A CHARACTERIZATION OF $M(G)$

BY
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Let G be a locally compact group (=locally compact Hausdorff topological group). By the measure algebra $M(G)$ of G , we mean the Banach $*$ -algebra of bounded regular Borel measures on G . By the group algebra $L^1(G)$ of G , we mean the algebra of (equivalence classes of) complex-valued functions on G , summable with respect to the Haar measure on G .

Recently work has been done on the problem of characterizing those Banach algebras which are isometric and isomorphic to group algebras or measure algebras. In particular, Rieffel [6] has obtained characterizations of $M(G)$ and $L^1(G)$ for G abelian, and Greenleaf [4] has characterized $L^1(G)$ for G compact.

The main result of this paper is Theorem 1, which characterizes those Banach algebras which are isomorphic and isometric to measure algebras. For definitions and basic results regarding $M(G)$ the reader is referred to [5], and for results concerning topological vector spaces to [1].

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THEOREM 1. *Let A be a Banach algebra, S its unit ball, and S^e the set of extreme points of S . Suppose that*

- (1) *there is a Banach space E such that A is the dual of E ,*
- (2) *for each x in A , the mappings $y \mapsto xy$ and $y \mapsto yx$ are $\sigma(A, E)$ -continuous,*
- (3) *if $x \in A$ is such that $xy=0$ for all y in A , then $x=0$,*
- (4) *if $x \in S^e$, then the mapping $y \mapsto xy$ is an isometry of A onto itself,*
- (5) *$S^e \cup \{0\}$ is $\sigma(A, E)$ -closed,*
- (6) *there is a nonzero multiplicative linear functional p on A with the following properties: if $G = \{x \in S^e : p(x) = 1\}$ then*
 - (i) *for each f in E there is a g in E such that $x(g) = x(f)^-$ (the complex conjugate of $x(f)$) for all x in G ,*
 - (ii) *for $f, g \in E$, there is an $h \in E$ such that $x(h) = x(f)x(g)$ for all $x \in G$.*

Then G is a locally compact group in the relative $\sigma(A, E)$ -topology and A is isomorphic and isometric to $M(G)$. If S^e is $\sigma(A, E)$ -closed then G is compact. G is unique to within topological isomorphism. Conversely if G is a locally compact group then $M(G)$ satisfies (1)–(6).

Proof. We begin by showing that the measure algebra of a locally compact group satisfies (1)–(6). It is well known that $M(G)$ is the dual of $C_0(G)$ (the Banach

space of continuous complex-valued functions which "vanish at infinity"). Thus (1) is satisfied. By Proposition 1 of [7], (2) is satisfied. $M(G)$ has a unit; consequently (3) holds.

For x in G let ε_x be the Dirac measure at x , and let G^ε be the set of all Dirac measures in $M(G)$. G^ε_σ is homeomorphic to G where σ is the relative $\sigma(M(G), C_0(G))$ -topology. Let T be the complex numbers of absolute value 1. By Proposition 3 of [7], $TG^\varepsilon = S^\varepsilon$, and an easy calculation shows that (4) is satisfied. If G is compact, then, since G^ε_σ is homeomorphic to G and T is compact, it follows that S^ε is $\sigma(M(G), C_0(G))$ -compact and hence $\sigma(M(G), C_0(G))$ -closed. If G is not compact, let G^∞ be the one point compactification of G . If $(x_j : j \in J)$ is a net in G^∞ which converges to ∞ , then $f(x_j) \rightarrow 0$ for each f in $C_0(G)$ so that $\varepsilon_{x_j} \xrightarrow{\sigma} 0$. Thus the mapping $x \mapsto \varepsilon_x$ has a continuous extension to G^∞ and this extension is one-one and therefore a homeomorphism of G^∞ onto $G^\varepsilon \cup \{0\}$. Thus $TG^\varepsilon \cup \{0\} = S^\varepsilon \cup \{0\}$ is $\sigma(M(G), C_0(G))$ -compact and hence $\sigma(M(G), C_0(G))$ -closed. Thus (5) is satisfied.

