, to $f(x, r)$, then the functions $f^*(x) = \limsup_{r \rightarrow 1} f(x, r)$ and $f_*(x) = \liminf_{r \rightarrow 1} f(x, r)$ are called the upper and

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Received by the editors July 17, 1967.

⁽¹⁾ This research was sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. AF-AFOSR 694-66.

⁽²⁾ This paper is a portion of the author's doctoral thesis. The author would like to express his sincere appreciation to his advisor, Professor Victor Shapiro.

lower Poisson sums respectively of the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$. Analogous to the Riemann function for trigonometric series [3, p. 319], we form the series

$$(6) \quad F(x, r) = - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^n \quad (0 \leq r < 1)$$

which under the given hypothesis converges to a function $F(x)$.

Major theorems. This paper is concerned with the problem of determining when a given series of Hermite functions, $\sum_{n=0}^{\infty} a_n \Phi_n(x)$, is a Hermite series in the above sense. In Chapter II we prove

THEOREM I. *Let the series $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ converge, for $0 \leq r < 1$, to $f(x, r)$. Suppose that*

- (i) $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \rightarrow 1$;
- (ii) there is a function $y_1 \in \mathcal{H}$ such that $-\infty < y_1(x) \leq f_*(x) \leq f^*(x) < +\infty$ for all x ;
- (iii) there is a function $y_2 \in \mathcal{H}$ such that $-\infty < y_2(x) \leq F(x)$ for all x .

Then the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ is Poisson summable almost everywhere, and is the Hermite series of its Poisson sum.

THEOREM II. *Let the series $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ converge, for $0 \leq r < 1$, to $f(x, r)$. Suppose that*

- (i) $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \rightarrow 1$;
- (ii) $\lim_{r \rightarrow 1} f(x, r) = 0$ for all x .

Then $a_n = 0$ for all n .

In Chapter II we shall show that these theorems are, in a certain sense, best possible. They extend several of the main results in Rudin's paper [2]. This paper was motivated by V. L. Shapiro's work on harmonic analysis and the heat equation, in particular [4].

CHAPTER I

Fundamental lemmas.

LEMMA 1. *Suppose*

- (i) $\mathcal{D}$ is the open rectangle: $-R < x < R$, $T_1 < t < T_2$ with boundary $\dot{\mathcal{D}}$. Let $\dot{\mathcal{D}}_1 = \{(x, t) \mid -R < x < R, t = T_2\}$ and let $\dot{\mathcal{D}}_2 = \dot{\mathcal{D}} \sim \dot{\mathcal{D}}_1$;
- (ii) $h(x, t)$ satisfies

$$(7) \quad h_{xx}(x, t) - (x^2 + 1)h(x, t) = 2h_t(x, t) \quad \text{in } \mathcal{D} \cup \dot{\mathcal{D}}_1,$$

where

$$h(x, t) \in \mathcal{C}^2(\mathcal{D} \cup \dot{\mathcal{D}}_1) \quad \text{and} \quad h(x, t) \in \mathcal{C}^0(\mathcal{D} \cup \dot{\mathcal{D}});$$

- (iii) $\liminf_{(x,t) \rightarrow (x_0, t_0)} h(x, t) \geq 0$ for all $(x_0, t_0) \in \dot{\mathcal{D}}_2$.

Then (a) $h(x, t) \geq 0$ in $\mathcal{D} \cup \dot{\mathcal{D}}$;

(b) $h(x, t)$ assumes its maximum in $\dot{\mathcal{D}}_2$.

Proof. Suppose $h(x_1, t_1) < 0$, where $(x_1, t_1) \in \mathcal{D}$. By (iii) and the continuity of $h(x, t)$ on the compact set $\mathcal{D} \cup \dot{\mathcal{D}}_1$, $h(x, t)$ must assume its minimum in $\mathcal{D} \cup \dot{\mathcal{D}}_1$, say at $(x_0, t_0) \in \mathcal{D} \cup \dot{\mathcal{D}}_1$, and $h(x_0, t_0) < 0$. This means $h_{xx}(x_0, t_0) \geq 0$ and $h_t(x_0, t_0) \leq 0$. Since $h(x, t)$ satisfies (7) in $\dot{\mathcal{D}} \cup \dot{\mathcal{D}}_1$, at (x_0, t_0) we have

$$\underbrace{h_{xx}(x_0, t_0)}_{\geq 0} - \underbrace{2h_t(x_0, t_0)}_{\geq 0} - \underbrace{(x_0^2 + 1)h(x_0, t_0)}_{> 0} = 0.$$

This is a contradiction in signs, proving (a).

Suppose $h(x, t)$ assumes its maximum in $\mathcal{D} \cup \dot{\mathcal{D}}_1$, at a point (x_2, t_2) . Then $h_t(x_2, t_2) \geq 0$ and $h_{xx}(x_2, t_2) \leq 0$. Again using (7) we have

$$\underbrace{h_{xx}(x_2, t_2)}_{\leq 0} - \underbrace{2h_t(x_2, t_2)}_{\leq 0} - \underbrace{(x_2^2 + 1)h(x_2, t_2)}_{\leq 0} = 0.$$

This means each term must be zero and $h(x_2, t_2) = 0$. Since $h(x, t) \geq 0$ in $\mathcal{D} \cup \dot{\mathcal{D}}$, the maximum in $\mathcal{D} \cup \dot{\mathcal{D}}_1$ is zero. This means $h(x, t) = 0$ for $(x, t) \in \mathcal{D} \cup \dot{\mathcal{D}}$, hence in any case $h(x, t)$ assumes its maximum in $\dot{\mathcal{D}}_2$.

LEMMA 2. Suppose

(i) $N(x, r)$ satisfies

$$(8) \quad N_{xx}(x, r) - (x^2 + 1)N(x, r) + 2[rN(x, r)]_r = 0$$

for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $N(x, r) \in \mathcal{C}^2$ for $-\infty < x < +\infty$ and $0 \leq r < 1$;

(ii) $\lim_{r \rightarrow 1} N(x, r) = 0$ uniformly on compact subsets;

(iii) $|N(x, r)| \leq K$, a positive constant for $-\infty < r < +\infty$ and $0 \leq r < 1$.

Then $N(x, r) \equiv 0$ for $-\infty < x < +\infty$ and $0 \leq r < 1$.

Proof. Define $h(x, t) = e^{-t}N(x, e^{-t})$ for $-\infty < x < \infty$ and $0 < t < \infty$. Then $h(x, t)$ satisfies (7). We note $|h(x, t)| = e^{-t}|N(x, e^{-t})| \leq K$, a constant for all x and $t > 0$. $\lim_{t \rightarrow 0} h(x, t) = \lim_{t \rightarrow 0} e^{-t} \lim_{t \rightarrow 0} N(x, e^{-t}) = 0$ uniformly on compact subsets. To show $N(x, r) \equiv 0$, it is sufficient to show $h(x, t) \equiv 0$. Consider the function

$$B(x, t) = \frac{1}{(\pi(1 - e^{-2t}))^{1/2}} \exp \left\{ \frac{x^2 - 2t}{2} - \frac{x^2}{1 - e^{-2t}} \right\} \quad \text{for } t > 0.$$

Claim B satisfies (7) for $t > 0$ and all x .

$$B_x(x, t) = \frac{-x}{\pi^{1/2}} \exp \left\{ \frac{x^2 - 2t}{2} - \frac{x^2}{1 - e^{-2t}} \right\} \frac{(1 + e^{-2t})}{(1 - e^{-2t})^{3/2}},$$

$$B_{xx}(x, t) = \frac{-1}{\pi^{1/2}} \frac{\exp \{ (x^2 - 2t)/2 - x^2/(1 - e^{-2t}) \}}{(1 - e^{-2t})^{5/2}} [(1 - e^{-4t}) - x^2(1 + e^{-2t})^2],$$

$$B_t(x, t) = \frac{-1}{\pi^{1/2}} \frac{\exp \{ (x^2 - 2t)/2 - x^2/(1 - e^{-2t}) \}}{(1 - e^{-2t})^{5/2}} [(1 - e^{-2t}) - 2x^2e^{-2t}].$$

Thus $B_{xx}(x, t) - 2B_t(x, t) = (x^2 + 1)B(x, t)$ which is (7). Set

$$U^R(x, t) = K[B(x + R, t) + B(x - R, t)]$$

for $R > 0$ and $t > 0$. Then $U^R(x, t)$ satisfies (7) since B does, for all x and $t > 0$. Note

$$B(0, t) = e^{-t}/(\pi(1 - e^{-2t}))^{1/2} = 1/(\pi(e^{2t} - 1))^{1/2} \quad \text{for } t > 0.$$

Thus $U^R(+R, t) = K[B(2R, t) + B(0, t)] \geq KB(0, t)$ since $B \geq 0$ and $U^R(-R, t) = K[B(0, t) + B(-2R, t)] \geq KB(0, t)$ since $B \geq 0$. Thus

$$U^R(\pm R, t) \geq K/(\pi(e^{2t} - 1))^{1/2} \quad \text{for } t > 0.$$

Let T be an arbitrary positive number. For $0 < t \leq T$ we have

$$(9) \quad |h(\pm R, t)| \leq K \leq K(\pi(e^{2T} - 1))^{1/2}/(\pi(e^{2t} - 1))^{1/2} \leq (\pi(e^{2T} - 1))^{1/2} U^R(\pm R, t) \quad \text{for } t > 0.$$

Hence the functions $(\pi(e^{2T} - 1))^{1/2} U^R(x, t) \pm h(x, t)$ which both satisfy (7) in the open rectangle $-R < x < R$ and $0 < t < T$ are nonnegative on the sides by (9). Call the open rectangle $D_{R,T}$. On the lower boundary of $D_{R,T}$ we have

$$\lim_{t \rightarrow 0} h(x, t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} U^R(x, t) = 0 \quad \text{uniformly.}$$

Thus Lemma 1 applies and we have

$$(\pi(e^{2T} - 1))^{1/2} U^R(x, t) \pm h(x, t) \geq 0 \quad \text{in } D_{R,T} \text{ closure.}$$

This means $|h(x, t)| \leq (\pi(e^{2T} - 1))^{1/2} U^R(x, t)$. Holding (x, t) fixed and letting $R \rightarrow \infty$ we have $\lim_{R \rightarrow \infty} U^R(x, t) = 0$.

Thus $|h(x, t)|$ can be made arbitrarily small which gives

$$h(x, t) \equiv 0 \quad \text{for } -\infty < x < +\infty \text{ and } 0 < t < T.$$

But T was arbitrary, thus $h(x, t) \equiv 0$ for $-\infty < x < \infty$ and all $t > 0$.

