COMPACT HOLOMORPHICALLY CONVEX SUBSETS OF A STEIN MANIFOLD[1]

BY

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1. **Introduction.** Let X be a complex manifold which is countable at infinity and let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on X. Let $\mathcal{O}(U)$ be the Fréchet algebra of holomorphic functions on an open set $U \subset X$, with the topology of uniform convergence on compact subsets of X. Let S be a subset of X, and define $\mathcal{O}(S)$ as the inductive limit

$$\mathcal{O}(S) = \operatorname*{ind\,lim}_{U \supset S} \mathcal{O}(U)$$

where U runs over open subsets of X which contain S, and suppose that $\mathcal{O}(S)$ has the inductive limit topology. Let $\tilde{S}$ be the spectrum[2] of the algebra $\mathcal{O}(S)$, and let $g: S \to \tilde{S}$ be the usual evaluation map defined by $g(x)f = f(x)$ for $f \in \mathcal{O}(S)$. We shall say that S is *holomorphically convex* if g is a bijective mapping onto $\tilde{S}$. If S is an open subset of a Stein manifold X, then it is well known that S is holomorphically convex if and only if S is a Stein manifold (see [6]). If S is compact we shall see (Proposition 2.4) that this definition of holomorphic convexity agrees with the definition given in [10] where various geometric characterizations are discussed.

The purpose of this paper is to prove Theorems A and B of Cartan for coherent analytic sheaves on a compact holomorphically convex subset of a Stein manifold X. We also give a characterization of holomorphic convexity of compact subsets of X in terms of a weakened form of Theorem B. In §II we introduce a concept of the envelope of holomorphy of arbitrary subsets of a Stein manifold, generalizing the envelope of holomorphy of open subsets of a Stein manifold, whose existence was shown by Rossi [6]. In §III we prove the main theorems of the paper, using some of our results from §II.

II. **Envelopes of holomorphy.** Let S be a subset of a Stein manifold X, and $\mathcal{O}(S)$ be as in §I. Let $\mathcal{O}'(S)$, the dual of $\mathcal{O}(S)$ as a topological vector space, have the weak topology induced by $\mathcal{O}(S)$. This induces the weak topology on $\tilde{S} \subset \mathcal{O}'(S)$. The restriction map from $\mathcal{O}(X)$ into $\mathcal{O}(S)$ has a continuous adjoint from $\mathcal{O}'(S)$ into $\mathcal{O}'(X)$ which obviously maps $\tilde{S}$ continuously into $\tilde{X}$. Since X is Stein we can identify X with $\tilde{X}$. This gives us a map from $\tilde{S}$ into X which will be denoted by π. Let g be the

[2] $\tilde{S}$ is the set of nonzero continuous homomorphisms of the algebra $\mathcal{O}(S)$ into C.

evaluation map from S into $\tilde{S}$. We then have the following commutative diagram

$$
\begin{array}{ccc}
 & & S \\
 & \nearrow_{g} & \uparrow_{\pi} \\
S & \longrightarrow & X
\end{array}
$$

We shall need the following important result due to Rossi [6] in the case where $S = U$ is open in X (see also [4, Chapter 1]).

THEOREM 2.1 (ROSSI). *Let U be an open set in a Stein manifold X, then U can be given the structure of a Stein manifold such that:*
  (a) *π is a locally biholomorphic map of $\tilde{U}$ into X.*
  (b) *g is an analytic isomorphism of U into $\tilde{U}$.*
  (c) *$g^*\colon \mathcal{O}(\tilde{U}) \to \mathcal{O}(U)$ is a Fréchet algebra isomorphism.*

