FREE DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY 11-SPHERES

BY

HSU-TUNG KU⁽¹⁾ AND MEI-CHIN KU

1. Introduction. By a homotopy n -sphere we mean a closed differentiable n -manifold having the homotopy of S^n . In [6] Montgomery and Yang have shown that there are exactly 10 homotopy 7-spheres not diffeomorphic to one another such that on each of them there is a free differentiable S^1 action. Moreover, on any homotopy 7-sphere, if there exists a free differentiable S^1 action, then there are infinitely many topologically distinct from one another. It is interesting to study an analogue for the free differentiable actions of S^1 and S^3 on the homotopy 11-spheres. The first example of free differentiable S^1 and S^3 action on an exotic homotopy 11-sphere is due to the Hsiangs [4]. In this paper, we succeed in finding out all possible homotopy 11-spheres which admit free differentiable S^3 actions. More precisely, we shall prove the following results.

THEOREM A. *Let Σ_M^{11} denote the Milnor sphere which is the generator of θ_{11} [5]. Then the homotopy sphere $k\Sigma_M^{11}$, k odd, admits no free differentiable S^1 actions.*

THEOREM B. *A homotopy 11-sphere Σ^{11} admits a free differentiable S^3 action if and only if $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16 \pmod{31}$. These all admit infinitely many topologically distinct actions which can be distinguished by the first Pontrjagin class of the orbit spaces.*

Throughout the paper, $\mathbb{Z}$ denotes the ring of integers; $\mathbb{Q}$ denotes the field of rational numbers. We let QP^2 be the quaternionic projective plane, and let CP^5 be the complex projective 5-space, all having the usual differentiable structure.

The main methods used to prove these results are based on the computation of the Eells-Kuiper μ -invariant [2] and the Montgomery-Yang ν -invariant [6].

2. Proof of Theorem A. First, we recall the definition of ν -invariant. Let Σ be a homotopy $(4k-1)$ -sphere which is the boundary of a compact, oriented, differentiable $4k$ -manifold W . The invariant $\nu(\Sigma)$ of Montgomery-Yang [6] is defined as follows: We assume that

(1) there is an element $\beta \in H^2(W; \mathbb{Z})$ whose reduction mod 2 is the second Stiefel-Whitney class $w_2(W)$ of W . Note that the homomorphism

$$j^*: H^q(W, \Sigma; \mathbb{Q}) \rightarrow H^q(W; \mathbb{Q})$$

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induced by the inclusion map is an isomorphism onto for $q=4l$, $0 < l < k$ and for $q=2k$. Define $\nu(\Sigma)$ by

$$\nu(\Sigma) \equiv \left\langle e^{i^{s-1}\beta/2} \sum_{i=0}^{k-1} \hat{A}_i(p_1(W), \dots, p_i(W)) + N_k(p_1(W), \dots, p_{k-1}(W)), [W] \right\rangle \\ + t_k \tau(W) \pmod{1},$$

where we use the notation in [2].

We now proceed with the proof of the Theorem A. In order to prove Theorem A it suffices to prove the following lemma because the Eells-Kuiper invariant $\mu(\Sigma_M^{11}) \equiv -1/992 \pmod{1}$ and $\theta_{11} = Z_{992}$ [5].

LEMMA 2.1. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then*

(2) $\mu(\Sigma^{11}) \equiv (18i + 204i^2 + 576i^3 + 296j + 1440ij + 496t)/992 \pmod{1}$, for some integers j , i and t .

Proof. Let α be the generator of $H^2(\Sigma^{11}/S^1; \mathbb{Z})$. Since Σ^{11} admits free S^1 action, we have the principal bundle $\xi: S^1 \rightarrow \Sigma^{11} \rightarrow \Sigma^{11}/S^1$ which is homotopically equivalent to the classical Hopf bundle $S^1 \rightarrow S^{11} \rightarrow CP^5$. Hence the total Pontrjagin class $p(\xi)$ of the bundle ξ is given by $p(\xi) = 1 + \alpha^2$. Let W be the total space of the associated disk bundle $D^2 \rightarrow W \xrightarrow{\pi} \Sigma^{11}/S^1$ of ξ . The space W is homotopically equivalent to Σ^{11}/S^1 with $w_2(W) \neq 0$, and index $\tau(W) = 1$. Let