LEMMA 3. Let $f(x, r) = \lim_{N \rightarrow \infty} \sum_{k=0}^N a_k \Phi_k(x) r^k$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$.

Then (a) for fixed r , $0 \leq r < 1$, $f(x, r)$ converges uniformly in x .

(b) $f(x, r) \in \mathcal{H}$ for $0 \leq r < 1$.

Proof. Since $f(x, r)$ converges, $\lim_{n \rightarrow \infty} a_n \Phi_n(x) r^n = 0$ for all x and fixed r , $0 \leq r < 1$. By [2, Theorem 7],

$$(10) \quad a_n r^n = o(n^{1/4}) \quad \text{for } 0 \leq r < 1.$$

By choosing δ such that $(1 + \delta)r < 1$ we see that $a_n [(1 + \delta)r]^n \leq K n^{1/4}$ for all n , where K is a positive constant. Since $|H_n(x)| < 1.1(n!)^{1/2} 2^{n/2} \exp(x^2/2)$ for all x and n [5, p.324], we get

$$(11) \quad |\Phi_n(x)| < 1 \quad \text{for all } x \text{ and } n.$$

Thus

$$|f(x, r)| \leq \sum_{n=0}^{\infty} |a_n r^n| |\Phi_n(x)| \leq K \sum_{n=0}^{\infty} \frac{n^{1/4}}{(1 + \delta)^n} = K_1 < +\infty,$$

by the Weierstrass M -test for $0 \leq r < 1$, which proves (a).

Since

$$\begin{aligned} \int_{-\infty}^{\infty} |x^p f(x, r)| \exp(-x^2/2) dx &\leq \int_{-\infty}^{\infty} |x|^p \sum_{n=0}^{\infty} \frac{Kn^{1/4}}{(1+\delta)^n} \exp(-x^2/2) dx \\ &\leq K_1 \int_{-\infty}^{\infty} |x|^p \exp(-x^2/2) dx < +\infty \end{aligned}$$

for all $p \geq 0$, we have (b).

The Poisson kernel for Hermite series [6] has the form

$$(12) \quad P(x, t, r) = \sum_{n=0}^{\infty} \Phi_n(x) \Phi_n(t) r^n = [\pi(1-r^2)]^{1/2} \exp \left\{ \frac{x^2 - t^2}{2} - \frac{(x-rt)^2}{1-r^2} \right\} \\ (0 \leq r < 1).$$

For ease in integration we write

$$(13) \quad P(x, t, r) = \frac{\exp \left(-\frac{x^2}{2} \left(\frac{1-r^2}{1+r^2} \right) \right)}{\pi^{1/2} (1-r^2)^{1/2}} \exp \left\{ -\frac{(1+r^2)}{2(1-r^2)} \left(t - \frac{2xr}{1+r^2} \right)^2 \right\} \\ (0 \leq r < 1)$$

and

$$(14) \quad P_x(x, t, r) = \left[\frac{-x(1+r^2) + 2rt}{1-r^2} \right] P(x, t, r) \quad (0 \leq r < 1).$$

LEMMA 4. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$. Then

$$f(x, r+r_0-1) = \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

for $0 < r_0 < 1$ and $1-r_0 < r < 1$.

Proof. By [2, p. 398], we can differentiate across the integral and show

$$\int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

satisfies (8). Using the substitution $z = (r+r_0-1)/r_0$ we have

$$\lim_{r \rightarrow 1} \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt = \lim_{z \rightarrow 1} \int_{-\infty}^{\infty} f(t, r_0) P(x, t, z) dt = f(x, r_0)$$

uniformly on compact subsets, since $f(x, r_0)$ is continuous and in $\mathcal{H}$.

Since

$$|f(t, r_0)| \leq \frac{\varepsilon(r_0)}{1-r_0} \quad \text{and} \quad \int_{-\infty}^{\infty} P(x, t, r) dt \leq 2,$$

we have

$$\left| \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt \right| \leq \frac{\varepsilon(r_0)}{1-r_0} < K$$

for all x , $1-r_0 \leq r < 1$.

The above three conditions are also met by $f(x, r+r_0-1)$. In fact,

$$|f(x, r+r_0-1)| \leq \sum_{n=0}^{\infty} |a_n r_0^n| |\Phi_n(x)| \leq K_1 \sum_{n=0}^{\infty} \frac{n^{1/4}}{(1+\delta)^n} = K_2$$

for all x and r , using (10) and choosing δ as in Lemma 3. $f(x, r+r_0-1)$ satisfies (8) since $\Phi_n(x)$ satisfies [7, p. 105]

$$\Phi_n''(x) - (x^2 + 1)\Phi_n(x) = -(2n+2)\Phi_n(x) \quad (n = 0, 1, 2, \dots).$$

The uniform continuity of $f_x(x, r)$ and $f_{xx}(x, r)$ follows from the M -test.

Thus

$$f(x, r+r_0-1) - \int_{-\infty}^{\infty} f(t, r_0) P\left(x, t, \frac{r+r_0-1}{r_0}\right) dt$$

satisfies all the conditions of Lemma 2, proving this lemma.

LEMMA 5. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$.

Then (a) $f_x(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n'(x) r^n$ for $0 \leq r < 1$;

(b) for fixed r , $0 \leq r < 1$, $f_x(x, r)$ converges uniformly in x ;

(c) $f_x(x, r) \in \mathcal{H}$ for $0 \leq r < 1$;

$$(d) \quad f_x(x, v+r-1) = \int_{-\infty}^{\infty} f(t, r) P_x\left(x, t, \frac{v+r-1}{r}\right) dt = \sum_{n=0}^{\infty} a_n \Phi_n'(x) (v+r-1)^n$$

for $0 < r < 1$ and $1-r < v < 1$;

(e) $|f_x(x, r)| \leq \varepsilon_1(r)/(1-r)^{3/2}$ for $-\infty < x < +\infty$ and $\frac{1}{2} \leq r_0 < r < 1$, where $\varepsilon_1(r)$ is bounded and $\varepsilon_1(r) = o(1)$ as $r \rightarrow 1$.

Proof. By the recursion formula for Hermite polynomials [1, p. 286],

$$(15) \quad 2\Phi_n'(x) = (2n)^{1/2}\Phi_{n-1}(x) - (2n+2)^{1/2}\Phi_{n+1}(x) \quad \text{for } n \geq 1.$$

Using (11) and (15), we have $|\Phi_n'(x)| \leq An^{1/2}$ for all n and x where A is a positive constant. Thus

$$\begin{aligned} \left| \sum_{n=0}^{\infty} a_n \Phi_n'(x) r^n \right| &\leq \sum_{n=0}^{\infty} |a_n r^n| |\Phi_n'(x)| \\ &\leq K \sum_{n=0}^{\infty} \frac{n^{1/4} A n^{1/2}}{(1+\delta)^n} = K_2 < +\infty, \end{aligned}$$

using (10) and the ratio test. By the Weierstrass M -test $f_x(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n'(x) r^n$ converges uniformly for all x and $0 \leq r < 1$, and the representation is justified, proving (a) and (b).

Since $\int_{-\infty}^{\infty} |x^p f_x(x, r)| \exp(-x^2/2) dx \leq K_2 \int_{-\infty}^{\infty} |x^p| \exp(-x^2/2) dx < +\infty$ for all $p \geq 0$ and $0 \leq r < 1$ we have (c).

By Lemma 4

$$f(x, v+r-1) = \int_{-\infty}^{\infty} f(t, r) P\left(x, t, \frac{v+r-1}{r}\right) dt$$

for $0 < r < 1$ and $1 - r < v < 1$. Since $f(t, r) \in \mathcal{H}$ we can differentiate across the integral sign [2, pp. 398, 399] to obtain (d).

Set $\rho = (v + r - 1)/r$. Using (d), (14) and the substitution

$$y = [(1 + \rho^2)/2(1 - \rho^2)]^{1/2}(t - 2x\rho/(1 + \rho^2)),$$

$$\begin{aligned} |f_x(x, v + r - 1)| &= \frac{\varepsilon(r)}{1 - r} \frac{\exp\left\{-\frac{x^2}{2} \left(\frac{1 - \rho^2}{1 + \rho^2}\right)\right\}}{\pi^{1/2}(1 - \rho^2)^{3/2}} \int_{-\infty}^{\infty} |-x(1 + \rho^2) + 2t\rho| \exp(-y^2) dt \\ &= \frac{\varepsilon(r)}{1 - r} \frac{\exp\left\{-\frac{x^2}{2} \left(\frac{1 - \rho^2}{1 + \rho^2}\right)\right\}}{\pi^{1/2}(1 - \rho^2)^{1/2}(1 + \rho^2)} 4\rho \\ &\quad \times \int_{-\infty}^{\infty} \left|y - \frac{x(1 - \rho^2)^{3/2}}{[8\rho^2(1 + \rho^2)]^{1/2}}\right| \exp(-y^2) dy. \end{aligned}$$

Set $g_1(x, \rho, r)$ equal to the term outside the integral, and

$$A(x, \rho) = \frac{x(1 - \rho^2)^{3/2}}{[8\rho^2(1 + \rho^2)]^{1/2}}.$$

Then

$$\begin{aligned} |f_x(x, v + r - 1)| &= g_1(x, \rho, r) \left\{ \int_{-\infty}^A (A - y) \exp(-y^2) dy \right. \\ &\quad \left. + \int_A^{\infty} (y - A) \exp(-y^2) dy \right\} \\ &= g_1(x, \rho, r) \left\{ A \int_{-A}^A \exp(-y^2) dy - \int_A^{\infty} (-2y) \exp(-y^2) dy \right\} \\ &\leq g_1(x, \rho, r) \left\{ A \int_{-A}^A \exp(-y^2) dy + 1 \right\}. \end{aligned}$$

Note $g_1(x, \rho, r) \leq \varepsilon(r)K_3/(1 - r)(1 - \rho)^{1/2}$, where K_3 is a positive constant. Recalling $\rho = (v + r - 1)/r$, set $v = r$ to get $1/(1 - \rho) < 1/(1 - r)$, and $g_1(x, (2r - 1)/r, r) = \varepsilon_1(r)/(1 - r)^{3/2}$ if $1 > \rho > 0$. To ensure this, choose $r_0 > \frac{1}{2}$. Then, for $r > r_0$, we have $2 - 1/r > 0$, $(2r - 1)/r > 0$, and $\rho = (2r - 1)/r > 0$. Thus for $r > r_0$,

$$g_1(x, (2r - 1)/r, r) = \varepsilon_1(r)/(1 - r)^{3/2},$$

where $\varepsilon_1(r)$ is bounded and $o(1)$ as $r \rightarrow 1$.

