A UNIQUENESS THEOREM FOR HAAR AND WALSH SERIES⁽¹⁾

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1. **Introduction.** It is well known that if a trigonometric series converges to an integrable function, except possibly in a countable set, and the series' coefficients converge to zero then that series is the Fourier series of the limit function [8, p. 329].

This problem for Walsh series was open for many years, but in 1965 two independent solutions were published: [2] and [3]. By combining the American and Soviet techniques we are able to obtain a theorem which contains the Walsh series result in [2] but which has a form similar to the theorem in [3]. Following the Soviet example, we will prove the result for Haar series and obtain the Walsh series result as a corollary.

In this paper E will represent a countable subset of $[0, 1]$, and $D.R.$ will represent the set of dyadic rationals. Given a Borel set A , $I_A(x)$ will denote the characteristic function of the set A .

The Haar system $\{\chi_k\}_{k=0}^\infty$ is defined as $\chi_0(x) = 1$, $\chi_1(x) = I_{[0, 1/2)}(x) - I_{[1/2, 1)}(x)$; in general writing $k' = 2^n + k$, $0 \leq k < 2^n$ where n is the largest power of 2 which is less than or equal to k' , we define

$$\begin{aligned}\chi_k(x) &= \chi_n^{(k)}(x) = (2^n)^{1/2} & 2k - 2/2^{n+1} < x < 2k - 1/2^{n+1}, \\ &= -(2^n)^{1/2} & 2k - 1/2^{n+1} < x < 2k/2^{n+1}, \\ &= (2^n/4)^{1/2} & x = k - 1/2^n, \\ &= -(2^n/4)^{1/2} & x = k/2^n, \\ &= 0 & \text{otherwise.}\end{aligned}$$

We will denote

$$(2k - 2/2^{n+1}, 2k - 1/2^{n+1}) = \Delta_{k'}^{(1)}, \quad (2k - 1/2^{n+1}, 2k/2^{n+1}) = \Delta_{k'}^{(2)}$$

and these two open intervals will be referred to as the positive (respectively negative) support of the k th Haar function. Alexits [1] proves this sequence is a complete orthonormal system.

We define the Walsh system $\{\Psi_k\}_{k=0}^\infty$ by letting $\Phi_0(x) = I_{[0, 1/2)}(x) - I_{[1/2, 1)}(x)$, $\Phi_n(x) = \Phi_0(2^n x)$ (where Φ_0 is extended by periodicity of period 1) and then defining $\Psi_0(x) = 1$, $\Psi_n(x) = \Phi_{n_1}(x) \cdots \Phi_{n_r}(x)$ where $n = \sum_{i=1}^r 2^{n_i}$ and the n_i are uniquely

Received by the editors October 1, 1968.

⁽¹⁾ This research was done in partial satisfaction for the Doctor of Philosophy degree at the University of California, Riverside, under the direction of Professor Victor L. Shapiro.

determined by $n_{i+1} < n_i$. Walsh [7] proves this sequence is a complete orthonormal system.

Both the Walsh and Haar systems are extended by periodicity of period 1 to the whole real line. Kacmarcz noted [1, p. 62] that if $k' = 2^n + k$, $0 \leq k < 2^n$ and n largest such that $2^n \leq k'$, then

$$(1) \quad \Psi_{k'}(x) = \frac{1}{(2^n)^{1/2}} \sum_{j=1}^{2^n} \varepsilon_{k'j} \chi_n^{(j)}(x) \quad \text{for } x \notin \text{D.R.}$$

where $\varepsilon_{jk'} = \pm 1$ and the matrix $[\varepsilon_{jk'}]$ has orthogonal rows. If S is a Haar or Walsh series, S_n will denote the partial sum of order $n-1$.

Given a Haar series $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$, we shall say S satisfies *condition G* if for every $x_0 \in [0, 1]$,

$$(2) \quad \lim_{j \rightarrow \infty} \frac{a_{k_j}}{\chi_{k_j}(x_0)} = 0$$

where k_j are all those integers p for which $\chi_p(x_0) \neq 0$. We note that this growth condition is essential to Theorem 1 since

$$\chi_0(x) + \sum_{n=0}^{\infty} (2^n)^{1/2} \chi_{2^n}(x) \equiv 0 \quad \text{on } (0, 1]$$

but the series fails to satisfy (2) only at the point $x_0 = 0$.