We now show that (6) is satisfied. For this let p be the linear functional on $M(G)$ defined by $p(\mu) = \mu(G)$. It follows easily that p is multiplicative. Note that $G^\varepsilon = \{\mu \in S^\varepsilon : p(\mu) = 1\}$. Thus, choosing f^- (the complex conjugate of f) for g , (i) is satisfied. (ii) is satisfied since $C_0(G)$ is an algebra.

We now prove the direct statements of the theorem. We shall divide the proof into a number of assertions.

I. A has a unit u ; $u \in S^\varepsilon$ and S^ε is a group.

Since A is the dual of a Banach space, S^ε is not empty. Let $x \in S^\varepsilon$. Then by (4) the mapping T_x defined by $T_x y = xy$ is an isometry of A onto itself, hence T_x^{-1} exists and is an isometry. Put $u = T_x^{-1}x$; it follows that u is a left unit, and it is easily seen using (3) that u is a unit and $\|u\| = 1$. Kakutani has shown that the unit of a Banach algebra is necessarily in S^ε . To show that S^ε is a group it suffices to show that if $x \in A$, and the mapping $T_x : y \mapsto xy$ is an isometry of A onto itself, then $x \in S^\varepsilon$. For this, suppose that $y, z \in S$ and $x = \alpha y + (1 - \alpha)z$, $0 < \alpha < 1$. Then $u = \alpha T_x^{-1}y + (1 - \alpha)T_x^{-1}z$. Since T_x is an isometry, T_x^{-1} is also an isometry, consequently $T_x^{-1}y, T_x^{-1}z \in S$, and since $u \in S^\varepsilon$, we therefore have $y = z = x$.

II. For any $x \in S^\varepsilon$, $p(x)^- = p(x^{-1})$ and $|p(x)| = 1$.

Since p is a multiplicative linear functional on a Banach algebra with a unit u , we have $p(u) = 1$ and $\|p\| = 1$. Let $x \in S^\varepsilon$. Then $|p(x)| \leq 1$, and, since $x^{-1} \in S^\varepsilon$, $|p(x^{-1})| \leq 1$. If $|p(x)| < 1$, then $p(u) = p(xx^{-1}) = |p(x)| |p(x^{-1})| < 1$, an absurdity, so that $|p(x)| = 1$. Then $p(x)p(x)^- = 1 = p(xx^{-1}) = p(x)p(x^{-1})$ and therefore $p(x)^- = p(x^{-1})$.

We now topologize G with the relative $\sigma(A, E)$ -topology and, with respect to this topology, show that G is a locally compact group.

III. G is a locally compact group and, if S^ε is $\sigma(A, E)$ -closed, G is compact.

Let T be the complex numbers of absolute value 1, and consider the mapping g of $T \times G$ into S^ε defined by $g(a, x) = ax$. It is easily verified that this mapping is one-one and onto. Let S^ε_g be S^ε taken with the relative $\sigma(A, E)$ -topology. Since g is

the restriction to $T \times G$ of the mapping $(\alpha, x) \rightarrow \alpha x$ of $C \times A_\sigma \rightarrow A_\sigma$, it follows that g is continuous. It is easily verified that g is open, and consequently a homeomorphism of $T \times G$ onto S_σ^ε . By (5) $S^\varepsilon \cup \{0\}$ is $\sigma(A, E)$ -closed, consequently, since S is $\sigma(A, E)$ -compact, $S^\varepsilon \cup \{0\}$ is $\sigma(A, E)$ -compact. Therefore S_σ^ε is locally compact. Since $T \times G$ is homeomorphic to S_σ^ε , G is locally compact. Clearly if S^ε is $\sigma(A, E)$ -closed then G is compact. By (2) multiplication is $\sigma(A, E)$ -continuous in each variable separately so, by a theorem of Ellis [3], G is a locally compact group.

For $f \in E$, let $\hat{f}$ be the function on G defined by $\hat{f}(x) = x(f)$.