Part (e) of Lemma 5 will be proven if we show that

$$g_1\left(x, \frac{2r - 1}{r}, r\right) A \int_{-A}^A \exp(-y^2) dy \leq \frac{\varepsilon_1(r)}{(1 - r)^{3/2}}$$

or that

$$\frac{\varepsilon(r)}{1 - r} K_4 \frac{\exp\left\{-\frac{x^2}{2} \left(\frac{1 - \rho^2}{1 + \rho^2}\right)\right\}}{(1 - \rho^2)^{1/2}} |x|(1 - \rho^2)^{3/2} \int_{-\infty}^{\infty} \exp(-y^2) dy \leq \frac{\varepsilon_1(r)}{(1 - r)^{3/2}}$$

for $r_0 < r < 1$.

It is sufficient to show

$$|x| \left(\frac{1-\rho^2}{1+\rho^2} \right) \exp \left\{ -\frac{x^2}{2} \left(\frac{1-\rho^2}{1+\rho^2} \right) \right\}$$

is uniformly bounded for ρ near 1. Setting $\eta = (1-\rho^2)/(1+\rho^2)$ and noting $\eta \rightarrow 0$ as $\rho \rightarrow 1$ we need only show $|x|\eta \exp((-x^2/2)\eta)$ is uniformly bounded. In fact, for $0 \leq \eta < \frac{1}{2}$, $|x|\eta \exp((-x^2/2)\eta) < 1$.

Generalized Hermite operators. $\Phi_n(x)$ satisfies [7, p. 105]

$$(16) \quad \Phi_n''(x) - (x^2 + 1)\Phi_n(x) = -(2n+2)\Phi_n(x) \quad (n = 0, 1, 2, \dots).$$

We consider the equation

$$(17) \quad y''(x) - (x^2 + 1)y(x) = 0.$$

Putting

$$\beta(x) = \exp(x^2/2) \int_{-\infty}^x \exp(-u^2) du,$$

we note that $\beta(x)$ and $\beta(-x)$ are linearly independent solutions to (17). Given a function $F(t)$, defined in a neighborhood of the point x , and $h > 0$, there exists a unique function $y(t)$ which is a solution of (17) and is such that $y(x+h) = F(x+h)$ and $y(x-h) = F(x-h)$. We define

$$(18) \quad \Lambda F(x) = \lim_{h \rightarrow 0} \frac{2[y(x) - F(x)]}{h^2}$$

provided the limit exists. $\Lambda^* F(x)$ and $\Lambda_* F(x)$ are defined likewise with $\limsup_{h \rightarrow 0}$ and $\liminf_{h \rightarrow 0}$ in place of $\lim$. By [2, p. 388],

$$(19) \quad \Lambda F(x) = F''(x) - (x^2 + 1)F(x) \quad \text{if } F''(x)$$

exists.

Setting

$$\begin{aligned} k(x, t) &= \pi^{-1/2} \beta(x) \beta(-t) & (x < t), \\ &= \pi^{-1/2} \beta(-x) \beta(t) & (x \geq t), \end{aligned}$$

we define

$$(20) \quad \Omega f(x) = - \int_{-\infty}^{\infty} f(t) k(x, t) dt,$$

provided $f \in \mathcal{H}_0$. The Ω operator is the inverse of Λ . In particular, for $f \in \mathcal{H}$, $f(x) \sim \sum_{n=0}^{\infty} a_n \Phi_n(x)$ iff $\Omega f(x) \sim \sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x)$ [2, p. 389].

LEMMA 6. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \epsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\epsilon(r)$ is bounded and $\epsilon(r) = o(1)$ as $r \rightarrow 1$. Set $F(x, r) = - \sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ for $0 \leq r < 1$.

Then (a) $|F(x, r)| \leq \varepsilon_2(r) \log(1/(1-r))$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon_2(r)$ is bounded and $\varepsilon_2(r) = o(1)$ as $r \rightarrow 1$;

(b) $F(x, r) = \Omega f(x, r)$ and $F(x, r) \in \mathcal{H}$ ($0 \leq r < 1$);

(c) $F_x(x, r) = \Omega f_x(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ ($0 \leq r < 1$);

(d) $|F_x(x, r)| \leq \varepsilon_3(r)/(1-r)^{1/2}$ for $-\infty < x < +\infty$ and $\frac{1}{2} \leq r_0 < r < 1$, where $\varepsilon_3(r)$ is bounded and $\varepsilon_3(r) = o(1)$ as $r \rightarrow 1$.

Proof.

$$\begin{aligned} |rF(x, r) - r_0F(x, r_0)| &= \left| -\sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^{n+1} + \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r_0^{n+1} \right| \\ &= \left| \sum_{n=0}^{\infty} \frac{a_n}{2} \Phi_n(x) \int_r^{r_0} \rho^n d\rho \right| = \left| \int_{r_0}^r \frac{1}{2} \sum_{n=0}^{\infty} a_n \Phi_n(x) \rho^n d\rho \right| \end{aligned}$$

since $f(x, r)$ is a power series in r and converges uniformly over compact subsets of r when r is bounded away from 1.

$$\frac{1}{2} \int_{r_0}^r |f(x, \rho)| d\rho \leq \frac{1}{2} \int_{r_0}^r \frac{\varepsilon_1(\rho)}{1-\rho} d\rho = \varepsilon_2(r) \log \frac{1}{1-r},$$

where $\varepsilon_2(r)$ is bounded and $o(1)$ as $r \rightarrow 1$.

Thus $F(x, r)$ satisfies all the hypotheses of Lemmas 3, 4, and 5 proving $F(x, r) \in \mathcal{H}$, for $0 \leq r < 1$, $F_x(x, r) \in \mathcal{H}$, and $F_x(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n'(x) r^n$. By [2, Theorems 2 and 8] we have $F(x, r) = \Omega f(x, r)$ and $F_x(x, r) = \Omega f_x(x, r)$.

It remains to prove (d). By Lemma 5, for $r_0 < r < 1$, we have $|f_x(x, r)| \leq \varepsilon_1(r)/(1-r)^{3/2}$. Thus

$$\begin{aligned} |rF_x(x, r) - r_0F_x(x, r_0)| &\leq \left| -\frac{1}{2} \sum_{n=0}^{\infty} a_n \Phi_n'(x) \left[\frac{r^{n+1}}{n+1} - \frac{r_0^{n+1}}{n+1} \right] \right| \\ &\leq \left| \frac{1}{2} \int_{r_0}^r \sum_{n=0}^{\infty} a_n \Phi_n'(x) \rho^n d\rho \right| \leq \frac{1}{2} \int_{r_0}^r |f_x(x, \rho)| d\rho \\ &\leq \frac{\varepsilon_3(r)}{2(1-r)^{1/2}} \quad \text{for } -\infty < x < +\infty \text{ and } \frac{1}{2} \leq r_0 < r < 1, \end{aligned}$$

where $\varepsilon_3(r)$ is bounded and $\varepsilon_3(r) = o(1)$ as $r \rightarrow 1$. Thus

$$|F_x(x, r)| \leq \frac{\varepsilon_3(r)}{(1-r)^{1/2}} \quad \text{for } 0 \leq r_0 < r < 1,$$

proving (d).

Smoothness. We shall say that G is smooth at the point x if $G(x)$ is defined and finite in a neighborhood of x and if

$$(21) \quad \lim_{h \rightarrow 0} [G(x+h) + G(x-h) - 2G(x)]/h = 0.$$

We shall say that G is smooth on $(-\infty, \infty)$ if it is smooth at every point x .

LEMMA 7. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$.

Set $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2))\Phi_n(x)r^n$ for $0 \leq r < 1$ and

$$G(x, r) = \sum_{n=0}^{\infty} (a_n/(2n+2))^2 \Phi'_n(x)r^n \quad \text{for } 0 \leq r < 1.$$

Then

- (a) $|G(x, r)| \leq K_5$, a positive constant, for all x and $0 \leq r_1 < r < 1$;
- (b) $G(x, r) = \Omega F_x(x, r)$ and $G(x, r) \in \mathcal{H}$;
- (c) $\lim_{r \rightarrow 1} G(x, r) = G(x)$ uniformly in x as $r \rightarrow 1$, where $G(x)$ is a bounded continuous function on $(-\infty, \infty)$;
- (d) $G(x)$ is smooth on $(-\infty, \infty)$.

Proof. Choose r_1 and r such that $1 > r > r_1 > r_0 > \frac{1}{2}$, where r_0 is chosen as in Lemma 6 such that

$$|F_x(x, r)| \leq \varepsilon_3(r)/(1-r)^{1/2} \quad \text{for } r_0 < r < 1.$$

Then

$$\begin{aligned} |rG(x, r) - r_1G(x, r_1)| &= \left| \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x)(r^{n+1} - r_1^{n+1}) \right| \\ &\leq \frac{1}{2} \left| \int_{r_1}^r \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi'_n(x) \rho^n d\rho \right| \\ &\leq \frac{1}{2} \int_{r_1}^r |F_x(x, \rho)| d\rho \quad \text{since } F_x(x, r) \text{ converges uniformly} \\ &\leq \frac{1}{2} \int_{r_1}^r \frac{\varepsilon_3(\rho)}{(1-\rho)^{1/2}} d\rho \leq K_5[(1-r)^{1/2} - (1-r_1)^{1/2}]. \end{aligned}$$

Thus, $G(x, r)$ is uniformly bounded for all x and $1 > r > r_0$, and $G(x, r)$ converges uniformly to a bounded continuous function $G(x)$.

Since $G(x, r)$ satisfies the hypothesis of Lemma 6, we have (b).

