THE C^* -ALGEBRA GENERATED BY AN ISOMETRY. II

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I. Introduction. In [2] it was shown that the C^* -algebra generated by any nonunitary isometry is $*$ -isomorphic to the C^* -algebra $\mathcal{A}(S)$ generated by the unilateral shift of multiplicity one, S . Further, the ideal theory of $\mathcal{A}(S)$ was determined. The main point was that $\mathcal{A}(S)$ contains the full algebra of compact operators $\mathcal{K}$ and $\mathcal{A}(S)/\mathcal{K}$ is $*$ -isomorphic to $C(T)$, the algebra of all complex-valued continuous functions on the unit circle, T .

Here, I examine the structure of $\mathcal{A}(S)$ and certain related C^* -algebras in greater detail. In particular, the irreducible $*$ -subalgebras of $\mathcal{A}(S)$ are characterized and reasonable necessary and sufficient conditions are given for an operator in $\mathcal{A}(S)$ to generate $\mathcal{A}(S)$. Finally, I provide an example of a C^* -algebra which is irreducible and has the same ideal structure as $\mathcal{A}(S)$, but which is not $*$ -isomorphic to $\mathcal{A}(S)$.

II. Preliminary results. Henceforth all Hilbert spaces are over the complex numbers. If $\mathcal{H}$ is a Hilbert space and B is a bounded operator on $\mathcal{H}$ then the smallest C^* -algebra of operators containing B and 1 is denoted by $\mathcal{A}(B)$. The full algebra of bounded operators on $\mathcal{H}$ with the operator norm topology is called $\mathcal{B}(\mathcal{H})$ and the subalgebra of all compact operators on $\mathcal{H}$ is called $\mathcal{K}(\mathcal{H})$ (or just $\mathcal{K}$). We let π denote the usual quotient map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}$. In this paper, all ideals are closed and two-sided. We write $\sigma(B)$ for the spectrum of B in $\mathcal{B}(\mathcal{H})$.

In portions of this paper, we will be concerned with the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space X with supremum norm. More particularly, let T be the unit circle and let A be the uniformly closed subalgebra of $C(T)$ consisting of those ϕ in $C(T)$ which are uniform limits of polynomials in the complex variable z . Further, let μ be normalized Haar measure on T and let L^2 denote the associated Hilbert space of square-integrable functions on T . As usual, we let H^2 denote the L^2 -closure of A . In the following sections we will be interested in the C^* -algebras generated by Toeplitz operators [1] on H^2 associated with functions ϕ in $C(T)$ by the relation $T_\phi f = P(\phi f)$, where P is the orthogonal projection from L^2 onto H^2 . We will also be concerned with the Laurent operators [1] on L^2 associated with ϕ in $C(T)$ by the relation $M_\phi f = \phi f$. It should be pointed out that in general [1], [3]

$$\|T_\phi\| = \|M_\phi\| = \|\pi(T_\phi)\| = \|\pi(M_\phi)\| = \|\phi\|.$$

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Further, the Toeplitz operator T_z is just the unilateral shift S on H^2 since $\{z^n \mid n=0, 1, 2, \dots\}$ is a basis for H^2 .

We will deal in the following sections with certain irreducible C^* -algebras and certain abelian C^* -algebras. The two propositions below will be among the necessary working tools.

PROPOSITION 1. *Let $\mathcal{A}$ be an irreducible C^* -algebra on $\mathcal{H}$ and suppose $\mathcal{A} \cap \mathcal{K} \neq (0)$. Then $\mathcal{K} \subset \mathcal{A}$ and $\mathcal{K} \subset \mathcal{I}$ for any nontrivial ideal $\mathcal{I}$ in $\mathcal{A}$.*

Proof. Immediate from [4, 4.1.10; 2.11.3 (i); and 1.8.2.]. $\square$

If $\mathcal{A}$ is a C^* -subalgebra of $C(X)$ which contains the constants then a natural equivalence relation is given on X by writing $x \sim y$ if and only if $f(x) = f(y)$ for all f in $\mathcal{A}$. Let Q be the quotient space $X/\sim$ and let q be the usual quotient map. To each f in $\mathcal{A}$ there corresponds a complex-valued function $\tilde{f}$ on Q by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & C \\ q \downarrow & \nearrow \tilde{f} & \\ Q & & \end{array}$$

We will consider two topologies on Q . The first (Q, ω) is the weak topology induced by all the $\tilde{f}$ for f in $\mathcal{A}$. The second is the usual quotient topology, (Q, q) . It is not hard to see that (Q, q) is stronger than (Q, ω) . Whether (Q, q) and (Q, ω) are equivalent will be left open.