Now let us return to the case where S is an arbitrary subset of the Stein manifold X. Suppose that $\{U_\alpha, \alpha \in I\}$ is a fundamental system of open neighborhoods of S, where $U_\alpha \subset U_\beta$ if and only if $\alpha > \beta$. If $\alpha > \beta$, then the restriction map $\mathcal{O}(U_\beta)$ into $\mathcal{O}(U_\alpha)$ has a continuous adjoint from $\mathcal{O}'(U_\alpha)$ into $\mathcal{O}'(U_\beta)$ which, when restricted to $\tilde{U}_\alpha$, defines a continuous map $\varphi_{\alpha\beta}$ from $\tilde{U}_\alpha$ into $\tilde{U}_\beta$. By Theorem 2.1(a), we see that $\varphi_{\alpha\beta}$ is the spreading map of $\tilde{U}_\alpha$ over $\tilde{U}_\beta$ obtained by considering U_α as an open subset of the Stein manifold $\tilde{U}_\beta$. This implies that $\varphi_{\alpha\beta}$ is locally biholomorphic.

In general, the weak dual of an inductive limit of locally convex spaces is the projective limit of the weak duals (see, for example, [8, p. 140]). Therefore

$$
\mathcal{O}'(S) = \operatorname*{proj\,lim}_{\alpha \in I} \mathcal{O}'(U_\alpha)
$$

and hence

(2.1)
$$
\tilde{S} = \operatorname*{proj\,lim}_{\alpha \in I} \tilde{U}_\alpha
$$

where the projective limits are taken with respect to the maps $\{\varphi_{\alpha\beta}\}$ defined above.

Let $_\alpha\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\tilde{U}_\alpha$. These sheaves induce, in a natural way, a sheaf of rings on $\tilde{S}$. This sheaf will be called the *holomorphic structure sheaf* on $\tilde{S}$ and will be denoted by $_{\tilde{S}}\mathcal{O}$. It is defined as follows. For each U_α, the obvious map φ_α from $\tilde{S}$ into $\tilde{U}_\alpha$ induces an inverse image sheaf $\varphi_\alpha^*(_\alpha\mathcal{O})$ on $\tilde{S}$, such that $(\varphi_\alpha^*(_\alpha\mathcal{O}))_x \cong {}_\alpha\mathcal{O}_{\varphi_\alpha(x)}$. Also, the maps $\varphi_{\alpha\beta}$, from $\tilde{U}_\alpha$ into $\tilde{U}_\beta$ induce compatible sheaf homomorphisms

$$
\psi_{\alpha\beta}\colon \varphi_\beta^*(_\beta\mathcal{O}) \to \varphi_\alpha^*(_\alpha\mathcal{O}).
$$

We then let

$$
_{\tilde{S}}\mathcal{O} = \operatorname*{ind\,lim}_{\alpha \in I} \varphi_\alpha^*(_\alpha\mathcal{O})
$$

where the inductive limit is taken with respect to the maps $\{\psi_{\alpha\beta}\}$.

We have the following analogues of parts (a) and (b) of Theorem 2.1.

PROPOSITION 2.2. (a) *The map π from $\tilde{S}$ into X induces an algebraic isomorphism from $\mathcal{O}_{\pi(x)}$ onto $_{\tilde{S}}\mathcal{O}_x$ for each $x \in \tilde{S}$.*

(b) *The evaluation map g is a homeomorphism from S onto g(S) which induces a sheaf isomorphism from $_\tilde{S}\mathcal{O}|_{g(S)}$ onto $\mathcal{O}|_S$, if S is identified with g(S).*

Proof. For each $x \in \tilde{S}$, $_\tilde{S}\mathcal{O}_x$ is the inductive limit of the stalks $\{_\alpha\mathcal{O}_{\varphi_a(x)}\}$. Since each $\varphi_{\alpha\beta}$ is a locally biholomorphic map, this implies that the map from $_\alpha\mathcal{O}_{\varphi_a(x)}$ into $_\tilde{S}\mathcal{O}_x$ is an isomorphism for each $\alpha \in I$. Now part (a) is just the special case where $U_\alpha = X$ and $\varphi_\alpha = \pi$. To prove part (b) first note that g^{-1} is equal to $g_\alpha^{-1} \circ \varphi_\alpha$ and hence it is continuous. The remainder of part (b) follows from the fact that for each $\alpha \in I$, $(\varphi_\alpha^*(_\alpha\mathcal{O}))|_{g(S)}$ is isomorphic to $\mathcal{O}|_S$ (under the identification of S with g(S)), where this isomorphism is a consequence of part (b) of Theorem 2.1.   Q.E.D.