To prove $G(x)$ is smooth on $(-\infty, \infty)$, let x_0 be given arbitrarily. Select $\delta < (1-r_0)^{1/2}$. Then for $0 < h < \delta$,

$$\begin{aligned} &[G(x_0+h) + G(x_0-h) - 2G(x_0)]/h \\ &= [G(x_0+h, 1-h^2) + G(x_0-h, 1-h^2) - 2G(x_0, 1-h^2)]/h \\ &\quad + [G(x_0+h) - G(x_0+h, 1-h^2)]/h \\ &\quad + [G(x_0-h) - G(x_0-h, 1-h^2)]/h \\ &\quad + 2[G(x_0, 1-h^2) - G(x_0)]/h. \end{aligned}$$

We conclude from (21) that to show G is smooth at x_0 , and consequently on $(-\infty, \infty)$, we need only establish

$$(22) \quad \lim_{h \rightarrow 0} [G(x, 1-h^2) - G(x)]/h = 0 \quad \text{uniformly for } |x - x_0| \leq \delta,$$

and

$$(23) \quad G(x_0+h, 1-h^2) + G(x_0-h, 1-h^2) - 2G(x_0, 1-h^2) = o(h) \quad \text{as } h \rightarrow 0.$$

To show that (22) holds we will show that given $\varepsilon > 0$ there is an h_0 , such that for $0 < h < h_0$ and all x such that $|x - x_0| \leq \delta$, we have $|G(x, 1-h^2) - G(x)| < h\varepsilon$. Select

r_1 as in Lemma 6 so that for $1 > r > r_1 > r_0$ we have $|F_x(x, r)| < \varepsilon/2(1-r)^{1/2}$. Choose h_0 such that

- (i) $h_0 < \delta < (1-r_1)^{1/2}$ and hence $1-h^2 > r_1$ for $0 < h < h_0$;
- (ii) for $0 < h < h_0$ we have

$$|G(x, 1-h^2)| < |G(x)| + 1 < \max_{|x-x_0| \leq \delta} |G(x)| + 1 = K_6,$$

a positive constant, by continuity of $G(x)$ and the uniform convergence of $G(x, r)$ to $G(x)$;

- (iii) $h_0 < \varepsilon/2K_6$;

- (iv) $h_0 < 1/(x_0^2 + 1)$.

Now $|G(x, 1-h^2) - G(x)| \leq |G(x) - (1-h^2)G(x, 1-h^2)| + h^2|G(x, 1-h^2)|$.

$$\begin{aligned} |G(x) - (1-h^2)G(x, 1-h^2)| &= \left| \lim_{r \rightarrow 1} rG(x, r) - \sum_{n=0}^{\infty} (a_n/(2n+2)^2) \Phi'_n(x)(1-h^2)^{n+1} \right| \\ &= \left| \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} \frac{a_n}{2(2n+2)} \Phi'_n(x) \left[\frac{r^{n+1}}{n+1} - \frac{(1-h^2)^{n+1}}{n+1} \right] \right| \\ &= \left| \lim_{r \rightarrow 1} \frac{1}{2} \int_{1-h^2}^r \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi'_n(x) \rho^n d\rho \right| \\ &\leq \lim_{r \rightarrow 1} \frac{1}{2} \int_{1-h^2}^r \frac{\varepsilon d\rho}{2(1-\rho)^{1/2}} \quad \text{since } 1-h^2 > r_1, \text{ for } r > r_1 \\ &\leq \lim_{r \rightarrow 1} \left[\frac{\varepsilon}{2} h - (1-r)^{1/2} \frac{\varepsilon}{2} \right] = \frac{\varepsilon}{2} h. \end{aligned}$$

Thus

$$\begin{aligned} |G(x, 1-h^2) - G(x)| &\leq \varepsilon h + h^2 |G(x, 1-h^2)|; \\ &\leq \varepsilon/2h + h^2 K_6 \quad \text{for } 0 < h < h_0; \\ &\leq \varepsilon/2h + h\varepsilon/2 = \varepsilon h, \quad \text{proving (22).} \end{aligned}$$

To prove (23) we will prove, for $\varepsilon > 0$ and $0 < h < h_0$, that $(1-h^2)|\nabla G| < \varepsilon h$, where $\nabla G = G(x_0 + h, 1-h^2) + G(x_0 - h, 1-h^2) - 2G(x_0, 1-h^2)$.

$$\begin{aligned} (1-h^2)|\nabla G| &= \left| \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x_0+h)(1-h^2)^{n+1} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x_0-h)(1-h^2)^{n+1} - 2 \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x_0)(1-h^2)^{n+1} \right| \\ &\leq \frac{1}{2} \left| \sum_{n=0}^{\infty} \left\{ \frac{a_n}{2n+2} \Phi'_n(x_0+h) \int_{r_0}^{1-h^2} \rho^n d\rho + \frac{a_n}{2n+2} \Phi'_n(x_0-h) \int_{r_0}^{1-h^2} \rho^n d\rho \right. \right. \\ &\quad \left. \left. - \frac{2a_n}{2n+2} \Phi'_n(x_0) \int_{r_0}^{1-h^2} \rho^n d\rho \right\} \right| \\ &\quad + \left| \frac{a_0}{4} [\Phi'_0(x_0+h) + \Phi'_0(x_0-h) - 2\Phi'_0(x_0)] \right| \\ &\leq \frac{1}{2} \left| \int_{r_0}^{1-h^2} [F_x(x_0+h, \rho) + F_x(x_0-h, \rho) - 2F_x(x_0, \rho)] d\rho \right| + o(h) \\ &\quad \text{since } \Phi'_0 \text{ is smooth} \\ &\leq \frac{1}{2} \left| \int_0^{1-h^2} \int_0^h [F_{xx}(x_0+y, \rho) - F_{xx}(x_0-y, \rho)] dy d\rho \right| + o(h) \end{aligned}$$

by the uniform convergence of F_x and F_{xx} . Using the fact that $F_{xx}(x_0, \rho) = (x_0^2 + 1)F(x_0, \rho) - 2\partial[\rho F(x_0, \rho)]/\partial\rho$ we have

$$(1-h^2)|\nabla G| \leq o(h) + \frac{1}{2} \left| \int_0^{1-h^2} \int_0^h \left\{ -2 \frac{\partial}{\partial \rho} [\rho F(x_0 + y, \rho)] + [(x_0 + y)^2 + 1]F(x_0 + y, \rho) \right. \right. \\ \left. \left. + 2 \frac{\partial}{\partial \rho} [\rho F(x_0 - y, \rho)] \right. \right. \\ \left. \left. + [(x_0 - y)^2 + 1]F(x_0 - y, \rho) \right\} dy d\rho \right|.$$

By Fubini's Theorem, we have

$$(1-h^2)|\nabla G| \leq o(h) + \frac{1}{2} \left| \int_0^h \int_0^{1-h^2} \left\{ -2 \frac{\partial}{\partial \rho} [\rho F(x_0 + y, \rho) - \rho F(x_0 - y, \rho)] d\rho dy \right\} \right. \\ (24) \quad \left. + \int_0^h \int_0^{1-h^2} \{ [(x_0 + y)^2 + 1]F(x_0 + y, \rho) \right. \\ \left. - [(x_0 - y)^2 + 1]F(x_0 - y, \rho) \} d\rho dy \right|.$$

The first integral in (24) is then

$$\left| \int_0^h (-2) [\rho F(x_0 + y, \rho) - \rho F(x_0 - y, \rho)]_0^{1-h^2} dy \right| \\ = \left| -2 \int_0^h (1-h^2) [F(x_0 + y, 1-h^2) - F(x_0 - y, 1-h^2)] dy \right| \\ = \left| -2 \int_0^h (1-h^2) \int_{x_0-y}^{x_0+y} F_x(x, 1-h^2) dx dy \right| \\ \leq 2 \int_0^h (1-h^2) \int_{x_0-y}^{x_0+y} |F_x(x, 1-h^2)| dx dy \\ \leq 2 \int_0^h \int_{x_0-y}^{x_0+y} (1-h^2) \frac{\varepsilon}{2(1-(1-h^2))^{1/2}} dx dy \quad \text{since } r_1 < 1-h^2 \\ \leq 2 \int_0^h \frac{\varepsilon(1-h^2)}{2h} 2y dy = \frac{\varepsilon}{h} (1-h^2)h^2 < \varepsilon h.$$

The second integral in (24) is

$$\frac{1}{2} \left| \int_0^h \int_0^{1-h^2} \{ [(x_0 + y)^2 + 1]F(x_0 + y, \rho) - [(x_0 - y)^2 + 1]F(x_0 - y, \rho) \} d\rho dy \right| \\ \leq \frac{1}{2} (x_0^2 + 1) \left| \int_0^h \int_0^{1-h^2} [F(x_0 + y, \rho) - F(x_0 - y, \rho)] d\rho dy \right| \\ + \left| \int_0^h y \int_0^{1-h^2} [F(x_0 + y, \rho) + F(x_0 - y, \rho)] d\rho dy \right| \\ + \frac{1}{2} \left| \int_0^h y^2 \int_0^{1-h^2} [F(x_0 + y, \rho) - F(x_0 - y, \rho)] d\rho dy \right| \\ \leq \frac{1}{2} (x_0^2 + 1) \left| \int_0^h \int_0^{1-h^2} \int_{-y}^y F_t(x_0 + t, \rho) dt d\rho dy \right. \\ \left. + \int_0^h y \int_0^{1-h^2} |F(x_0 + y, \rho) + F(x_0 - y, \rho)| d\rho dy \right. \\ \left. + \frac{1}{2} \int_0^h y^2 \int_0^{1-h^2} \int_{-y}^y F_t(x_0 + t, \rho) dt d\rho dy \right|.$$

Now using the fact that $|F_i(x, r)| < \varepsilon/(1-r)^{1/2}$ for $r > r_1$, $|F_i(x, r)| < K_7$ for $0 \leq r < r_1$,

$$|F(x, r)| < \varepsilon \log 1/(1-r) \quad \text{for } r > r_2, \quad |F(x, r)| < K_8 \quad \text{for } 0 \leq r < r_2,$$

we obtain that the second integral in (24) is bounded by

$$\begin{aligned} & \frac{1}{2}(x_0^2 + 1) \int_0^h \left[\int_0^{r_1} \int_{-y}^y K_7 dt d\rho + \int_{r_1}^{1-h^2} \int_{-y}^y \frac{\varepsilon}{(1-\rho)^{1/2}} dt d\rho \right] dy \\ & \quad + \int_0^h y \int_0^{r_2} 2K_8 d\rho dy + \varepsilon \int_0^h y \int_{r_2}^{1-h^2} \log(1/(1-\rho)) d\rho dy \\ & \quad + \frac{1}{2} \int_0^h y^2 \left[\int_0^{r_1} \int_{-y}^y K_7 dt d\rho + \int_{r_1}^{1-h^2} \int_{-y}^y \frac{\varepsilon}{(1-\rho)^{1/2}} dt d\rho \right] dy \\ & \leq \frac{1}{2}(x_0^2 + 1) \int_0^h \left[\int_0^{r_1} K_7 2y d\rho + \int_{r_1}^{1-h^2} \frac{2\varepsilon y}{(1-\rho)^{1/2}} d\rho \right] dy \\ & \quad + 2K_8 \int_0^h y r_2 dy + \varepsilon \int_0^h y \int_{1-r_2}^{h^2} \log s ds dy \\ & \quad + \frac{1}{2} K_7 \int_0^h y^2 \left[\int_0^{r_1} 2y d\rho + \int_{r_1}^{1-h^2} \frac{2y\varepsilon}{(1-\rho)^{1/2}} d\rho \right] dy \\ & \leq \frac{1}{2}(x_0^2 + 1) \int_0^h [2K_7 y r_1 - \varepsilon y (1-\rho)^{1/2} |_{r_1}^{1-h^2}] dy \\ & \quad + 2K_8 r_2 \frac{y^2}{2} \Big|_0^h + \varepsilon \int_0^h y [s \log s - s]_{1-r_2}^{h^2} dy \\ & \quad + K_7 r_1 \int_0^h y^3 dy - K_7 \varepsilon \int_0^h y^3 (1-\rho)^{1/2} \Big|_{r_1}^{1-h^2} dy \\ & \leq \frac{1}{2}(x_0^2 + 1) [K_7 r_1 h^2 - 2\varepsilon(h - (1-r_1)^{1/2})h^2] \\ & \quad + K_8 r_2 h^2 + \varepsilon h^2 [(h^2 \log h^2 - h^2) - (1-r_2) \log(1-r_2) + (1-r_2)] \\ & \quad + K_7 r_1 (h^4/4) - K_7 \varepsilon [h - (1-r_1)^{1/2}] h^4/4 = O(h^2) = o(h). \end{aligned}$$