IV. $f \mapsto \hat{f}$ is a norm decreasing linear mapping of E into $C_0(G)$.

It is clear that this mapping is linear and, since $\|x\| \leq 1$ for x in G , we have $|\hat{f}(x)| \leq \|f\|$ so that $\|\hat{f}\| \leq \|f\|$. Note that $\hat{f}$ is continuous. To show that $\hat{f}$ is in $C_0(G)$, first note that if S_σ^ε is compact then G is compact so that $C_0(G) = C(G)$. If G is not compact then S^ε is not compact, so that 0 is a $\sigma(A, E)$ -adherence point of S^ε (since $S^\varepsilon \cup \{0\}$ is $\sigma(A, E)$ -compact). Now let $\varepsilon > 0$ be given and suppose $f \neq 0$. Clearly $U = \{x \in A : |x(f)| < \varepsilon\} \cap (S^\varepsilon \cup \{0\})$ is an open $\sigma(A, E)$ -neighborhood of 0 in $S^\varepsilon \cup \{0\}$ so that $W = S^\varepsilon \setminus U$ is compact in S^ε . Since $T \times G$ is homeomorphic to S^ε , $g^{-1}(W)$ is compact in $T \times G$. Let K be the image of $g^{-1}(W)$ by the projection mapping $T \times G \rightarrow G$, then K is compact in G . Thus for $f \in E$, we have found a compact set K such that $|\hat{f}(x)| < \varepsilon$ for $x \notin K$ because $G \setminus K \subseteq U$.

V. Let $\hat{E}$ be the image of E in $C_0(G)$ under the mapping $f \mapsto \hat{f}$. $\hat{E}$ is dense in $C_0(G)$.

If for $x, y \in G$, $\hat{f}(x) = \hat{f}(y)$ for all $\hat{f} \in \hat{E}$, then $x(f) = y(f)$ for all $f \in E$; hence $x = y$, so that $\hat{E}$ separates the points of G . If $x \in G$, then $x \neq 0$, so there is an $f \in E$ such that $x(f) \neq 0$, i.e., $\hat{f}(x) \neq 0$. Thus, given $x \in G$, we can find an $\hat{f} \in \hat{E}$ such that $\hat{f}(x) \neq 0$. Further, for any $f \in E$, by (6)(i), there is a $g \in E$ such that $x(g) = x(f)^{-}$; i.e., $\hat{g}(x) = \hat{f}(x)^{-}$ for all $x \in G$. By (6)(ii), $\hat{E}$ is a subalgebra of $C_0(G)$. Thus $\hat{E}$ is a subalgebra of $C_0(G)$ which separates the points of G , does not vanish at any point of G , and is closed under complex conjugation; hence the Stone-Weierstrass theorem applies, and we may conclude that $\hat{E}$ is dense in $C_0(G)$.

Let θ be the adjoint of the mapping of $f \mapsto \hat{f}$, i.e., $\theta\mu(f) = \mu(\hat{f})$ for $\mu \in C_0(G)'$ $= M(G)$ and $f \in E$. Note that $\theta\varepsilon_x = x$, so that by the linearity of θ we have

$$(*) \quad \theta\left(\sum_1^n a_i \varepsilon_{x_i}\right) = \sum_1^n a_i x_i.$$

VI. θ is a norm decreasing one-one linear mapping of $M(G)$ into A , and θ is continuous as a mapping of $M(G)_\sigma$ into A_σ .

This follows from IV, V, and the general properties of adjoint mappings.

Let S^M be the unit ball of $M(G)$.

VII. $(S^M)_\sigma$ is homeomorphic to S_σ , and θ is an isometry of $M(G)$ onto A .

Since $(S^M)_\sigma$ is compact, and θ is one-one and continuous, to prove the first assertion, it suffices to show that $\theta(S^M) = S$. By VI, $\theta(S^M) \subseteq S$ so it suffices to show that $\theta(S^M) \supseteq S$. For this let $x \in S$, then since S is convex and $\sigma(A, E)$ -compact, the