$$p_1(\Sigma^{11}/S^1) = r_1\alpha^2, \quad p_2(\Sigma^{11}/S^1) = r_2\alpha^4, \\ p_1(W) = r_3\beta^2, \quad p_2(W) = r_4\beta^4,$$

where $\beta = \pi^*\alpha$, and $r_1, r_2, r_3, r_4 \in \mathbb{Z}$. We can see that $r_3 = r_1 + 1$, and $r_4 = r_1 + r_2$ because

$$p^*(W) = \pi^*\{(1 + r_1\alpha^2 + r_2\alpha^4)(1 + \alpha^2)\} = 1 + r_3\beta^2 + r_4\beta^4.$$

Since β reduction mod 2 is $w_2(W)$, the invariant $\nu(\Sigma^{11})$ is well defined. By substituting the above data into the formula for ν , we obtain

$$\nu(\Sigma^{11}) \equiv \frac{1}{2^{10} \cdot 3^2 \cdot 5} - \frac{r_3}{2^{10} \cdot 3^2} + \frac{-4r_4 + 7r_3^2}{2^{10} \cdot 3^2 \cdot 5} + \frac{1260r_3r_4 - 945r_3^3}{2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31} - \frac{1}{2^7 \cdot 31} \pmod{1}.$$

But $\nu(\Sigma^{11}) = 2\mu(\Sigma^{11})$ [6]; hence

$$496\nu(\Sigma^{11}) \equiv 992\mu(\Sigma^{11}) \\ \equiv \{-329 - 155r_3 - 124r_4 + 217r_3^2 + 60r_3r_4 - 45r_3^3\}/2^5 \cdot 3^2 \cdot 5 \pmod{496} \\ \equiv \{-312 + 80r_1 - 64r_2 + 142r_1^2 + 60r_1r_2 - 45r_1^3\}/2^5 \cdot 3^2 \cdot 5 \pmod{496}.$$

As μ is a complete invariant for homotopy 11-spheres, $992\mu(\Sigma^{11})$ is an integer modulo 496. Thus we must have $r_1 = 2s$ for some integer s , and

$$992\mu(\Sigma^{11}) \equiv \{-39 + 20s - 8r_2 + 71s^2 + 15r_2s - 45s^3\}/2^2 \cdot 3^2 \cdot 5 \pmod{496}.$$

Again we can see that both r_2 and s must be odd, say, $r_2 = 2h + 1$ and $s = 2n + 1$ for some integers h and n . Then

$$992\mu(\Sigma^{11}) \equiv \{7 + 42n + 7h - 128n^2 + 30hn - 180n^3\}/2 \cdot 3^2 \cdot 5 \pmod{496}.$$

The integer h must be odd, that is, $h = 2a + 1$ for some integer a . Hence we have

$$(3) \quad 992\mu(\Sigma^{11}) \equiv \{7 + 36n + 7a - 64n^2 + 30an - 90n^3\}/45 \pmod{496}.$$

On the other hand 2α (resp. 4α) reduction mod 2 is equal to $0 = w_2(\Sigma^{11}/S^1)$, hence

$$A(\Sigma^{11}/S^1, \alpha) = \langle e^a \hat{A}(\Sigma^{11}/S^1), [\Sigma^{11}/S^1] \rangle$$

(respectively, $A(\Sigma^{11}/S^1, 2\alpha)$) is an integer [3], where $[\Sigma^{11}/S^1]$ denotes the fundamental homology class of Σ^{11}/S^1 . By a simple calculation

$$\begin{aligned} A(\Sigma^{11}/S^1, \alpha) &= \left\langle \frac{\alpha^5}{5!} - \frac{\alpha^3 p_1(\Sigma^{11}/S^1)}{6 \cdot 24} + \frac{\alpha}{2^7 \cdot 45} \{-4p_2(\Sigma^{11}/S^1) + 7p_1^2(\Sigma^{11}/S^1)\}, [\Sigma^{11}/S^1] \right\rangle \\ &= \frac{1}{5!} - \frac{r_1}{6 \cdot 24} + \frac{-4r_2 + 7r_1^2}{2^7 \cdot 45} \\ &= (7n^2 - 3n - 1 - a)/360; \end{aligned}$$