LEMMA 8. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$. Set $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ for $0 \leq r < 1$, and suppose that there is $y_1 \in \mathcal{H}$ such that $-\infty < y_1(x) \leq f_*(x) \leq f^*(x) < +\infty$ for all x . Then $\lim_{r \rightarrow 1} F(x, r) = F(x)$ exists and is finite for all x .

Proof. It is sufficient to show $\lim_{r \rightarrow 1} rF(x, r)$ exists and is finite.

$$rF(x_0, r) = - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x_0) r^{n+1} = -\frac{1}{2} \int_0^r f(x_0, \rho) d\rho.$$

By hypothesis $f_*(x_0)$ and $f^*(x_0)$ are finite, hence $f(x_0, r)$ is bounded in absolute value by $A(x_0) = \sup \{|f^*(x_0)| + 1, |f_*(x_0)| + 1\}$ for $0 < r_3 < r < 1$. If r_4 and r_5 are such that $0 < r_3 < r_4 < r_5 < 1$, then

$$|r_5 F(x_0, r_5) - r_4 F(x_0, r_4)| \leq \frac{1}{2} \int_{r_4}^{r_5} |f(x_0, \rho)| d\rho \leq \frac{A(x_0)}{2} (r_5 - r_4)$$

which approaches zero as r_4 and r_5 approach 1.

Generalized derivatives. If G is defined and finite in a neighborhood of the point x , we set $D_1^*G(x) = \limsup_{h \rightarrow 0} [G(x+h) - G(x-h)]/2h$. $D_{1*}G(x)$ will designate the corresponding $\liminf_{h \rightarrow 0}$.

LEMMA 9. Under the same hypothesis as Lemma 8, we have

$$\begin{aligned} D_{1*}G(x) &\leq \liminf_{r \rightarrow 1} [F(x, r) + (x^2 + 1)\Omega F(x, r)] \\ &\leq \limsup_{r \rightarrow 1} [F(x, r) + (x^2 + 1)\Omega F(x, r)] \leq D_1^*G(x). \end{aligned}$$

Proof. $\Omega F(x, r)$ is well defined since $F(x, r) \in \mathcal{H}$. By Lemmas 7 and 4,

$$G(x, v+r-1) = \int_{-\infty}^{\infty} G(t, r) P\left(x, t, \frac{v+r-1}{r}\right) dt \text{ for } 1 > r > r_0, 1 > v+r-1 > r_0.$$

Taking limits as $r \rightarrow 1$,

$$G(x, v) = \int_{-\infty}^{\infty} G(t) P(x, t, v) dt \text{ for } 1 > v > r_0.$$

It is sufficient to show

$$D_{1*}G(x_0) = \liminf_{h \rightarrow 0} \frac{G(x_0+h) - G(x_0-h)}{2h} \leq \liminf_{r \rightarrow 1} [F(x_0, r) + (x_0^2 + 1)\Omega F(x_0, r)].$$

If $D_{1*}G(x_0) = -\infty$ we are done. Assume $D_{1*}G(x_0) > q > -\infty$. Observe

$$\begin{aligned} G_x(x_0, r) &= \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n''(x_0) r^n \\ &= - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x_0) r^n + (x_0^2 + 1) \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x_0) r^n \end{aligned}$$

since $\Phi_n(x)$ satisfies (16). Therefore

$$G_x(x_0, r) = F(x_0, r) + (x_0^2 + 1)\Omega F(x_0, r) = \int_{-\infty}^{\infty} G(t) P_x(x_0, t, r) dt,$$

and we have to show

$$D_{1*}G(x_0) \leq \liminf_{r \rightarrow 1} \int_{-\infty}^{\infty} G(t) P_x(x_0, t, r) dt = \liminf_{r \rightarrow 1} G_x(x_0, r).$$

Since $D_{1*}G(x_0) > q > -\infty$ it is sufficient to show

$$(25) \quad \liminf_{r \rightarrow 1} G_x(x_0, r) \geq q.$$

To prove (25) note that

$$\begin{aligned} G_x(x_0, r) &= \int_{-\infty}^{\infty} G(t) P_x(x_0, t, r) dt \quad \text{which by (14)} \\ &= \frac{\exp\left\{-\frac{x_0^2}{2} \left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} \int_{-\infty}^{\infty} G(t) [-x_0(1+r^2) + 2tr] e^{-u^2} dt \end{aligned}$$

where

$$u = \left[\frac{1+r^2}{2(1-r^2)} \right]^{1/2} \left(t - \frac{2x_0r}{1+r^2} \right).$$

Rearranging we find

$$\begin{aligned}
 G_x(x_0, r) &= \frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} (-x_0)(1-r)^2 \\
 (26) \quad &\times \left[\int_0^\infty G(x_0+s) \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right. \\
 &\quad \left. + \int_0^\infty G(x_0-s) \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right] \\
 &+ \frac{\exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} 2r \\
 &\times \left[\int_0^\infty G(x_0+s)s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right. \\
 &\quad \left. - \int_0^\infty G(x_0-s)s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right].
 \end{aligned}$$

The claim is that (26) is $o(1)$ as $r \rightarrow 1$. Because $G(x)$ is bounded by K_5 and using the substitutions $t = [(1+r^2)/2(1-r^2)]^{1/2}[s \pm x_0(1-r)^2/(1+r^2)]$, we find

$$\lim_{r \rightarrow 1} |(26)| \leq \frac{|x_0|}{2\pi^{1/2}} K_5 \lim_{r \rightarrow 1} (1-r)^{1/2} \left(\frac{1-r^2}{1+r^2}\right)^{1/2} \left\{ 2 \int_{-\infty}^{\infty} e^{-t^2} dt \right\} = 0.$$

Therefore $G_x(x_0, r)$ can be written as

$$\begin{aligned}
 o(1) &+ \frac{2r \exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} \\
 (27) \quad &\times \left[\int_0^\infty \{G(x_0+s) - G(x_0-s)\}s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right] \\
 &+ \frac{2r \exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} \\
 &\times \left[\int_0^\infty G(x_0-s)s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right. \\
 &\quad \left. - \int_0^\infty G(x_0-s)s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s-x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right].
 \end{aligned}$$

Using the same substitutions, we get $\lim_{r \rightarrow 1} |(27)| = 0$, since the exponential term dominates. Thus

$$\begin{aligned}
 G_x(x_0, r) &= o(1) + \frac{2r \exp\left\{-\frac{x_0^2}{2}\left(\frac{1-r^2}{1+r^2}\right)\right\}}{\pi^{1/2}(1-r^2)^{3/2}} \\
 &\times \left[\int_0^\infty \{G(x_0+s) - G(x_0-s)\}s \exp\left\{-\left[\frac{1+r^2}{2(1-r^2)}\right]\left[s+x_0\left(\frac{1-r}{1+r^2}\right)^2\right]^2\right\} ds \right].
 \end{aligned}$$

Note it is sufficient to consider the integral only over the interval $s \in [0, \delta]$, for any fixed $\delta > 0$, because the exponential term carries the integral to zero as $r \rightarrow 1$, when s is bounded away from zero. Therefore

$$(28) \quad G_x(x_0, r) = o(1) + \frac{2r \exp \left\{ -\frac{x_0^2}{2} \left(\frac{1-r^2}{1+r^2} \right) \right\}}{\pi^{1/2}(1-r^2)^{3/2}} \\ \times \left[\int_0^\delta \{G(x_0+s) - G(x_0-s)\} s \exp \left\{ -\left[\frac{1+r^2}{2(1-r^2)} \right] \left[s+x_0 \frac{(1-r)^2}{(1+r^2)} \right]^2 \right\} ds \right].$$

Since $D_{1*}G(x_0) > q$, we choose δ such that, for $|s| < \delta$, we have

$$\{G(x_0+s) - G(x_0-s)\} > 2qs.$$

Then (28) dominates

$$\frac{\exp \left\{ -\frac{x_0^2}{2} \left(\frac{1-r^2}{1+r^2} \right) \right\} 2r2q}{\pi^{1/2}(1-r^2)^{3/2}} \left[\int_0^\delta s^2 \exp \left\{ -\left[\frac{1+r^2}{2(1-r^2)} \right] \left[s+x_0 \frac{(1-r)^2}{(1+r^2)} \right]^2 \right\} ds \right].$$

By the usual substitution, and letting $r \rightarrow 1$,

$$\liminf_{r \rightarrow 1} G_x(x_0, r) \geq \frac{4q}{\pi^{1/2}} \int_0^\infty t^2 e^{-t^2} dt = q.$$

LEMMA 10. Let $f(x, r) = \sum_{n=0}^\infty a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \varepsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\varepsilon(r)$ is bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$, and that there is $y_1 \in \mathcal{H}$ such that $-\infty < y_1 \leq f_*(x) \leq f^*(x) < +\infty$ for all x . Set $F(x, r) = -\sum_{n=0}^\infty (a_n/(2n+2)) \Phi_n(x) r^n$ for $0 \leq r < 1$, and suppose $F(x, r) \rightarrow F(x)$, where $F(x)$ is a continuous function on (a, b) . Then

(a) $F(x, r) \rightarrow F(x)$ uniformly on compact subsets of (a, b) as $r \rightarrow 1$;

(b) $\Lambda^* F(x) \geq f_*(x)$ and $f^*(x) \geq \Lambda_* F(x)$ in (a, b) .