Krein-Milman theorem [1, Chapitre 2] applies and there is a net $(x_j : j \in J)$ such that $x_j \xrightarrow{\sigma} x$ and $x_j = \sum_{i=1}^{n_j} a_{i,j} x_{i,j}$, where $\sum_{i=1}^{n_j} a_{i,j} = 1$, $a_{i,j} > 0$ and the $x_{i,j}$ are extreme points of the unit ball in A . Putting $y_{i,j} = x_{i,j}/p(x_{i,j})$, we have $y_{i,j} \in G$ for any i, j . Considering $\mu_j = \sum_{i=1}^{n_j} a_{i,j} p(x_{i,j}) \varepsilon_{y_{i,j}}$ we see that μ_j is an element of the unit ball S^M of $M(G)$ for each j and $T\mu_j = x_j$ by (*). Since S^M is $\sigma(M(G), C_0(G))$ -compact, there is a $\mu \in S^M$ and there is a subnet $(\mu_{j(i)}) \subseteq (\mu_j)$ such that $\mu_{j(i)} \xrightarrow{\sigma} \mu$. Since θ is continuous as a map of $M(G)_\sigma$ into A_σ we have that $\theta\mu_{j(i)} \xrightarrow{\sigma} \theta\mu$, $(x_{j(i)}) \subseteq (x_j)$, and $\theta\mu_{j(i)} = x_{j(i)} \xrightarrow{\sigma} x$, it follows that $\theta\mu = x$. This shows θ maps S^M onto S and hence is a homeomorphism because S^M and S are compact. Hence θ maps $M(G)$ onto A .

To show that θ is an isometry suppose there is a $\mu \in M(G)$ such that $\|\theta\mu\| < \|\mu\|$. Since S^M is mapped onto S and θ is one-to-one, $\|\theta^{-1}\| \leq 1$. Thus $\|\mu\| = \|\theta^{-1}\theta\mu\| \leq \|\theta\mu\| < \|\mu\|$, a contradiction.

Finally in order to show that A is isometric and isomorphic to $M(G)$, we have to show the following:

VIII. For $\mu, \lambda \in M(G)$, $\theta(\mu * \lambda) = \theta\mu\theta\lambda$.

First let $\mu, \lambda \in V$ (the linear span of the Dirac measures). Then $\mu = \sum_{i=1}^n a_i \varepsilon_{x_i}$ and $\lambda = \sum_{i=1}^n b_i \varepsilon_{y_i}$, where a_i, b_i are complex numbers. We have

$$\mu * \lambda = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \varepsilon_{x_i} * \varepsilon_{y_j} = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \varepsilon_{x_i y_j}$$

so that by (*);

$$(*) \quad \theta(\mu * \lambda) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j x_i y_j = \left(\sum a_i x_i\right) \left(\sum b_j y_j\right) = (\theta\mu)(\theta\lambda).$$

Now let $\mu, \lambda \in M(G)$, then by Proposition 1 of [7] there are nets $(\mu_j : j \in J)$, $(\lambda_k : k \in K)$ in V such that $\mu_j \xrightarrow{\sigma} \mu$ and $\lambda_k \xrightarrow{\sigma} \lambda$. Then since multiplication in $M(G)$ is separately $\sigma(M(G), C_0(G))$ -continuous, we have $\mu * \lambda = \lim_j (\lim_k \mu_j * \lambda_k)$. Since θ is continuous $M(G)_\sigma \rightarrow A_\sigma$ we have

$$\begin{aligned} \theta(\mu * \lambda) &= \lim_j \theta \left(\lim_k \mu_j * \lambda_k \right) = \lim_j \left(\lim_k \theta(\mu_j * \lambda_k) \right) \\ &= \lim_j \left(\lim_k \theta\mu_j \theta\lambda_k \right) \end{aligned}$$

by the (*) above. By hypothesis (2) of the theorem, multiplication in A is $\sigma(A, E)$ -continuous in each variable separately, thus $\lim_j (\lim_k T\mu_j T\lambda_k) = \lim_j T\mu_j T\lambda = T\mu T\lambda$ which proves the assertion.

Thus A is isometric and isomorphic to $M(G)$. The uniqueness of G is a consequence of Theorem 1 of [7].

We now examine those Banach algebras which satisfy conditions (1)–(5) of Theorem 1. We begin with some preliminary results. The following proposition is due to Greenleaf, and has appeared in [4] in a less general form.