PROPOSITION 2. *The maximal ideal space of $\mathcal{A}$ is just (Q, ω) and $\mathcal{A}$ is $*$ -isomorphic to $C(Q, \omega)$. A necessary and sufficient condition that (Q, q) and (Q, ω) are equivalent is that (Q, q) be Hausdorff.*

Proof. Since $\mathcal{A}$ is a closed symmetric subalgebra of $C(X)$, every irreducible representation of $\mathcal{A}$ extends to an irreducible representation of $C(X)$ [6, p. 305]. Since complex-homomorphisms of $\mathcal{A}$ are irreducible representations and the only irreducible representations of $C(X)$ are point-evaluations at points of X , we see that every complex-homomorphism of $\mathcal{A}$ is a point-evaluation at a point of X . Now since x and y in X induce the same homomorphism on $\mathcal{A}$ if and only if $x \sim y$, it is clear that Q can be identified with the maximal ideal space of $\mathcal{A}$. Under this identification, $\tilde{f}$ is the Gelfand transform of f in $\mathcal{A}$ and so (Q, ω) is the usual maximal ideal space with the weak topology. The mapping $f \rightarrow \tilde{f}$ is clearly a $*$ -isomorphism from $\mathcal{A}$ into $C(Q, \omega)$. This mapping is also an isometry so the image $\tilde{\mathcal{A}}$ is a C^* -subalgebra of $C(Q, \omega)$. Since the $\tilde{f}$ separate points of Q , the Stone-Weierstrass theorem implies that $\tilde{\mathcal{A}} = C(Q, \omega)$.

If (Q, q) and (Q, ω) are to be equivalent then (Q, q) must be Hausdorff since (Q, ω) is Hausdorff. But the map $f \rightarrow \tilde{f}$ is an isometric $*$ -isomorphism from $\mathcal{A}$ into

$C(Q, q)$ and if (Q, q) is Hausdorff the Stone-Weierstrass theorem again applies to show that $\mathcal{A} = C(Q, q)$. Hence $(Q, q) = (Q, \omega)$. $\square$

III. The algebra $\mathcal{A}(T_z)$. We now focus on certain C^* -algebras on H^2 . Since T_z is the unilateral shift on H^2 , we will be interested in $\mathcal{A}(T_z)$ and its $*$ -subalgebras. In this section we write $\mathcal{K} = \mathcal{K}(H^2)$ for the algebra of compact operators on H^2 . We begin with a "functional" characterization of $\mathcal{A}(T_z)$. Recall (cf. [1]) that for Ψ in $C(T)$, $T_\Psi^* = T_{\bar{\Psi}}$. Also, if Ψ is in $C(T)$ and ϕ is in A , then $T_\Psi T_\phi = T_{\Psi\phi}$.

LEMMA 1. *If Ψ_1, Ψ_2 are in $C(T)$ then $T_{\Psi_1} T_{\Psi_2} = T_{\Psi_1 \Psi_2} + K$ for some K in $\mathcal{K}$.*

Proof. It follows from standard results on polynomial approximation that there are sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ of polynomials in A such that

$$\alpha_n + \bar{\beta}_n \rightarrow \Psi_1, \quad \gamma_n + \bar{\delta}_n \rightarrow \Psi_2.$$

It follows from the fact that $\|T_\phi\| = \|\phi\|$ that

$$T_{\alpha_n} + T_{\bar{\beta}_n} \rightarrow T_{\Psi_1} \quad \text{and} \quad T_{\gamma_n} + T_{\bar{\delta}_n} \rightarrow T_{\Psi_2}.$$

Hence, we have

$$T_{\alpha_n} T_{\gamma_n} + T_{\bar{\beta}_n} T_{\bar{\delta}_n} + T_{\alpha_n} T_{\bar{\delta}_n} + T_{\bar{\beta}_n} T_{\gamma_n} \rightarrow T_{\Psi_1} T_{\Psi_2}.$$