It might happen that for some S, the set $\tilde{S}$ is embedded in a complex manifold in a natural way, then the stalk isomorphism in Proposition 2.2 implies that π is "locally biholomorphic", by the inverse function theorem. The following proposition is an analogue of Theorem 2.1(c), and shows that holomorphic functions on S can be "continued analytically" to the part of the spectrum connected to S.

Let E(S) denote the components of $\tilde{S}$ which intersect S.

PROPOSITION 2.3. *The restriction map $g^*: \Gamma(E(S), _\tilde{S}\mathcal{O}) \to \Gamma(S, \mathcal{O})$ is an algebraic isomorphism.*

Proof. Suppose f is a section of $\mathcal{O}$ over S. Then f is the restriction of a section f^1 of $\mathcal{O}$ on some neighborhood U_α of S (see [3]). By part (c) of Theorem 2.1, f^1 extends to a section f^2 of $_\alpha\mathcal{O}$ on $\tilde{U}_\alpha$ (where we identify U_α with its image in $\tilde{U}_\alpha$). Then f^2 induces a section f^3 of $_\tilde{S}\mathcal{O}$ over $\tilde{S}$, which when restricted to S is just f (again identifying S with g(S)). This proves g^* is onto. The fact that g^* is one-to-one follows easily from the following principle of unique continuation for sections of $_\tilde{S}\mathcal{O}$ and the assumption that E(S) is connected to S. Suppose $\{x_\nu\}$ is a net in $\tilde{S}$ which converges to $x \in \tilde{S}$, and suppose that f is a section of $_\tilde{S}\mathcal{O}$ which is defined in a neighborhood of $\{x_\nu\} \cup x$, such that $f_{x_\nu} = 0$ (as a germ) for all ν, then $f_x = 0$ [3]. To show this, let $h_{\varphi_a(x)}$ be the germ at $\varphi_a(x)$ which induces the germ f_x (see Proposition 2.2). The germ $h_{\varphi_a(x)}$ is induced by a section h of $_\alpha\mathcal{O}$ in some neighborhood of $\varphi_a(x)$, and h induces the section f in some neighborhood of x. Therefore there exists a μ such that $h_{\varphi_a(x_\nu)}$ induces f_x if $\nu > \mu$. This implies that $h_{\varphi_a(x_\nu)}$ is the zero germ at $\varphi_a(x_\nu)$ for $\nu > \mu$, since $_\tilde{S}\mathcal{O}_{x_\nu} \cong \mathcal{O}_{\varphi_a(x_\nu)}$ (see the proof of part (a) of Proposition 2.2). By the uniqueness of analytic continuation on $\tilde{U}_\alpha$ we obtain that $h_{\varphi_a(x)}$, and hence f_x, is the zero germ.   Q.E.D.

Whether E(S) is the same as S or not is related to the following question. Let A(S) denote the closure of $\mathcal{O}(S)$ in C(S), where C(S) is the space of continuous functions on S with the topology of uniform convergence on compact subsets, and let $\mathscr{S}(A(S))$ be the spectrum of A(S). Then there is a continuous map ρ from $\mathscr{S}(A(S))$ into the set of $x \in \tilde{S}$ satisfying $|x(f)| \leq \sup_K |f|$, for all $f \in \mathcal{O}(S)$ and for some compact set $K \subset S$. The Šilov idempotent theorem can be used to prove that

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[3] If f is a section of a sheaf, then f_x denotes the germ of f at x.