PROPOSITION 1. *Let E be a Banach space and let N be a $\sigma(E', E)$ -closed subspace of the dual E' of E . Let π be the canonical mapping $E' \rightarrow E'/N$; then π maps the unit ball of E' onto the unit ball of E'/N .*

Proof. Since N is $\sigma(E', E)$ -closed, N is norm closed so that E'/N is a Banach space with the norm of an element $\pi(x)$ given by

$$(1) \quad \|\pi(x)\| = \inf \{\|x+n\| : n \in N\} \leq \|x\|.$$

Hence $\|x\| \leq 1$ implies $\|\pi(x)\| \leq 1$, i.e. π maps the unit ball of E' into the unit ball of E'/N . To show that π is onto, let $\pi(x) \in E'/N$ with $\|\pi(x)\| \leq 1$. By (1) there are $x_j \in x + N$ such that

$$\|x_j\| \leq \|\pi(x)\| + 1/j\|x\|, \quad j = 1, 2, \dots$$

The sequence $(x_j : j=1, 2, \dots)$ is then norm bounded and therefore contained in a $\sigma(E', E)$ -compact subset of E' . Thus there is a $y \in E'$ and a subsequence $(x_{j(i)}) \subseteq (x_j)$ such that $x_{j(i)} \xrightarrow{\sigma} y$. Since N and hence $x+N$ are $\sigma(E', E)$ -closed, $y \in x+N$ and therefore $\pi(y) = \pi(x)$. Since the norm is $\sigma(E', E)$ lower semicontinuous, we have $\|y\| \leq \liminf \|x_{j(i)}\| \leq \|\pi(x)\| \leq 1$. Thus for each element $\pi(x)$ of the unit ball of E'/N there is an element y of the unit ball of E' such that $\pi(y) = \pi(x)$.

The next proposition is also due to Greenleaf [4] in the case where G is a compact group. It should be noted that our proof is new and somewhat simpler than his.

PROPOSITION 2. *Let G be a locally compact group; N a $\sigma(M(G), C_0(G))$ -closed two-sided ideal in $M(G)$; S^ε (resp. S^π) the set of extreme points of the unit sphere of $M(G)$ (resp. $M(G)/N$), and π the canonical mapping $M(G) \rightarrow M(G)/N$. Then $\pi(S^\varepsilon) = S^\pi$.*

Proof. We first show that $S^\pi \subseteq \pi(S^\varepsilon)$. Recall that $M(G)/N$ can be identified with the dual of N^0 the polar of N in $C_0(G)$, [1, Chapitre IV]. If G is not compact, let G^∞ be the one point compactification of G , and if G is compact put $G^\infty = G$. Consider $N^0 \subseteq C(G^\infty)$; N^0 is $\sigma(C_0(G), M(G))$ -closed and hence norm closed. Let $\mu \in S^\pi$, then by [2, V.8.6], there is a complex number c , $|c|=1$ and an x in G such that $\mu(f) = c\varepsilon_x(f)$ for f in N^0 , and this means that $c\varepsilon_x \in \pi^{-1}(\mu)$. By [7, Proposition 3] $c\varepsilon_x \in S^\varepsilon$, thus $S^\pi \subseteq \pi(S^\varepsilon)$.

If $\mu \in S^\varepsilon$, then the mapping $\lambda \mapsto \mu * \lambda$ is an isometry of $M(G)$ onto itself and it follows that $\pi(\mu)$ has the analogous property in $M(G)/N$ since π is norm decreasing. It follows as in the proof of I of Theorem 1 that $\pi(\mu)$ is in S^π . Thus $\pi(S^\varepsilon) = S^\pi$.

THEOREM 2. *Let A be a Banach algebra which satisfies conditions (1) to (5) of Theorem 1. Then there is a locally compact group G , and a $\sigma(M(G), C_0(G))$ -closed two-sided ideal N in $M(G)$ such that A is isometric and isomorphic to $M(G)/N$. Conversely if N is a $\sigma(M(G), C_0(G))$ -closed two-sided ideal in $M(G)$, then $M(G)/N$ satisfies (1) to (5).*