We shall say a function g is in the class $\mathcal{A}$ if there is a closed countable set $A_g \subseteq [0, 1]$ such that g is locally integrable in $(0, 1) \sim A_g$. We note that $L^1[0, 1]$ is a proper subset of $\mathcal{A}$ since $A_{\{1/x\}} = \{0\}$.

We shall prove:

THEOREM 1. Let $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ be a Haar series satisfying condition G such that for some function $g \in \mathcal{A}$ and subsequence of natural numbers $\{n_j\}$

- (i) $\lim_{j \rightarrow \infty} S_{2^{n_j}}(x) = g(x)$ in measure;
- (ii) $\limsup_{j \rightarrow \infty} |S_{2^{n_j}}(x)| < \infty$, $x \notin E$;
- (iii) $\limsup_{j \rightarrow \infty} S_{2^{n_j}}(x)$ dominates an integrable function $f(x)$ for $x \notin E$.

Then g is integrable and S is the Haar Fourier series of g .

THEOREM 2. Let $S(x) = \sum_{k=0}^{\infty} a_k \Psi_k(x)$ be a Walsh series such that for a function $g \in \mathcal{A}$ and some subsequence of natural numbers $\{n_j\}$

- (i) $\lim_{j \rightarrow \infty} S_{2^{n_j}}(x) = g(x)$ in measure;
- (ii) $\limsup_{j \rightarrow \infty} |S_{2^{n_j}}(x)| < \infty$ for $x \notin E$;
- (iii) $\limsup_{j \rightarrow \infty} S_{2^{n_j}}(x)$ dominates an integrable function $f(x)$ for $x \notin E$;
- (iv) $\lim_{k \rightarrow \infty} a_k = 0$.

Then g is integrable and S is the Walsh Fourier series of g .

2. Fundamental lemmas. By $\alpha_n = \alpha_n(x)$ and $\beta_n(x) = \beta_n$ we shall mean

$$(3) \quad \alpha_n = P \cdot 2^{-n} \leq x < (P+1)2^{-n} = \beta_n$$

and $\alpha'_n(x) = \alpha_n(x)$ if $x \notin D.R.$, $\alpha'_n(x) = \alpha_n(x) - 2^{-n}$ otherwise. Given a Haar series $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ we define

$$(4) \quad L(S, x, n_j) = \lim_{j \rightarrow \infty} \sum_{k=0}^{2^{n_j}-1} a_k \int_0^x \chi_k(u) du = \lim_{j \rightarrow \infty} L_{2^{n_j}}(S, x)$$

when this limit exists. In case $L(S, x, n)$ exists we will write it as $L(S, x)$ following the notation in [5].

LEMMA 1. If $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ is a Haar series satisfying condition G , then for every x and $n \geq 0$ both $L(S, \alpha_n(x))$ and $L(S, \beta_n(x))$ exist and are finite. Furthermore

$$(5) \quad L(S, \beta_n(x)) - L(S, \alpha_n(x)) = 2^{-n} S_{2^n}(x),$$

$$(6) \quad L(S, \beta_n(x)) - L(S, \alpha_n(x)) = o(1) \quad \text{as } n \rightarrow \infty.$$

Proof. We use [3, Lemma 1] as an outline of the proof of this lemma. To show (6), we use (5) and the fact that condition G implies $2^{-n} S_{2^n}(x) = o(1)$.

To see this, given $\varepsilon > 0$ and $x_0 \in [0, 1]$ we choose N sufficiently large so that $j \geq N$ implies

$$(7) \quad |a_{k_j}| < \varepsilon \cdot |\chi_{k_j}(x_0)|$$

where k_j are defined in the definition of condition G . We then recall from the very definition of the Haar functions that the set

$$(8) \quad \{\chi_k^{(1)}(x_0), \chi_k^{(2)}(x_0), \dots, \chi_k^{(2^k)}(x_0)\}$$

has at most two nonzero elements for each k , and that $|\chi_k^{(p)}(x_0)|^2 \leq 2^k$.

Combining (7) and (8) if $n > N$,

$$(9) \quad \left| \frac{S_{2^n}(x_0)}{2^n} \right| \leq \sup_{0 \leq j \leq N} \frac{|a_{k_j}| \cdot N}{(2^n)^{1/2}} + 2\varepsilon \cdot \frac{\sum_{k=0}^n 2^k}{2^n}.$$

Hence taking lim sup of (9) as $n \rightarrow \infty$ we obtain $S_{2^n}(x_0) = o(2^n)$.