here we use the fact that $r_1 = 4n + 2$, and $r_2 = 4a + 3$. Similarly

$$A(\Sigma^{11}/S^1, 2\alpha) = (7n^2 - 33n + 29 - a)/180.$$

Hence we have

$$(4) \quad a \equiv 7n^2 - 3n - 1 \pmod{360},$$

$$(5) \quad a \equiv 7n^2 - 33n + 29 \pmod{180}.$$

Solving (4) and (5) we obtain the solution $n \equiv 1 \pmod{6}$, thus we can write

$$n = 6i + 1, \quad a = 7n^2 - 3n - 1 + 360j$$

for some integers i and j . Substituting these values into (3), we get

$$992\mu(\Sigma^{11}) \equiv 18i + 204i^2 + 576i^3 + 296j + 1440ij \pmod{496}.$$

This completes the proof of Lemma 2.1.

Notice that $r_1 = 4n + 2 = 24i + 6$, hence as a by-product of the proof, we have the following corollary:

COROLLARY 2.2. *If S^1 acts freely and differentiably on a homotopy 11-sphere Σ^{11} , then*

$$p_1(\Sigma^{11}/S^1) = (24i + 6)\alpha^2, \quad p_2(\Sigma^{11}/S^1) = (1008i^2 + 264i + 15 + 1440j)\alpha^4,$$

for some integers i and j .

3. Proof of Theorem B. First we will demonstrate a number of lemmas which lead to the proof of Theorem B.

LEMMA 3.1. *Let S^3 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then $\mu(\Sigma^{11}) \equiv n/124 \pmod{1}$ for some integer n .*

Proof. The proof is almost the same as the proof of Lemma 2.1. We have the principal bundle $\eta: S^3 \rightarrow \Sigma^{11} \rightarrow \Sigma^{11}/S^3$ with associated disk bundle $D^4 \rightarrow W \xrightarrow{\pi} \Sigma^{11}/S^3$. The bundle η is homotopically equivalent to the Hopf bundle $S^3 \rightarrow S^{11} \rightarrow QP^2$, hence the total Pontrjagin class $p(\eta) = 1 + 2\bar{\alpha} + \bar{\alpha}^2$, where $\bar{\alpha}$ is the generator of $H^4(\Sigma^{11}/S^3; \mathbb{Z})$. Since W and Σ^{11}/S^3 are of the same homotopy type, it is not difficult to see that W is a spin manifold with index $\tau(W) = 0$. Let $p_1(\Sigma^{11}/S^3) = r\bar{\alpha}$, $r \in \mathbb{Z}$, and $\beta = \pi^*\bar{\alpha} \in H^4(W; \mathbb{Z})$. By Hirzebruch's index theorem [3] we have

$$p_2(\Sigma^{11}/S^3) = (45 + r^2)\bar{\alpha}^2/7.$$

The total Pontrjagin class $p(W)$ is given by

$$p(W) = \pi^*\{(1 + r\bar{\alpha} + (45 + r^2)\bar{\alpha}^2/7)(1 + 2\bar{\alpha} + \bar{\alpha}^2)\}.$$

This implies

$$p_1(W) = (r+2)\beta, \quad p_2(W) = (r^2 + 14r + 52)\beta^2/7.$$

We calculate the Eells-Kuiper μ -invariant

$$\begin{aligned} \mu(\Sigma^{11}) &\equiv \{4p_2(W)p_1(W) - 3p_1^3(W) - 24\tau\}[W]/2^{11} \cdot 3 \cdot 31 \pmod{1} \\ &\equiv -(r+2)(r-2)(17r+62)/2^{11} \cdot 3 \cdot 7 \cdot 31 \pmod{1}. \end{aligned}$$

The number $992\mu(\Sigma^{11})$ is an integer modulo 992, hence the integer r must be in the form $r = 4m + 2$ for some integer m so that

$$992\mu(\Sigma^{11}) \equiv -m(m+1)(17m+24)/21 \pmod{992},$$

and

$$(6) \quad m(m+1)(17m+24) \equiv 0 \pmod{21}.$$

Note that $\hat{A}_2(\Sigma^{11}/S^3)$ is an integer because Σ^{11}/S^3 is a closed spin manifold [3]. As

$$\begin{aligned} \hat{A}_2(\Sigma^{11}/S^3) &= \{-4p_2(\Sigma^{11}/S^3) + 7p_1^2(\Sigma^{11}/S^3)\}/2^7 \cdot 45 \\ &= (r^2 - 4)/2^7 \cdot 7 = m(m+1)/56, \end{aligned}$$