Proof. Let S and S^ε be as in Theorem 1, and take G to be S^ε , then G is a locally compact group (see the proof of Theorem 1). For f in E , let $\hat{f}$ be the function on G given by $\hat{f}(x) = x(f)$. Then $f \mapsto \hat{f}$ is a norm decreasing linear mapping of E into $C_0(G)$ (see the proof of IV in Theorem 1). Let θ be the adjoint of $f \mapsto \hat{f}$, then we have that θ is a norm decreasing and continuous linear mapping of $M(G)_\sigma$ onto A_σ . The arguments to show that $\theta(\mu * \lambda) = \theta\mu\theta\lambda$ and $\theta(S^M) = S$ are similar (and easier) than those

used in the proofs of VII and VIII of Theorem 1. Now let $N = \ker \theta$, then N is a weakly closed two-sided ideal in $M(G)$. Let π be the canonical mapping $M(G) \rightarrow M(G)/N$ and let θ' be the mapping $M(G)/N \rightarrow A$ such that $\theta = \theta' \circ \pi$. Clearly θ' is one-one and onto. We now show that θ' is an isometry. By Proposition 1, $\pi(S^M)$ is the unit sphere in $M(G)/N$ and since $\theta(S^M) = S$ we have $\theta'(\pi(S^M)) = S$, i.e., θ' maps the unit sphere of $M(G)/N$ onto the unit sphere of A . Thus $\|\theta'\| \leq 1$ and $\|\theta'^{-1}\| \leq 1$, and this means that θ' is an isometry (see the calculation used in the proof of Theorem 1). This completes the proof of the first assertion.

Now let N be a $\sigma(M(G), C_0(G))$ -closed two-sided ideal in $M(G)$. We shall show that $M(G)/N$ satisfies (1) to (5) of Theorem 1. $M(G)/N$ may be identified with the dual of N^0 . Since N^0 is $\sigma(C_0(G), M(G))$ -closed in $C_0(G)$, N^0 is norm closed and therefore a Banach space. Thus (1) is satisfied. To show that (2) is satisfied note that since N^0 is $\sigma(C_0(G), M(G))$ -closed and since $N^{00} = N$, the $\sigma(M(G)/N, N^0)$ -topology equals the quotient weak topology on $M(G)/N$. Thus it suffices to show that for $\lambda \in M(G)/N$, the mapping $\mu \mapsto \lambda\mu$ is continuous in the quotient weak topology and this is true because the quotient of a topological algebra is a topological algebra. Let S^ε (resp. S^π) be the set of extreme points of the unit ball of $M(G)$ (resp. $M(G)/N$). Let $\mu \in S^\pi$, then by Proposition 2 there is a $\mu \in S^\varepsilon$ such that $\pi(\mu) = \mu$. It follows that (3) and (4) are satisfied. To show that $S^\pi \cup \{0\}$ is $\sigma(M(G)/N, N^0)$ -closed, note that π is weakly continuous, hence since $S^\varepsilon \cup \{0\}$ is weakly compact, and since $\pi(S^\varepsilon) = S^\pi$; $\pi(S^\varepsilon \cup \{0\}) = S^\pi \cup \{0\}$ is $\sigma(M(G)/N, N^0)$ -compact and hence $\sigma(M(G)/N, N^0)$ -closed, so that (5) is satisfied.

REMARK. We now give an example of a Banach algebra satisfying conditions (1)–(5) of Theorem 1 but which does not satisfy (6). Let T be the circle group and let

$$N = H^1(T) = \left\{ \mu \in M(T) : \int e^{-in\theta} d\mu(\theta) = 0 \text{ for all } n > 0 \right\}.$$

Then N is a $\sigma(M(T), C(T))$ -closed ideal, so that by Theorem 2, $M(T)/N$ satisfies (1)–(5). Since $M(G)/N$ is also semisimple and commutative, there are multiplicative linear functionals on $M(T)/N$. Let π be the canonical mapping $M(T) \rightarrow M(T)/N$. If there were a locally compact group G and an isometric isomorphism $\theta: M(T)/N \rightarrow M(G)$ then $\theta \circ \pi$ would be a norm decreasing homomorphism of $M(T)$ onto $M(G)$. It is shown in [7; §4] that $\theta \circ \pi(L^1(T)) = L^1(G)$. In [4; §2] Greenleaf has shown that $\pi(L^1(T))$ is not isometrically isomorphic to the group algebra of any locally compact group. Consequently $M(T)/N$ cannot be the measure algebra of a locally compact group.

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