The following lemma is the Haar series analogue of a lemma in [3], and the proof is essentially the same.

LEMMA 2. If $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ is a Haar series satisfying condition G and if $L(S, x, n_j)$ exists and is finite, then

$$(10) \quad L(S, \alpha'_{n_j}) \rightarrow L(S, x, n_j) \quad \text{as } j \rightarrow \infty,$$

$$(11) \quad L(S, \beta_{n_j}) \rightarrow L(S, x, n_j) \quad \text{as } j \rightarrow \infty.$$

The following lemma is proved in a similar manner to Lemma 3 of [3].

LEMMA 3. Let $G(x)$ be defined in $a < x < b$ and satisfy:

(i) Except perhaps on a countable set Z , $\liminf_{j \rightarrow \infty} 2^{n_j} [G(\beta_{n_j}) - G(\alpha_{n_j})] \leq 0$.

(ii) For all $x \in (a, b)$, both $G(\alpha'_{n_j}) \rightarrow G(x)$, $G(\beta_{n_j}) \rightarrow G(x)$ as $j \rightarrow \infty$.

Then $G(x)$ is monotone nonincreasing in (a, b) .

LEMMA 4. Let $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ be a Haar series satisfying condition G , let f be a finite valued function, integrable over $(a-\varepsilon, b+\varepsilon)$ for $\varepsilon > 0$, and let $\{n_j\}$ be a subsequence of the natural numbers. Suppose further that Z is a countable subset of $(a-\varepsilon, b+\varepsilon)$ and that

(i) $L(S, x, n_j)$ exists and is finite for every $x \in (a-\varepsilon, b+\varepsilon)$;

(ii) $+\infty > \limsup_{j \rightarrow \infty} S_{2^{n_j}}(x) \geq f(x)$ for $x \notin Z$.

Then

$$(12) \quad \limsup_{j \rightarrow \infty} S_{2^{n_j}}(x) = \liminf_{j \rightarrow \infty} S_{2^{n_j}}(x) \quad \text{a.e. in } (a, b)$$

and this function is integrable over (a, b) .

Proof. From hypothesis (ii) and (5) we have

$$(13) \quad \limsup_{j \rightarrow \infty} 2^{n_j} [L(S, \beta_{n_j}) - L(S, \alpha_{n_j})] \geq f(x) \quad \text{for } x \in (a-\varepsilon, b+\varepsilon) \sim Z.$$

Using the Vitali-Carathéodory Theorem [5, p. 75] we choose an upper semi-continuous function $f_1 \leq f$ in $L^1(a-\varepsilon, b+\varepsilon)$.

Let $F_1(x) = \int_{a-\varepsilon/2}^x f_1(u) du$ for $x \in (a-\varepsilon/2, b+\varepsilon)$. Clearly since $|\alpha_{n_j} - \beta_{n_j}| = 2^{-n_j}$,

$$\liminf_{j \rightarrow \infty} 2^{n_j} [F_1(\beta_{n_j}) - F_1(\alpha_{n_j})] \leq f(x) \quad \text{for } x \in (a-\varepsilon/2, b+\varepsilon).$$

Thus by (13), for $x \in (a-\varepsilon/2, b+\varepsilon) \sim Z$,

$$(14) \quad \liminf_{j \rightarrow \infty} 2^{n_j} \{F_1(\beta_{n_j}) - L(S, \beta_{n_j}) - [F_1(\alpha_{n_j}) - L(S, \alpha_{n_j})]\} \leq 0.$$

Now Lemma 2 and continuity of F_1 imply hypothesis (ii) of Lemma 3 is also satisfied by the function $F_1(x) - L(S, x, n_j)$, so by Lemma 3 this function is monotone nonincreasing in $(a-\varepsilon/2, b+\varepsilon)$.

Thus $L(S, x, n_j)$ has derivatives almost everywhere which are integrable on every closed subinterval of $(a-\varepsilon/2, b+\varepsilon)$. Hence (12) is a consequence of the equality

$$(15) \quad \frac{L(S, \beta_{n_j}) - L(S, \alpha_{n_j})}{\beta_{n_j} - \alpha_{n_j}} = S_{2^{n_j}}(x).$$