hence

$$(7) \quad m(m+1) \equiv 0 \pmod{56}.$$

Thus there exists an integer c such that $m(m+1) = 56c$. Substituting this into (6) we obtain

$$8c(17m+24) \equiv 0 \pmod{3},$$

or $c(17m+24)/3$ is an integer. Therefore

$$\begin{aligned}\mu(\Sigma^{11}) &\equiv -m(m+1)(17m+24)/992 \cdot 21 \pmod{1} \\ &\equiv -8\{c(17m+24)/3\}/992 \pmod{1} \\ &\equiv -\{c(17m+24)/3\}/124 \pmod{1}.\end{aligned}$$

LEMMA 3.2. *If S^3 acts freely and differentiably on Σ^{11} , then $p_1(\Sigma^{11}/S^3) = (96\bar{i} + 2)\bar{\alpha}$, for some $\bar{i} \in \mathbb{Z}$, where $\bar{\alpha}$ is the generator of $H^4(\Sigma^{11}/S^3; \mathbb{Z})$.*

Proof. Since S^1 is a subgroup of S^3 , the free action of S^3 on Σ^{11} induced a free action of S^1 on Σ^{11} . It follows from (2) and Lemma 3.1 that

$$18i + 204i^2 + 576i^3 + 296j + 1440ij \equiv 0 \pmod{8},$$

or

$$2i(9 + 102i) \equiv 0 \pmod{8}.$$

Hence $i \equiv 0 \pmod{4}$. This shows that $i = 4\bar{i}$ for some integer $\bar{i}$.

Next consider the fibre bundle $\zeta: S^2 \rightarrow \Sigma^{11}/S^1 \xrightarrow{\pi} \Sigma^{11}/S^3$ which is homotopically equivalent to the standard fibration $S^2 \rightarrow CP^5 \xrightarrow{\pi} QP^2$. By using the well-known fact that $p(\zeta) = 1 + 4\bar{\alpha}$ [1],

$$p(\Sigma^{11}/S^1) = \pi^*\{[1 + r\bar{\alpha} + (45 + r^2)\bar{\alpha}^2/7][1 + 4\bar{\alpha}]\},$$

and so $r_1 = r + 4 = 24i + 6 = 96\bar{i} + 6$ by Corollary 2.2. Thus

$$(8) \quad r = 96\bar{i} + 2.$$

Recall that $r = 4m + 2$, and m satisfies (6) and (7).

LEMMA 3.3. *The solutions m of (6) and (7) satisfying (8) are the following:*

$$(9) \quad m \equiv 0 \pmod{168}, \text{ hence } r \equiv 2 \pmod{672},$$

or

$$(10) \quad m \equiv 48 \pmod{168}, \text{ hence } r \equiv 194 \pmod{672}.$$

The proof is an elementary number theory calculation.

Summing up, we have therefore proved

LEMMA 3.4. *Let S^3 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then one of the following holds:*

$$(11) \quad p_1(\Sigma^{11}/S^3) = (672k + 2)\bar{\alpha} \text{ for some integer } k, \text{ and}$$

$$\mu(\Sigma^{11}) \equiv -(13k + 1)(k + 6)k/31 \pmod{1}.$$

$$(12) \quad p_1(\Sigma^{11}/S^3) = (672k + 194)\bar{\alpha} \text{ for some integer } k, \text{ and}$$

$$\mu(\Sigma^{11}) \equiv -7(-k + 1)(7k + 2)(k - 7)/31 \pmod{1}.$$

Now we can easily verify Theorem B. By direct computation, a free differentiable S^3 action on a homotopy 11-sphere Σ^{11} with $p_1(\Sigma^{11}/S^3)$ and $\mu(\Sigma^{11})$ satisfying (11)

(respectively (12)) implies $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16 \pmod{31}$. But W. C. Hsiang and W. Y. Hsiang [4] have constructed examples satisfying (11) for all integers $k \in \mathbb{Z}$. This clearly completes the proof of Theorem B.

It is of interest to find out what can be said about the existence of free differentiable S^3 actions on Σ^{11} satisfying (12).

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INSTITUTE FOR ADVANCED STUDY,
PRINCETON, NEW JERSEY