But $T_{\alpha_n} T_{\gamma_n} = T_{\alpha_n \gamma_n}$, $T_{\bar{\beta}_n} T_{\gamma_n} = T_{\bar{\beta}_n \gamma_n}$ and $T_{\bar{\beta}_n} T_{\bar{\delta}_n} = (T_{\delta_n} T_{\bar{\beta}_n})^* = T_{\bar{\delta}_n \bar{\beta}_n}$. Further $T_{\bar{\delta}_n} T_{\alpha_n} = T_{\bar{\delta}_n \alpha_n}$ and

$$T_{\alpha_n} T_{\bar{\delta}_n} - T_{\bar{\delta}_n} T_{\alpha_n}$$

is in $\mathcal{K}$ since $\mathcal{A}(T_z)/\mathcal{K}$ is abelian. Putting the above remarks together, we see that there is a sequence of $\{K_n\}$ in $\mathcal{K}$ so that

$$T_{(\alpha_n + \bar{\beta}_n)(\gamma_n + \bar{\delta}_n)} + K_n \rightarrow T_{\Psi_1} T_{\Psi_2}.$$

But

$$T_{(\alpha_n + \bar{\beta}_n)(\gamma_n + \bar{\delta}_n)} \rightarrow T_{\Psi_1 \Psi_2}$$

and it follows that there is a K in $\mathcal{K}$ with $K_n \rightarrow K$ such that

$$T_{\Psi_1 \Psi_2} + K = T_{\Psi_1} T_{\Psi_2}. \quad \square$$

THEOREM 1. *Let $\mathcal{B} = \{T_\Psi + K : \Psi \in C(T), K \in \mathcal{K}\}$. Then $\mathcal{A}(T_z) = \mathcal{B}$. Further, the representation as sums $T_\Psi + K$ is unique.*

Proof. Clearly, if Ψ is in $C(T)$ and $\varepsilon > 0$ is arbitrary then there are polynomials α, β in A so that $\|\Psi - \alpha - \bar{\beta}\| < \varepsilon$. Since T_α and $T_{\bar{\beta}}$ are in $\mathcal{A}(T_z)$ it follows that T_Ψ is in $\mathcal{A}(T_z)$. Further, by Proposition 1 or directly as in [2] it is easily seen that $\mathcal{K} \subset \mathcal{A}(T_z)$. Thus, we have $\mathcal{B} \subset \mathcal{A}(T_z)$. To show the reverse inclusion, it suffices since T_z is in $\mathcal{B}$ to show that $\mathcal{B}$ is a closed $*$ -subalgebra of bounded operators. Clearly, we have 1 in $\mathcal{B}$. Also, if B_1 and B_2 are in $\mathcal{B}$ then B_1^* , λB_1 and $B_1 + B_2$ are in $\mathcal{B}$. It follows immediately from Lemma 1 that $B_1 B_2$ is in $\mathcal{B}$. Thus, it remains

to check that $\mathcal{B}$ is closed. To this end, suppose we have $T_{\Psi_n} + K_n \rightarrow B$. Then $\{\pi(T_{\Psi_n})\}$ is a Cauchy sequence. But $\|\Psi\| = \|T_{\Psi}\| = \|\pi(T_{\Psi})\|$ so $\{\Psi_n\}$ is a Cauchy sequence in $C(T)$ and hence $\Psi_n \rightarrow \Psi$ for some Ψ in $C(T)$. It follows that $T_{\Psi_n} \rightarrow T_{\Psi}$ and hence $K_n \rightarrow K$ for some K in $\mathcal{K}$. Hence $B = T_{\Psi} + K$ and so $\mathcal{B}$ is closed. To check uniqueness, suppose $T_{\Psi} + K = T_{\Psi_1} + K_1$ then $\pi(T_{\Psi - \Psi_1}) = 0$ so $\Psi = \Psi_1$ and $K = K_1$. $\square$

COROLLARY (1.1). *If B is in $\mathcal{A}(T_z)$ then either $\sigma(B)$ contains a nontrivial component or $\sigma(B)$ has at most one limit point.*

Proof. By Theorem 1, B has the form $B = T_{\Psi} + K$. Now, it is known [8] that $\sigma(T_{\Psi})$ is connected and [1] $\sigma(T_{\Psi})$ consists of one point only if Ψ is constant. Further, $\sigma(T_{\Psi}) \subset \sigma(T_{\Psi} + K')$ for all compact K' [3]. Thus, if $\sigma(T_{\Psi})$ is nontrivial then $\sigma(B)$ has a nontrivial component containing $\sigma(T_{\Psi})$. If $\sigma(T_{\Psi})$ is a one-point set, then $B = \lambda I + K$ so $\sigma(B)$ has at most one limit point. $\square$