Proof. Let $[a_1, b_1] \subset (a, b)$. We wish to show $F(x, r) \rightarrow F(x)$ uniformly in $[a_1, b_1]$ as $r \rightarrow 1$. Let

$$(29) \quad a < a_4 < a_3 < a_2 < a_1 < b_1 < b_2 < b_3 < b_4 < b.$$

Let $\lambda(x)$ be a $\mathcal{C}^\infty$ function such that

$$(30) \quad \begin{aligned} \lambda(x) &= 1 && \text{in } [a_3, b_3], \\ \lambda(x) &= 0 && \text{for } x \notin [a_4, b_4], \\ \lambda(x) &\leq 1. \end{aligned}$$

$$(31) \quad \begin{aligned} &\text{For } x \in (a, b) \text{ define } F_1(x) = \lambda(x)F(x) \\ &\text{and for } x \notin (a, b) \text{ define } F_1(x) = 0. \end{aligned}$$

Then by (29), (30), (31), and the hypothesis we have $F_1(x)$ is continuous, in $\mathcal{L}^1, \mathcal{H}$ and bounded by a constant, say K . Set

$$(32) \quad F_1(x, r) = \int_{-\infty}^\infty P(x, t, r) F_1(t) dt \quad \text{for } 0 \leq r < 1.$$

Then $F_1(x, r)$ satisfies (8) for $0 \leq r < 1$. Since $F_1(x)$ is continuous and in $\mathcal{H}$, we have $F_1(x, r) \rightarrow F_1(x)$ as $r \rightarrow 1$ [2, pp. 398–399]. We will show

(33) $F_1(x, r) \rightarrow F_1(x)$ uniformly on compact subsets of $(-\infty, \infty)$ as $r \rightarrow 1$. Let $[c, d]$ be any closed interval in $(-\infty, \infty)$. For $\varepsilon > 0$ choose r' such that when $r > r'$,

$$\left| \int_{-\infty}^{\infty} P(x, t, r) dt - 1 \right| < \frac{\varepsilon}{2K} \quad \text{on } [c, d].$$

This can be done since $\int_{-\infty}^{\infty} P(x, t, r) dt \rightarrow 1$ uniformly on compact subsets. Hence for $r > r'$ we have

$$\begin{aligned} |F_1(x, r) - F_1(x)| &\leq \left| \int_{-\infty}^{\infty} F_1(t) P(x, t, r) dt - \int_{-\infty}^{\infty} F_1(x) P(x, t, r) dt \right| \\ &\quad + \left| F_1(x) \int_{-\infty}^{\infty} P(x, t, r) dt - F_1(x) \right| \\ &\leq \int_{-\infty}^{\infty} |F_1(t) - F_1(x)| P(x, t, r) dt + K \left| \int_{-\infty}^{\infty} P(x, t, r) dt - 1 \right| \\ &\leq \frac{\varepsilon}{2} + \int_{-\infty}^{x-\delta} + \int_{x-\delta}^x + \int_x^{x+\delta} + \int_{x+\delta}^{\infty} |F_1(t) - F_1(x)| P(x, t, r) dt \end{aligned}$$

where $\delta > 0$ is to be chosen. Since $\lim_{r \rightarrow 1} P(x, t, r) = 0$ uniformly for $|x - t| \geq \delta$ and $|F_1(t) - F_1(x)| \leq 2K$, the integrals $\int_{-\infty}^{x-\delta}$ and $\int_{x+\delta}^{\infty}$ go to zero uniformly on $[c, d]$. Choose δ by the uniform continuity of $F_1(x)$ on $[c, d]$ such that, for the given $\varepsilon > 0$, we have that when $|x - t| < \delta$ then $|F_1(t) - F_1(x)| < \varepsilon/4$. Then

$$\int_x^{x+\delta} |F_1(t) - F_1(x)| P(x, t, r) dt \leq \frac{\varepsilon}{4} \int_x^{x+\delta} P(x, t, r) dt \leq \frac{\varepsilon}{4} \int_{-\infty}^{\infty} P(x, t, r) dt$$

which converges to $\varepsilon/4$ uniformly on compact subsets. The integral $\int_{x-\delta}^x$ similarly becomes smaller than $\varepsilon/4$, proving (33).

Since $F_1(x) = F(x)$ on $[a_1, b_1]$, we have $F_1(x, r) \rightarrow F(x)$ uniformly on $[a_1, b_1]$. It remains to show $F(x, r) - F_1(x, r) \rightarrow 0$ uniformly on $[a_1, b_1]$.

$$H(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n \quad (0 \leq r < 1).$$

Claim:

(34) $H(x, r) \rightarrow H(x)$, a bounded continuous function, and the convergence is uniform in x .

To see this, choose $r_1 > r_2 > 1/2$. Then

$$\begin{aligned} |r_1 H(x, r_1) - r_2 H(x, r_2)| &\leq \frac{1}{2} \left| \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) \int_{r_2}^{r_1} \rho^n d\rho \right| \\ &\leq \frac{1}{2} \int_{r_2}^{r_1} |F(x, \rho)| d\rho \rightarrow 0 \end{aligned}$$

uniformly in x as $r_1, r_2 \rightarrow 1$ since $|F(x, r)| \leq \varepsilon_2(r) \log 1/(1-r)$ by Lemma 6. Thus $H(x, r)$ is bounded for all x and is in $\mathcal{H}$. By [2, Theorems 2 and 8] we also have

$$H(x, r) = \Omega F(x, r) \sim \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n.$$

Observe $H(x, r) - \int_{-\infty}^{\infty} P(x, t, r) H(t) dt$ satisfies all the hypothesis of Lemma 2, which proves $H(x, r) = \int_{-\infty}^{\infty} P(x, t, r) H(t) dt$. Define

$$H_1(x, r) = \Omega F_1(x, r) \quad (0 \leq r < 1).$$

(35) Note $H_1(x, r) \rightarrow H_1(x)$ uniformly for x on compact subsets of $(-\infty, \infty)$ where $H_1(x)$ is a bounded and continuous function, since

$$\begin{aligned} \lim_{r \rightarrow 1} \Omega F_1(x, r) &= -\lim_{r \rightarrow 1} \int_{-\infty}^{\infty} F_1(t, r) k(x, t) dt \\ &= -\int_{-\infty}^{\infty} F_1(t) k(x, t) dt = \Omega F_1(x) \end{aligned}$$

and $F_1(x, r) \rightarrow F_1(x)$ uniformly for x on compact subsets. By Lemma 2 applied to $H_1(x, r) - \int_{-\infty}^{\infty} P(x, t, r) H_1(t) dt$ we find

$$H_1(x, r) = \int_{-\infty}^{\infty} P(x, t, r) H_1(t) dt \quad (0 \leq r < 1).$$

Set

$$H_2(x, r) = H(x, r) - H_1(x, r) \quad (0 \leq r < 1).$$

Then by (34) and (35), setting $H_2(x) = H(x) - H_1(x)$ we have

$H_2(x, r) \rightarrow H_2(x)$ uniformly for x in compact subsets, where $H_2(x)$ is a bounded continuous function in $\mathcal{H}$.

Set

$$F_2(x, r) = F(x, r) - F_1(x, r) \quad (0 \leq r < 1).$$

(36) Then $F_2(x, r) \rightarrow F_2(x)$, where $F_2(x) = 0$ for $x \in [a_3, b_3]$.

Setting

$$F_1(x, r) \sim -\sum_{n=0}^{\infty} \frac{a'_n}{2n+2} \Phi_n(x) r^n,$$

we have

$$H_1(x, r) = \Omega F_1(x, r) \sim \sum_{n=0}^{\infty} \frac{a'_n}{(2n+2)^2} \Phi_n(x) r^n$$

and

$$H_2(x, r) \sim \sum_{n=0}^{\infty} \frac{(a_n - a'_n)}{(2n+2)^2} \Phi_n(x) r^n = \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x) r^n.$$

Claim:

$$(37) \quad H_2(x) \sim \sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x).$$

Since $H_2(x) \in \mathcal{H}$ we have

$$H_2(x) \sim \sum_{n=0}^{\infty} \frac{b_n}{(2n+2)^2} \Phi_n(x)$$

and by the continuity of $H_2(x)$,

$$\sum_{n=0}^{\infty} \frac{b_n}{(2n+2)^2} \Phi_n(x) r^n \rightarrow H_2(x)$$

uniformly on compact subsets. We also have

$$\sum_{n=0}^{\infty} \frac{\tilde{a}_n}{(2n+2)^2} \Phi_n(x) r^n \rightarrow H_2(x)$$

uniformly on compact subsets. Thus applying Lemma 2 to

$$\sum ((b_n - \tilde{a}_n)/(2n+2)^2) \Phi_n(x) r^n$$

we have

$$\tilde{a}_n = b_n \quad \text{for all } n, \text{ proving (37).}$$

By [2, Theorem 4], $\Lambda_* H_2(x) \leq F_2^*(x)$ and $F_{2*}(x) \leq \Lambda^* H_2(x)$. By (36) we have $\Lambda_* H_2(x) \leq 0 \leq \Lambda^* H_2(x)$ on $[a_3, b_3]$. By [2, p. 395], $H_2(x)$ satisfies (17) in $[a_3, b_3]$.

(38) That is, $H_2(x)$ is twice differentiable in $[a_3, b_3]$ and $\Lambda H_2(x) = 0$ in $[a_3, b_3]$. Now $\Lambda H_2(x, r) = \int_{-\infty}^{\infty} \Lambda P(x, t, r) H_2(t) dt$ since $H_2(x) \in \mathcal{H}$ by [2, p. 398]. We will show

$$(39) \quad \Lambda H_2(x, r) \rightarrow 0 \quad \text{uniformly in } [a_1, b_1] \text{ as } r \rightarrow 1.$$

Now $\Lambda P(x, t, r) = -2[rP(x, t, r)]_r = P_{tt}(x, t, r) - (t^2 + 1)P(x, t, r)$. Since $H_2(t)$ is bounded, $\int_{x+\delta}^{\infty} \Lambda P(x, t, r) dt = -2 \int_{x+\delta}^{\infty} [rP(x, t, r)]_r dt \rightarrow 0$ uniformly on compact subsets for any fixed $\delta > 0$, and similarly $\int_{-\infty}^{x-\delta} \Lambda P(x, t, r) dt \rightarrow 0$ uniformly in compact subsets, it is sufficient to restrict ourselves to $\int_{a_2}^{b_2} \Lambda P(x, t, r) H_2(t) dt$.

$$\begin{aligned} \int_{a_2}^{b_2} \Lambda P(x, t, r) H_2(t) dt &= \int_{a_2}^{b_2} [P_{tt}(x, t, r) - (t^2 + 1)P(x, t, r)] H_2(t) dt \\ (40) \quad &= \int_{a_2}^{b_2} P_{tt}(x, t, r) H_2(t) dt - \int_{a_2}^{b_2} (t^2 + 1)P(x, t, r) H_2(t) dt \\ &= H_2(b_2)P_t(x, b_2, r) - H_2(a_2)P_t(x, a_2, r) - H_2'(b_2)P(x, b_2, r) \\ &\quad + H_2'(a_2)P(x, a_2, r) + \int_{a_2}^{b_2} P(x, t, r) \Lambda H_2(t) dt. \end{aligned}$$

The integral in (40) is zero on $[a_1, b_1]$ by (38). The first two terms in (40) go to zero uniformly on $[a_1, b_1]$ by (29), since b_2 is not contained in $[a_1, b_1]$, and $P_t(x, t, r) \rightarrow 0$ uniformly for $|x - t| \geq \delta > 0$.