$E(S)$ contains $\rho(\mathscr{S}(A(S)))$ (see [4]). Therefore, if $\rho(\mathscr{S}(A(S)))$ is dense in S, then $E(S) = \tilde{S}$. If S is open then trivially we have that $\rho(\mathscr{S}(A(S))) = \tilde{S}$.

PROPOSITION 2.4. *If K is a compact set in a complex manifold X, then the spectra of the algebras $A(K)$ and $\mathcal{O}(K)$ are homeomorphic.*

REMARK. $\mathscr{S}(A(K))$ was called the holomorphic hull of K in [10].
Proof. Suppose $x \in \tilde{K}$. We must show that

$$|x(f)| \leqq \sup_K |f|, \quad \text{for all } f \in \mathcal{O}(K).$$

Suppose this were not so, then there would be a function $f \in \mathcal{O}(K)$ such that $x(f) = 1$ but $\sup_K |f| < 1$. These two facts imply that $x(1-f) = 0$ and $(1-f)^{-1} \in \mathcal{O}(K)$. This is impossible, since then we would have

$$1 = x((1-f)(1-f)^{-1}) = x(1-f)x(1-f)^{-1} = 0. \qquad \text{Q.E.D.}$$

In analogy to Theorem 2.1(a) one could ask whether π were "locally biholomorphic" in the sense of being a local homeomorphism which induces an isomorphism (locally) between $\mathcal{O}$ and $_{\tilde{S}}\mathcal{O}$. The following example shows that this is not necessarily the case. Let

$$B_0 = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leqq 1\},$$
$$B_n = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leqq 1 + 1/n\}, \qquad n \geqq 1,$$

and letting ∂B_n be the boundary of B_n we set

$$S = \bigcup_{n=0}^{\infty} \partial B_n.$$

Then it is easy to see that set-theoretically (disjoint union)

$$\tilde{S} = \bigcup_{n=0}^{\infty} B_n.$$

However $\tilde{S}$, which is equipped with the weak topology induced by $\mathcal{O}(S)$, is not mapped locally homeomorphically by π into $\mathbf{C}^2$. Suppose it were (which would be the case if $\tilde{S}$ were equipped with the disjoint union topology, for example), then, letting p_n be the center of the ball B_n in $\tilde{S}$, $\{p_n\}$ would be a discrete set. But it follows easily from Hartogs' theorem that, in the weak topology on $\tilde{S}$, the points p_n converge to the point p_0, which is a contradiction. Thus $\tilde{S}$ is not a subset of a complex manifold in any natural way. The authors would like to thank the referee for supplying this example.

III. **Compact holomorphically convex sets.** In order to compute cohomology it will be convenient to express $\tilde{K}$ as the projective limit of compact sets, in the case where K is compact. Considering K as a subset of $\tilde{U}_\alpha$, where we use the same notation as before, let $\hat{K}_\alpha$ denote the $\mathcal{O}(\tilde{U}_\alpha)$-hull of K, i.e.,

$$\hat{K}_\alpha = \left\{ x \in \tilde{U}_\alpha : |f(x)| \leqq \sup_K |f|, \forall f \in {}_\alpha\mathcal{O}(\tilde{U}_\alpha) \right\}.$$

One can easily check that $\varphi_\alpha(\tilde{K}) \subset \hat{K}_\alpha$ and $\varphi_{\alpha\beta}(\hat{K}_\alpha) \subset \hat{K}_\beta$. Since $\tilde{U}_\alpha$ is a Stein manifold, it follows that $\hat{K}_\alpha$ is compact. Thus we have the important

LEMMA 3.1. *If K is compact in X, then $\tilde{K}$ is homeomorphic to the projective limit of the compact Hausdorff spaces $\hat{K}_\alpha$.*

The projective limit of compact spaces retains one of the important properties of a fundamental neighborhood system.