COROLLARY (1.2). *The mapping $\phi \rightarrow \pi(T_{\phi})$ gives a *-isomorphism from $C(T)$ onto $\mathcal{A}(T_z)/\mathcal{K}$.*

Proof. Using Theorem 1 and the fact that $\|\phi\| = \|\pi(T_{\phi})\|$, it is clear that the mapping $\phi \rightarrow \pi(T_{\phi})$ is an additive *-isomorphism and isometry from $C(T)$ onto $\mathcal{A}(T_z)/\mathcal{K}$. To check multiplicativity, recall that by Lemma 1 $T_{\Psi_1}T_{\Psi_2} = T_{\Psi_1\Psi_2} + K$. Hence, we have

$$\pi(T_{\Psi_1})\pi(T_{\Psi_2}) = \pi(T_{\Psi_1}T_{\Psi_2}) = \pi(T_{\Psi_1\Psi_2}),$$

which is the desired result. $\square$

COROLLARY (1.3). *We have $\sigma(\pi(T_{\phi})) = \text{range } \phi$ for ϕ in $C(T)$.*

Proof. In $C(T)$, $\sigma(\phi) = \text{range } \phi$. This fact and the isomorphism of Corollary (1.2) give the desired result. $\square$

We are now in a position to classify the irreducible C^* -subalgebras of $\mathcal{A}(T_z)$. First note that if a closed *-subalgebra $\mathcal{D}$ is to be irreducible then $\mathcal{D}$ is clearly non-abelian so $\mathcal{D}$ contains a nontrivial commutator (i.e. element of the form $BD - DB$). But $\mathcal{A}(T_z)/\mathcal{K}$ is abelian so all commutators in $\mathcal{A}(T_z)$ are compact. Hence, $\mathcal{D}$ contains a nontrivial compact operator and it follows from Proposition 1 that $\mathcal{K} \subset \mathcal{D}$. For any irreducible C^* -subalgebra $\mathcal{D}$ of $\mathcal{A}(T_z)$, define $\mathcal{D}'$ by

$$\mathcal{D}' = \{\phi \text{ in } C(T) : T_{\phi} \text{ is in } \mathcal{D}\}.$$

THEOREM 2. *$\mathcal{D}'$ is a closed *-subalgebra of $C(T)$.*

Proof. If Ψ_1, Ψ_2 are in $\mathcal{D}'$ then clearly $\lambda\Psi_1, \bar{\Psi}_1$, and $\Psi_1 + \Psi_2$ are in $\mathcal{D}'$. Using Lemma 1 and the fact that $\mathcal{K} \subset \mathcal{D}$, we also see that $\Psi_1\Psi_2$ is in $\mathcal{D}'$. Finally, it follows from the fact that $\|T_{\Psi}\| = \|\Psi\|$ that $\mathcal{D}'$ is closed. $\square$

In fact, the correspondence between irreducible C^* -subalgebras of $\mathcal{A}(T_z)$ and C^* -subalgebras of $C(T)$ induced by the map $\mathcal{D} \rightarrow \mathcal{D}'$ is 1-1 and onto.

THEOREM 3. *Let $\mathcal{A}$ be any C^* -subalgebra of $C(T)$. Then there is an irreducible C^* -subalgebra $\mathcal{D}$ of $\mathcal{A}(T_z)$ such that $\mathcal{D}' = \mathcal{A}$.*

Proof. Let $\mathcal{D} = \{T_\phi + K : \phi \in \mathcal{A}, K \in \mathcal{K}\}$. We need only check that $\mathcal{D}$ is closed under multiplication. This is a direct consequence of Lemma 1. $\square$

The ideal theory of $\mathcal{D}$ can be determined by combining Proposition 1 and the following result.

THEOREM 4. *The mapping $\phi \rightarrow \pi(T_\phi)$ gives a $*$ -isomorphism from $\mathcal{D}'$ onto $\mathcal{D}/\mathcal{K}$.*

Proof. This follows from Corollary (1.2). $\square$

From now on, we will assume that $\mathcal{D}$ is an irreducible C^* -subalgebra of $\mathcal{A}(T_z)$ and 1 is in $\mathcal{D}$.