Thus (39) will be proved if we show

(41) $H'_2(b_2)P(x, b_2, r)$ and $H'_2(a_2)P(x, a_2, r) \rightarrow 0$ uniformly on $[a_1, b_1]$. Since $G(x, r) \rightarrow G(x)$ uniformly in x , and $G(x)$ is bounded, we have

$$\lim_{r \rightarrow 1} \int_0^x G(t, r) dt = \int_0^x \lim_{r \rightarrow 1} G(t, r) dt = \int_0^x G(t) dt.$$

Hence

$$\begin{aligned} \lim_{r \rightarrow 1} \int_0^x \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(t) r^n &= \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} \int_0^x \frac{\Phi'_n(t) r^n}{(2n+2)^2} dt \\ &= \lim_{r \rightarrow 1} \left[\sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n - \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(0) r^n \right] \\ &= H(x) - H(0). \end{aligned}$$

This means $H(x) - H(0) = \int_0^x G(t) dt$ or $H'(x) = G(x)$. Defining

$$G_1(x, r) = \Omega \left[\frac{\partial}{\partial x} F_1(x, r) \right]$$

we get $G_1(x, r) \rightarrow G_1(x)$ uniformly, where $G_1(x)$ is bounded and continuous.

Setting $G_2(x) = G(x) - G_1(x)$ we see that $G_2(x)$ is a bounded and continuous function since $G(x)$ and $G_1(x)$ are. Thus (41) is proved if we can show

$G_2(b_2)P(x, b_2, r)$ and $G_2(a_2)P(x, a_2, r)$ converge uniformly to zero on $[a_1, b_1]$ as $r \rightarrow 1$.

But $G_2(x)$ is bounded and $b_2 \notin [a_1, b_1]$. Thus $P(x, b_2, r)$ and $P(x, a_2, r) \rightarrow 0$ uniformly on $[a_1, b_1]$. This proves (39). Since $\Delta H_2(x, r) = F_2(x, r)$ we have:

$F(x, r) - F_1(x, r) \rightarrow 0$ uniformly on $[a_1, b_1]$, which completes the proof that $F(x, r) \rightarrow F(x)$ uniformly on compact subsets of (a, b) .

Since $F_1(x) \in \mathcal{H}$, we have

$$F_1(x) \sim \sum_{n=0}^{\infty} \frac{a'_n}{2n+2} \Phi_n(x)$$

where $a'_n = \int_{-\infty}^{\infty} F_1(t) \Phi_n(t) dt$. Then

$$\begin{aligned} F_1(x, r) &= \int_{-\infty}^{\infty} F_1(t) P(x, t, r) dt = \int_{-\infty}^{\infty} F_1(t) \sum_{n=0}^{\infty} \Phi_n(x) \Phi_n(t) r^n dt \\ &= \sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} \Phi_n(t) F_1(t) dt \right] \Phi_n(x) r^n \\ &= - \sum_{n=0}^{\infty} \frac{a'_n}{2n+2} \Phi_n(x) r^n, \end{aligned}$$

which is consistent with our previous notation.

By [2, Theorem 4] we have

$$(42) \quad \Lambda^* F_1(x) \geq \liminf_{r \rightarrow 1} \sum_{n=0}^{\infty} a'_n \Phi_n(x) r^n = \liminf_{r \rightarrow 1} [-2r F_1(x, r)]_r.$$

Since $F_1(x) = F(x)$ on $[a_3, b_3]$, we have

$$(43) \quad \Lambda^* F_1(x) = \Lambda^* F(x) \quad \text{for } x \in [a_1, b_1].$$

Repeating the identical argument which showed $\Lambda H_2(x, r) \rightarrow 0$ uniformly on $[a_1, b_1]$ as $r \rightarrow 1$, and using the properties that $\Lambda P(x, t, r)$ and $[\Lambda P(x, t, r)]_t$ converge uniformly to zero for $|x - t| \geq \delta > 0$, we have $\Lambda[\Lambda H_2(x, r)] \rightarrow 0$ uniformly on $[a_1, b_1]$.

This means $[rF(x, r)]_r - [rF_1(x, r)]_r \rightarrow 0$ uniformly on compact subsets of (a, b) .

This gives, for $x_0 \in (a, b)$,

$$(44) \quad \liminf_{r \rightarrow 1} -2[rF(x_0, r)]_r = \liminf_{r \rightarrow 1} -2[rF_1(x_0, r)]_r.$$

But

$$(45) \quad \liminf_{r \rightarrow 1} -2[rF(x_0, r)]_r = f_*(x_0).$$

Thus combining (42), (43), (44), and (45) we have $\Lambda^* F(x) \geq f_*(x)$. The proof that $f^*(x) \geq \Lambda^* F(x)$ follows by a change of sign.

Λ -convex functions. The function $F(x)$ is said to be Λ -convex in (a, b) if the equations $y(c) = F(c)$ and $y(d) = F(d)$ for $a < c < d < b$ imply that $F(x) \leq y(x)$ for $c < x < d$, where $y(x)$ is a solution of (17).

Generalized convex functions of this type have been studied by Beckenbach and Bing [8][9]. In particular, if F is Λ -convex in (a, b) , then F is continuous in (a, b) and $F(a+)$, $F(b-)$ exist (as finite numbers or $\pm\infty$).

LEMMA 11. *Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \epsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\epsilon(r)$ is bounded and $\epsilon(r) = o(1)$ as $r \rightarrow 1$, and that there is $y_1 \in \mathcal{H}$ such that $-\infty < y_1(x) \leq f_*(x) \leq f^*(x) < +\infty$ for all x . Set $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ for $0 \leq r < 1$, and suppose $F(x, r) \rightarrow F(x)$, where $F(x)$ is a continuous function on (a, b) . Then $F(x)$ is continuous in $[a, b]$.*

Proof. It will be shown that:

$$(46) \quad F(a+) \text{ exists (finite or } \pm\infty) \text{ and}$$

$$(47) \quad F(a) = F(a+) \text{ and is finite.}$$

By symmetry we then have $F(x)$ is continuous at b . By hypothesis we have $-\infty < y(x) \leq f_*(x) \leq f^*(x) < +\infty$. Let $u(x)$ be a function chosen as in [2, p. 396] such that

$u(x)$ is upper semicontinuous,

$u(x) \leq y(x) < +\infty$ for all x , and

$u(x) \in \mathcal{H}$.

Set $W(x) = F(x) - \Omega u(x)$. Then $\Lambda^* W(x) \geq \Lambda^* F(x) - \Lambda^* \Omega u(x) \geq f_*(x) - u(x) \geq y(x) - u(x) \geq 0$ by [2, p. 392] and Lemma 10.

Applying [2, p. 395], we have $W(x)$ is Λ -convex in (a, b) . Since W is Λ -convex to the right of a , this means

$$W(a+) \text{ exists as a finite number or } \pm\infty.$$

By the continuity of $\Omega u(x)$, this proves (46). To prove (47), we select $a < a' < b' < b$. Since

$$G(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x) r^n \quad \text{for } 0 \leq r < 1,$$

and $\Phi_n(x)$ satisfies (16),

$$\begin{aligned} G_x(x, r) &= \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi'_n(x) r^n \\ &= - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^n + (x^2+1) \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n \\ &= F(x, r) + (x^2+1)H(x, r). \end{aligned}$$

Thus

$$G(a+h, r) = \int_{a'}^{a+h} F(t, r) dt + \int_{a'}^{a+h} (t^2+1)H(t, r) dt + G(a', r).$$

Taking the limit as $r \rightarrow 1$, and applying Lemmas 7 and 10 we have

$$G(a+h) - G(a') = \int_{a'}^{a+h} F(t) dt + \int_{a'}^{a+h} (t^2+1)H(t) dt.$$

Since $F(a+)$ exists (as finite or $\pm\infty$) we can take the limit as $a' \rightarrow a$ and get

$$G(a+h) - G(a) = \int_a^{a+h} F(t) dt + \int_a^{a+h} (t^2+1)H(t) dt.$$

Thus

$$\lim_{h \rightarrow 0^+} \frac{G(a+h) - G(a)}{h} = F(a+) + (a^2+1)H(a)$$

and this must be finite or $\pm\infty$.

But G is smooth by Lemma 7, which means that the right-hand and left-hand limits must agree. Therefore $G'(a) = F(a+) + (a^2+1)H(a)$. Applying Lemma 9

$$\begin{aligned} G'(a) &= \lim_{r \rightarrow 1} [F(a, r) + (a^2+1)H(a, r)] \\ &= F(a) + (a^2+1)H(a). \end{aligned}$$

This means $F(a) = F(a+)$. By Lemma 8, $F(a)$ is everywhere finite, proving (47).

LEMMA 12. Let $f(x, r) = \sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ exist for $-\infty < x < +\infty$ and $0 \leq r < 1$. Suppose that $|f(x, r)| \leq \epsilon(r)/(1-r)$ for $-\infty < x < +\infty$ and $0 \leq r < 1$, where $\epsilon(r)$ is

bounded and $\varepsilon(r) = o(1)$ as $r \rightarrow 1$, and that there is $y_1 \in \mathcal{H}$ such that $-\infty < y_1(x) \leq f_*(x) \leq f^*(x) < +\infty$ for all x . Set

$$F(x, r) = - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^n \quad \text{for } 0 \leq r < 1.$$

Then $F(x)$ is continuous in $(-\infty, \infty)$.

Proof. Let $R > 0$. It is sufficient to show $F(x)$ is continuous on $(-R, R)$. In Lemma 8 we saw that the function

$$F(x, r) = - \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x) r^n, \quad 0 \leq r < 1,$$

was Poisson summable at all points x to a function $F(x)$. Let E be the set of points at which $F(x)$ is discontinuous. We propose to show that E is the empty set.