COROLLARY 3.2. *Suppose N is a neighborhood of $\varphi_\beta(\tilde{K})$ in $\hat{K}_\beta$. Then there exists an $\alpha > \beta$ such that $\varphi_{\alpha\beta}(\hat{K}_\alpha) \subset N$.*

Proof. Suppose this were not so. Then $\varphi_{\alpha\beta}^{-1}(\hat{K}_\beta - N)$ is a compact nonempty set for all $\alpha \geqq \beta$, which implies that there exists a point

$$x \in \operatorname*{proj\,lim}_\alpha \varphi_{\alpha\beta}^{-1}(\hat{K}_\beta - N) \subset \operatorname*{proj\,lim}_\alpha \hat{K}_\alpha = \tilde{K}$$

such that $\varphi_\beta(x) \notin N$. But this is a contradiction.   Q.E.D.

We now have the following analogue to the Fundamental Theorem of Cartan for Stein manifolds (see [1], [2]).

THEOREM 3.3. *Let K be a compact holomorphically convex subset of a Stein manifold X, and let $\mathscr{F}$ be a coherent analytic sheaf on K. Then:*
   (A) *For each $x \in K$ the $\mathcal{O}_x$-module $\mathscr{F}_x$ is generated by the germs at x of $\Gamma(K, \mathscr{F})$.*
   (B) *$H^q(K, \mathscr{F}) = 0$ for all $q > 0$.*

Proof. By Proposition 2 of Cartan [2] there exists a neighborhood U of K in X and a coherent analytic sheaf $\mathscr{G}$ on U which restricts to $\mathscr{F}$ on K. By Corollary 3.2 there exists a $\hat{K}_\beta$ such that $\pi_\beta(\hat{K}_\beta) \subset U$. The inverse image $\pi_\beta^* \mathscr{G}$ is a coherent analytic sheaf on a neighborhood of $\hat{K}_\beta$ in $\tilde{U}_\beta$ because π_β is locally biholomorphic. Now, as is well known, $\hat{K}_\beta$ has a fundamental neighborhood system in $\tilde{U}_\beta$ consisting of Stein manifolds. In particular, there is an open Stein manifold Y in $\tilde{U}_\beta$ which contains K such that $\pi_\beta^* \mathscr{G}$ is a coherent analytic sheaf on Y. Since $\pi_\beta^* \mathscr{G}$, when restricted to K, is just $\mathscr{F}$ (under the identification map $g_\beta : U_\beta \to \tilde{U}_\beta$), we see that part (A) follows immediately from Cartan's Theorem A applied to $\pi_\beta^* \mathscr{G}$ on Y. Since, for each α, $\hat{K}_\alpha$ has a fundamental neighborhood system consisting of Stein manifolds it follows as in [2] from Cartan's Theorem B that $H^q(\hat{K}_\alpha, \pi_\alpha^* \mathscr{G}) = 0$, for all $q > 0$ and all $\alpha > \beta$. Therefore, once we show that

$$(*) \qquad\qquad H^q(K, \mathscr{F}) = \operatorname*{ind\,lim}_{\alpha \geqq \beta} H^q(\hat{K}_\alpha, \pi_\alpha^* \mathscr{G})$$

part (B) will be proved.

To show $(*)$ we proceed as follows. Suppose q is an integer $\geqq 0$. Let $\{\mathscr{C}^{\alpha,q}\}$ denote the canonical flabby resolution[4] of $\pi_\alpha^* \mathscr{G}$ restricted to $\hat{K}_\alpha$. Let $\{\mathscr{C}^q\}$ denote

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[4] See, for example [3].

the canonical flabby resolution of $\mathscr{F}$ on K. Then there is a natural map

$$\rho\colon \operatorname*{ind\,lim}_{\alpha} \Gamma(\hat{K}_\alpha, \mathscr{C}^{\alpha,q}) \to \Gamma(K, \mathscr{C}^q).$$