THEOREM 5. *Let $\sim$ be the equivalence relation on T induced by $\mathcal{D}'$ and let (Q, ω) be the quotient space $T/\sim$ with the weak topology of §II. Then $\mathcal{D}/\mathcal{K}$ is $*$ -isomorphic to $C(Q, \omega)$.*

Proof. This follows from Theorem 4 and Proposition 2. $\square$

THEOREM 6. *If $\mathcal{D} = \mathcal{A}(T_\phi + K)$ for some fixed ϕ and K , then $\mathcal{D}' = \{\psi : \psi \text{ in the } C^*\text{-subalgebra of } C(T) \text{ generated by } \phi\}$.*

Proof. Since $\mathcal{D}$ is irreducible, $\mathcal{K} \subset \mathcal{D}$ and so T_ϕ is in $\mathcal{D}$. Hence, ϕ is in $\mathcal{D}'$ and so $\mathcal{D}'$ contains the C^* -subalgebra of $C(T)$ generated by ϕ because of Theorem 2. Conversely, for any "polynomial" $p(r, s)$ in two noncommuting indeterminates r, s , it follows from Lemma 1 that

$$p(T_\phi + K, T_\phi^* + K^*) - T_{p(\phi, \bar{\phi})}$$

is in $\mathcal{K}$. Hence, for B in $\mathcal{A}(T_\phi + K)$ there is a sequence of elements $\{\psi_n\}$ in the C^* -algebra generated by ϕ and a sequence of compact operators K_n such that $T_{\psi_n} + K_n \rightarrow B$. The proof of Theorem 1 then shows that there is a ψ in the C^* -algebra generated by ϕ such that $B = T_\psi + K'$ for K' in $\mathcal{K}$. Since representation in the form $T_\psi + K'$ is unique for elements of $\mathcal{A}(T_z)$, we see that $\mathcal{D}'$ is contained in the C^* -algebra generated by ϕ . $\square$

COROLLARY (6.1). *If $\mathcal{D} = \mathcal{A}(T_\phi + K)$ then $\mathcal{D}/\mathcal{K}$ is $*$ -isomorphic to $C(\text{range } \phi)$.*

Proof. This follows from Corollary (1.3) since

$$\pi(T_\phi + K) = \pi(T_\phi)$$

and $\pi(T_\phi + K)$ generates $\mathcal{D}/\mathcal{K}$ as a C^* -algebra. $\square$

COROLLARY (6.2). *$\mathcal{A}(T_\phi + K) = \mathcal{A}(T_z)$ if and only if $T_\phi + K$ is irreducible and ϕ is 1-1 on T .*

Proof. Clearly $\mathcal{A}(T_\phi + K) = \mathcal{D} = \mathcal{A}(T_z)$ if and only if $T_\phi + K$ is irreducible and $\mathcal{D}' = C(T)$. By Theorem 6, $\mathcal{D}'$ is the C^* -subalgebra of $C(T)$ generated by ϕ . Hence, by the Stone-Weierstrass theorem, $\mathcal{D}' = C(T)$ if and only if ϕ is 1-1. $\square$

COROLLARY (6.3). *If ϕ is in A then $\mathcal{A}(T_\phi) = \mathcal{A}(T_z)$ if and only if ϕ is 1-1 on T .*

Proof. If ϕ is in A and ϕ is 1-1 on T then by a result in [7], T_ϕ is irreducible. Hence, by Corollary (6.2) $\mathcal{A}(T_\phi) = \mathcal{A}(T_z)$. Conversely, if $\mathcal{A}(T_\phi) = \mathcal{A}(T_z)$ then ϕ is 1-1 on T by Corollary (6.2). $\square$

IV. **Extension of $\mathcal{K}$ by $C(T)$.** On the basis of the ideal theory developed in [2] and in the previous section, we can think of $\mathcal{A}(S)$ (or $\mathcal{A}(T_z)$) as an extension of $\mathcal{K}$ by $C(T)$. In this section, I construct a different extension of $\mathcal{K}$ by $C(T)$ and show that the new extension is not $*$ -isomorphic to $\mathcal{A}(S)$. The construction is motivated by Theorem 1. Henceforth, we will deal with the Hilbert space L^2 described in §II and $\mathcal{K} = \mathcal{K}(L^2)$. Since H^2 and L^2 have the same dimension, we can think of $\mathcal{A}(S)$ as represented on L^2 by some spatial isomorphism.