Choose increasing sequence $\{r_n\}_{n=1}^{\infty}$ with $r_1 = 0$ and $r_n \rightarrow 1$ as $n \rightarrow \infty$, and with the property that if $r_n \leq r \leq r_{n+1}$, then $|f(x, r) - f(x, r_n)| \leq 1$ for $|x| \leq 2R$. Since $\limsup_{n \rightarrow \infty} |f(x, r_n)| \leq f^*(x) < +\infty$, given any closed nondegenerate interval $\bar{J}$ contained in the interior of $(-R, R)$ there exists, by [10, Lemma 4, p. 645], a constant M and a closed nondegenerate subinterval $\bar{J}_1$ of $\bar{J}$ such that $|f(x, r_n)| \leq M$ for x in $\bar{J}_1$. But then $|f(x, r)| \leq M + 1$ for $0 \leq r < 1$ and x in $\bar{J}_1$. This fact implies that $F(x, r) \rightarrow F(x)$ uniformly for x in $\bar{J}_1$, and therefore that $F(x)$ is continuous in $\bar{J}_1$. We conclude that the set E is nondense in $(-R, R)$.

Suppose E contains an isolated point z_0 . Then there is an $h > 0$ such that $-R < z_0 - h < z_0 + h < R$ and such that $F(x)$ is continuous in each of the open intervals $(z_0 - h, z_0)$ and $(z_0, z_0 + h)$. Applying Lemma 11, $F(x)$ is continuous at z_0 , thus E contains no isolated points.

If E is not vacuous, its closure $\bar{E}$ is perfect. We again apply [10, Lemma 4, p. 645]. Let $\pi = J \cdot \bar{E}$, where J is a segment, be a portion of $\bar{E}$ on which $F(x)$ is continuous. Let (a, b) be a segment contiguous to π . On (a, b) , $F(x)$ is continuous. Thus by Lemma 11, $F(x)$ is continuous on $[a, b]$. If $x_0 \in E$ is a left-hand endpoint or right-hand endpoint of one of the contiguous intervals of π , this gives $F(x)$ is continuous at x_0 . Thus we may suppose x_0 is the limit of left-hand endpoints of contiguous intervals. Applying Lemma 11 to each contiguous interval (c, d) , we have $F(c) = F(c+)$ and $F(d) = F(d-)$. Defining $W(x) = F(x) - \Omega u(x)$ as in Lemma 11, we have $W(x)$ is Λ -convex in each interval and $W(c+) = W(c)$, $W(d-) = W(d)$. It follows that if x_0 is any point of π , then $W(x)$ is upper semicontinuous at x_0 . Then $W(x)$ is upper semicontinuous in J , which implies that $W(x)$ is Λ -convex and therefore continuous in J . Thus E is vacuous and $F(x)$ is continuous in $(-R, R)$.

The above proof is very similar to the analogous theorem for trigonometric series [3, p. 355].

CHAPTER II

Major theorems.

THEOREM I. *Let the series $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ converge, for $0 \leq r < 1$, to $f(x, r)$. Suppose that*

- (i) $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \rightarrow 1$;
- (ii) there is a function $y_1 \in \mathcal{H}$ such that $-\infty < y_1(x) \leq f_*(x) \leq f^*(x) < +\infty$ for all x ;
- (iii) there is a function $y_2 \in \mathcal{H}$ such that $-\infty < y_2(x) \leq F(x)$ for all x .

Then the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ is Poisson summable almost everywhere, and is the Hermite series of its Poisson sum.

Proof. As in Lemma 10, we set

$$H(x, r) = \Omega F(x, r) = \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n.$$

In (34) we showed $H(x, r)$ converged to the continuous and bounded function $H(x)$ as $r \rightarrow 1$. Repeating the argument of (37), we have

$$H(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x).$$

By [2, Theorem 5] and hypothesis (iii), the series $-\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x)$ is Poisson summable almost everywhere (we already proved in Lemma 8 this was everywhere to $F(x)$) and

$$(48) \quad F(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{2n+2} \Phi_n(x).$$

Again applying [2, Theorem 5] to (48), because of Lemma 12 and hypothesis (ii), we have that the series $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ is Poisson summable almost everywhere and is the Hermite series of its Poisson sum.

This result extends Rudin's results [2, Theorem 6], because the condition $a_n = o(n^{1/4})$ implies $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \rightarrow 1$. In fact, since

$$P(x, x, r) = \sum_{n=0}^{\infty} \Phi_n(x) \Phi_n(x) r^n = O(1/(1-r)^{1/2})$$

uniformly in x as $r \rightarrow 1$, and $\sum_{n=0}^{\infty} o(n^{1/2}) r^n = o((1/(1-r))^{3/2})$ as $r \rightarrow 1$, we have

$$\begin{aligned} |f(x, r)| &= \left| \sum_{n=0}^{\infty} (a_n r^{n/2}) (\Phi_n(x) r^{n/2}) \right| \leq \left(\sum_{n=0}^{\infty} a_n^2 r^n \right)^{1/2} \left(\sum_{n=0}^{\infty} \Phi_n^2(x) r^n \right)^{1/2} \\ &\leq \left[o\left(\frac{1}{1-r}\right)^{3/2} \right]^{1/2} \left[O\left(\frac{1}{(1-r)^{1/2}}\right) \right]^{1/2} = o\left(\frac{1}{1-r}\right) \end{aligned}$$

uniformly in x as $r \rightarrow 1$. If $a_n = o(n^{1/4})$, the condition that $F(x) \in \mathcal{H}$ follows from the Riesz-Fischer Theorem.

THEOREM II. Let the series $\sum_{n=0}^{\infty} a_n \Phi_n(x) r^n$ converge, for $0 \leq r < 1$, to $f(x, r)$. Suppose that

(i) $|f(x, r)| = o(1/(1-r))$ uniformly in x as $r \rightarrow 1$;

(ii) $\lim_{r \rightarrow 1} f(x, r) = 0$ for all x .

Then $a_n = 0$ for all n .

Proof. By Lemmas 10 and 12, $F(x, r) = -\sum_{n=0}^{\infty} (a_n/(2n+2)) \Phi_n(x) r^n$ converges as $r \rightarrow 1$ uniformly on compact subsets. By Lemma 10(b) we also have that $\Lambda^* F(x) \geq 0 \geq \Lambda_* F(x)$. Applying [2, Corollary 6.3], $F(x)$ is a solution to (17) for all x , that is $F''(x) - (x^2 + 1)F(x) = 0$. By (16),

$$\Phi_n(x) = (x^2 + 1) \frac{\Phi_n(x)}{2n+2} - \frac{\Phi_n''(x)}{2n+2}.$$

Thus

$$-(x^2 + 1) \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n + \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n''(x) r^n \rightarrow F(x)$$

uniformly on compact subsets. Integrating twice,

$$\sum_{n=0}^{\infty} \frac{a_n r^n}{(2n+2)^2} \int_0^x \int_0^s \Phi_n''(t) dt ds - \int_0^x \int_0^s (t^2 + 1) H(t, r) dt ds$$

converges to $\int_0^x \int_0^s F(t) dt ds$, where $H(x, r) = \Omega F(x, r)$, ($0 \leq r < 1$), as defined in Lemma 10. Since

$$\begin{aligned} \int_0^x \int_0^s \Phi_n''(t) dt ds &= \int_0^x [\Phi_n'(s) - \Phi_n'(0)] ds = \Phi_n(x) - \Phi_n(0) - x\Phi_n'(0), \\ (49) \quad \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(x) r^n - \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n(0) r^n - x \sum_{n=0}^{\infty} \frac{a_n}{(2n+2)^2} \Phi_n'(0) r^n \\ &\quad - \int_0^x \int_0^s (t^2 + 1) H(t, r) dt ds \quad \text{converges to} \quad \int_0^x \int_0^s F(t) dt ds. \end{aligned}$$

But (49) is known to converge to

$$H(x) - H(0) - xG(0) - \int_0^x \int_0^s (t^2 + 1) H(t) dt ds.$$

Therefore

$$(50) \quad H(x) - H(0) - xG(0) - \int_0^x \int_0^s (t^2 + 1) H(t) dt ds = \int_0^x \int_0^s F(t) dt ds.$$

Since H and G are bounded (Lemmas 7 and 10), the left-hand side of (50) is $O(x^4)$ for x large.

Since the function $\beta(x) = e^{x^2/2} \int_{-\infty}^x e^{-u^2} du$ and $\beta(-x) = e^{x^2/2} \int_{-\infty}^{-x} e^{-u^2} du$ are linearly independent solutions to (17), we have $F(x) = c_1 \beta(x) + c_2 \beta(-x)$. Let

$$g_1(x) = \frac{c_1}{x^4} \int_0^x \int_0^s \beta(t) dt ds \quad \text{and} \quad g_2(x) = \frac{c_2}{x^4} \int_0^x \int_0^s \beta(-t) dt ds.$$

As $x \rightarrow +\infty$, $g_1(x) \rightarrow +\infty$ and $g_2(x) \rightarrow 0$. So if $\int_0^x \int_0^s F(t) dt ds$ is $O(x^4)$, then $c_1=0$. As $x \rightarrow -\infty$, $g_2(x) \rightarrow +\infty$ and $g_1(x) \rightarrow 0$. So if $\int_0^x \int_0^s F(t) dt ds$ is $O(x^4)$, then $c_2=0$. Thus $F(x)=0$ for all x . Theorem I now applies and gives us $0 \sim \sum_{n=0}^{\infty} a_n \Phi_n(x)$ or $a_n=0$ for all n .

We cannot replace the condition $o(1/(1-r))$ by $O(1/(1-r))$ in Theorems I and II. To illustrate this, if we differentiate $e^{x^2/2}P(x, 0, r)$ we get

$$f(x, r) = \sum_{n=0}^{\infty} (2n+2)^{1/2} \Phi_{n+1}(0) \Phi_n(x) r^n = \frac{-2rx}{\pi^{1/2}(1-r^2)^{3/2}} \exp \left\{ -\frac{x^2}{2} \left(\frac{1-r^2}{1+r^2} \right) \right\} \quad (0 \leq r < 1).$$

This is a series of Hermite functions for which $f(x, r) \rightarrow 0$ for all x , $F(x) \in \mathcal{H}$, and $f(x, r) = O(1/(1-r))$ uniformly in x as $r \rightarrow 1$. This series is clearly not the zero series, and we conclude the above condition is in a certain sense best possible.

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