We shall show that ρ is an isomorphism. Suppose f is a section of $\mathscr{C}^q$ on K, then (identifying K with a subset of $\hat{K}_\beta$) it is easy to see that f extends to a section of $\mathscr{C}^{\beta,q}$ on $\hat{K}_\beta$. Thus ρ is onto. Suppose $f \in \operatorname{ind\,lim}_\alpha \Gamma(\hat{K}_\alpha, \mathscr{C}^{\alpha,q})$ is represented by $f^\beta \in \Gamma(\hat{K}_\beta, \mathscr{C}^{\beta,q})$ and f^β vanishes on K (considered as a subset of $\hat{K}_\beta$). Then f^β vanishes on a neighborhood N of K in $\hat{K}_\beta$. By Corollary 3.2 there exists an $\alpha \geq \beta$ such that $\varphi_{\alpha\beta}(\hat{K}_\alpha) \subset N$. Therefore f is represented in $\Gamma(\hat{K}_\alpha, \mathscr{C}^{\alpha,q})$ by zero. This proves ρ is one to one, and thus (*) is established, since cohomology commutes with inductive limits.   Q.E.D.

The following theorem gives a characterization of compact holomorphically convex sets, but first we need to make a definition. If $S \subset X$, then we shall say that $\mathscr{R}$ is a *sheaf of relations* on S if $\mathscr{R}$ is the kernel of a sheaf homomorphism

$$\mathcal{O}^p|_S \to \mathcal{O}^q|_S.$$

THEOREM 3.4. *Suppose K is a compact subset of a Stein manifold X. Then the following conditions are equivalent:*

(a) *K is holomorphically convex.*

(b) *$H^q(K, \mathscr{F}) = 0$ for all $q > 0$ and for all coherent analytic sheaves $\mathscr{F}$ on K.*

(c) *$H^1(K, \mathscr{R}) = 0$ for all sheaves of relations $\mathscr{R}$ on K.*

Proof. The only thing we have to show is that (c) implies (a).

Assume (c) and suppose that $y \in \tilde{K}$. If there exists a point $x \in K$ such that $y(f) = 0$ implies $f(x) = 0$ for any $f \in \mathcal{O}(K)$, then $y = x$. This can be seen as follows: the function $f - y(f)$ satisfies $y(f - y(f)) = 0$, hence $f(x) - y(f) = 0$, and by our identification we have $y = x$. Now assume that $y \in \tilde{K}$, but $y \notin K$. Then for each $x \in K$ there exists a function $f \in \mathcal{O}(K)$ such that $y(f) = 0$, but $f(x) \neq 0$. By compactness of K, this implies that there exists functions $f_1, \ldots, f_r \in \mathcal{O}(K)$ such that $y(f_j) = 0$, $j = 1, \ldots, r$, and such that these functions do not vanish simultaneously at any point in K. This means that the map $\mu\colon \mathcal{O}^r|_K \to \mathcal{O}|_K$ defined by $\mu(g_1, \ldots, g_r) = \sum_j f_j g_j$ is surjective. We then define $\mathscr{R}$ by the exact sequence

$$0 \to \mathscr{R} \to \mathcal{O}^r|_K \overset{\mu}{\to} \mathcal{O}|_K \to 0.$$

By hypothesis, $H^1(H, \mathscr{R}) = 0$, and thus $\mu\colon \mathcal{O}(K)^r \to \mathcal{O}(K)$ is surjective. Hence there exists $g_1, \ldots, g_r \in \mathcal{O}(K)$ such that $\sum_j f_j g_j = 1$. But this is impossible since $1 = y(1) = \sum_j y(f_j) y(g_j) = 0$.   Q.E.D.