THEOREM 7. *Let $\mathcal{C} = \{M_\phi + K : \phi \in C(T), K \in \mathcal{K}\}$. Then $\mathcal{C}$ is a C^* -subalgebra of $\mathcal{B}(L^2)$ and $\mathcal{C}/\mathcal{K}$ is $*$ -isomorphic to $C(T)$.*

Proof. Immediate from [4, 1.8.4]. $\square$

Theorem 7 combined with Proposition 1 shows that $\mathcal{C}$ is an irreducible C^* -algebra with the same ideal theory as $\mathcal{A}(S)$. To show that $\mathcal{A}(S)$ and $\mathcal{C}$ are not $*$ -isomorphic we need some machinery from the theory of Fredholm operators [5]. Recall that an operator B is semi-Fredholm if B has closed range $R(B)$ and either the null space $n(B)$ or $R(B)^\perp$ is of finite dimension. If B is semi-Fredholm, the index $\kappa(B)$ is defined by

$$\kappa(B) = \dim R(B)^\perp - \dim n(B).$$

It is well known that if B is semi-Fredholm and K is compact then $K+B$ is semi-Fredholm and $\kappa(B) = \kappa(B+K)$.

THEOREM 8. *There is no $*$ -isomorphism from $\mathcal{A}(S)$ onto $\mathcal{C}$.*

Proof. Suppose Φ is a $*$ -isomorphism from $\mathcal{A}(S)$ onto $\mathcal{C}$. Then $\Phi(S)^*\Phi(S) = 1$ so $\Phi(S)$ must be a noninvertible isometry. Thus $\Phi(S)$ must be a semi-Fredholm operator with index other than zero. But M_ϕ is a normal operator for all ϕ in $C(T)$ so

$$\dim n(M_\phi) = \dim R(M_\phi)^\perp.$$

It follows that if $\Phi(S) = M_\phi + K$ then M_ϕ must be semi-Fredholm and

$$\kappa(\Phi(S)) = \kappa(M_\phi + K) = \kappa(M_\phi) = 0.$$

This contradiction completes the proof. $\square$

V. **Some remarks.** It is clear from the ideal theory of $\mathcal{A}(S)$ that $\mathcal{A}(S)$ is post-liminaire [4, 4.3.1] with a composition series of length 2. It is not hard to describe the usual spaces of primitive ideals, irreducible representations and pure states of $\mathcal{A}(S)$. Denoting these spaces as usual [4, §3] by $\text{Prim}(\mathcal{A}(S))$, $\mathcal{A}(S)^\wedge$, and $P(\mathcal{A}(S))$,

we note that since $\mathcal{A}(S)$ is postliminaire, $\mathcal{A}(S)^\wedge = \text{Prim } (\mathcal{A}(S))$ [4, 4.4.1]. Further, $\text{Prim } (\mathcal{A}(S))$ is the union of a circle and a point, the point being dense in the Jacobson topology. This is because $\{0\}$ is primitive and by Proposition 1, any other primitive ideal contains $\mathcal{K}$ and so corresponds to a maximal ideal in $C(T)$. The relative topology on the circle part of $\text{Prim } (\mathcal{A}(S))$ is the usual one for T . Note also that for $\mathcal{C}$ as in §IV, $\hat{\mathcal{C}} = \text{Prim } (\mathcal{C}) = \text{Prim } (\mathcal{A}(S))$ since $\mathcal{C}$ has the same ideal structure as $\mathcal{A}(S)$. Similarly, it is not hard to see that $P(\mathcal{A}(S))$ consists of vector states of $\mathcal{K}$ extended to $\mathcal{A}(S)$ [4, 2.11.8] and multiplicative states corresponding to the points of T . Again $P(\mathcal{C}) = P(\mathcal{A}(S))$.

REFERENCES

1. A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1964), 89–102.
2. L. A. Coburn, *The C^* -algebra generated by an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726.
3. L. A. Coburn, *Weyl's theorem for non-normal operators*, Michigan Math. J. **13** (1966), 285–288.
4. J. Dixmier, *Les C^* -algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
5. J. C. Gohberg and M. G. Krein, *The basic propositions on defect numbers, root numbers, and indices of linear operators*, Amer. Math. Soc. Transl. (2) **13** (1960), 185–264.
6. M. A. Naimark, *Normed rings*, Noordhoff, Groningen, 1959.
7. E. A. Nordgren, *Reducing subspaces of analytic Toeplitz operators*, Duke Math. J. **34** (1967), 175–182.
8. H. Widom, *On the spectrum of a Toeplitz operator*, Pacific J. Math. **14** (1964), 365–375.

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