As in the case of Stein manifolds, holomorphically convex sets will generally be topologically restricted. More precisely we have

THEOREM 3.5. *Suppose K is a compact holomorphically convex subset of a Stein manifold X, then $H^q(K, \mathbf{Z}) = 0$, for $q > \dim_{\mathbf{C}} X$.*

Proof. If Y is a Stein manifold of complex dimension n, then $H^q(Y, \mathbf{Z}) = 0$ for $q > n$ (see [11]). The theorem then follows from this fact in exactly the same way that (B) of Theorem 3.3 followed from Cartan's Theorem B.   Q.E.D.

The following corollary generalizes the result that a Stein manifold of complex dimension ≥ 2 is connected at the boundary (see [4, p. 227]).

Corollary 3.6. *Suppose K is a compact holomorphically convex subset of a Stein manifold X of complex dimension n, then*

$$H^p(X, \mathbf{Z}) \cong H^p(X{-}K, \mathbf{Z}), \qquad p < n - 1.$$

Proof. It follows from Alexander duality and Theorem 3.5 that $H_K^p(X, \mathbf{Z}) = 0$ for $p < n$ ($H_K^p(X, \mathbf{Z})$ is frequently denoted by $H^p(X, X{-}K; \mathbf{Z})$). The corollary then follows from the exact sequence

$$\cdots \to H_K^p(X, \mathbf{Z}) \to H^p(X, \mathbf{Z}) \to H^p(X{-}K, \mathbf{Z}) \to \cdots \qquad \text{Q.E.D.}$$

REMARKS. (1) Suppose that K is a holomorphically convex compact subset of an n-dimensional Stein manifold X and $\mathscr{F}$ is a locally free analytic sheaf on K. By utilizing part (B) of Theorem 3.3 above one can show that (see [5] and [7]):

   (i) $H_K^q(X, \mathscr{F}) = 0$ for $q \neq n$.

   (ii) $H_K^n(X, \mathscr{F}) = H^0(K, \mathscr{F}^* \otimes \Omega^n)'$.

(2) Suppose K is a compact subset of a complex manifold X with a fundamental system of Stein neighborhoods. Then K is holomorphically convex (this follows immediately from (2.1) and the fact that $U = \tilde{U}$ if U is Stein). It is unknown whether the converse is true.

(3) Suppose M is a compact real C^∞ k-dimensional submanifold of an n-dimensional complex manifold X, and let $h_x = \dim_{\mathbf{C}} T_x(M) \cap JT_x(M)$, the complex dimension of the holomorphic tangent space to M at x. In this case we have a geometric characterization of the holomorphic convexity of M. Namely, under certain "generic" conditions on M, holomorphic convexity of M is equivalent to h_x being identically zero on M, and this in turn is equivalent to the apparently stronger assertion that $A(M) = C(M)$ (see [9] for a more complete discussion of these results).

REFERENCES

   1. H. Cartan, "Variétés analytiques complexes et cohomologie," pp. 41–55, in *Colloque sur les fonctions de plusiers variables*, Brussels, 1953.

   2. ———, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France **85** (1957), 77–99.

   3. R. Godement, *Topologie algébrique et théorie de faisceaux*, Hermann, Paris, 1958.

   4. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

   5. A. Martineau, *Les hyperfunctions de M. Sato*, Séminaire Bourbaki, **13** (1961).

   6. H. Rossi, *On envelopes of holomorphy*, Comm. Pure Appl. Math. **16** (1963), 9–17.

   7. M. Sato, *Theory of hyperfunctions*. II, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 387–437.

8. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.

9. R. O. Wells, Jr., "Holomorphic hulls and holomorphic convexity," in *Proceedings of the conference on complex analysis*, Rice University, 1967, Rice Univ. Studies, Vol. 54, Houston, Texas, (to appear).

10. ———, *Holomorphic hulls and holomorphic convexity of differentiable submanifolds*, Trans. Amer. Math. Soc. **132** (1968), 245–262.

11. A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. **69** (1959), 713–717.

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