LEMMA 5. If $S'(x) = \sum_{k=0}^{\infty} c_k \chi_k(x)$ is the Haar Fourier series of an integrable function $f(x)$ then

$$(16) \quad S' \text{ satisfies condition } G,$$

$$(17) \quad \lim_{n \rightarrow \infty} \int_0^1 |S'_n(x) - f(x)| dx = 0.$$

Proof. By assumption $c_k = \int_0^1 f(x) \chi_k(x) dx$. Thus

$$|c_k| \leq \max_x |\chi_k(x)| \left\{ \int_{\Delta_k^{(1)} \cup \Delta_k^{(2)}} |f(x)| dx \right\}.$$

But f is integrable, so $m(\Delta_k^{(1)} \cup \Delta_k^{(2)}) \rightarrow 0$ implies

$$\frac{|c_k|}{\max_x |\chi_k(x)|} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

But if $\chi_k(x_0) \neq 0$ then $|\chi_k(x_0)| \geq \frac{1}{2} \max_x |\chi_k(x)|$ so (16) is proved. (17) is the theorem appearing in [6].

We quote the main lemma of [2].

LEMMA 6. Suppose $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ is a Haar series satisfying condition G and f is an integrable function whose Haar Fourier series is $S'(x) = \sum_{k=0}^{\infty} c_k \chi_k(x)$. Suppose also that for some subsequence of natural numbers $\{n_j\}$, $S_{2^{n_j}}(x)$ converges to $f(x)$ in measure.

Let $x_0 \in [0, 1]$ and k_0 be an integer such that

(α) inside $\Delta_{k_0}^{(i_0)}$, $i_0 = 1$ or 2 , $S_{k_0+1}(x)$, $S'_{k_0+1}(x)$ have different constant values,

(β) $k > k_0$ implies χ_k is nonzero either in $\Delta_{k_0}^{(i_0)}$ or outside it.

Then for any $M > 0$ and any positive integer N we can find an n_j , a natural number p and an interval of the form $\Delta_p^{(i_p)}$, $i_p = 1$ or 2 such that

(1) $2^{n_j} > N$,

(2) $x_0 \notin \Delta_p^{(i_p)-}$ and $\Delta_p^{(i_p)-} \subset \Delta_{k_0}^{(i_0)}$,

(3) $|S_{2^{n_j}}(x)| > M$ for $x \in \Delta_p^{(i_p)}$ and is constant there,

(4) p , $\Delta_p^{(i_p)}$ satisfy (α), (β).

We close this section by recalling a theorem of Vitali [4, p. 152].

LEMMA 7. If $\{u_n\}_{n=0}^{\infty}$ is a sequence of functions which have equiabsolutely continuous integrals; i.e., given $\varepsilon > 0$ there is a δ such that for $E \subseteq [0, 1]$, $m(E) < \delta$ implies

$$\left| \int_E u_n(x) dx \right| < \varepsilon \quad \text{for all } n;$$

and if $u_n(x) \rightarrow f(x)$ in measure, then $f(x)$ is integrable and

$$\lim_{n \rightarrow \infty} \int_0^1 u_n(x) dx = \int_0^1 f(x) dx.$$

3. **Proof of Theorem 1.** Since $g \in \mathcal{A}$, given $x_0 \notin A_g$ we can find an interval of the form $\Delta_{j_0}^{(i_0)}$ containing x_0 such that $\Delta_{j_0}^{(i_0)} \cap A_g = \emptyset$, and g is integrable over $\Delta_{j_0}^{(i_0)}$. We claim that $L(S, x_0, n_j)$ exists and is finite. By Lemma 1, we may assume x_0 is not a dyadic rational.

Consider the function

$$(18) \quad g^*(x) = [g(x) - S_{j_0+1}(x)]I_{\Delta_{j_0}^{(i_0)}}(x).$$

By choice of j_0 , g^* is integrable in $[0, 1]$. By hypothesis (i), $S_{2^{n_j}}(x)I_{\Delta_{j_0}^{(i_0)}}(x) \rightarrow g(x) \cdot I_{\Delta_{j_0}^{(i_0)}}(x)$ in measure, so if we define

$$(19) \quad T(x) = [S(x) - S_{j_0+1}(x)]I_{\Delta_{j_0}^{(i_0)}}(x)$$

we have by (18)

$$(20) \quad \lim_{j \rightarrow \infty} T_{2^{n_j}}(x) = g^*(x) \quad \text{in measure.}$$

We note that T is actually a Haar series, say $T(x) = \sum_{k=0}^{\infty} \alpha_k \chi_k(x)$, and since $|\alpha_k| \leq |a_k|$, T satisfies *condition G*. Let $S'(x) = \sum_{k=0}^{\infty} c_k \chi_k(x)$ be the Haar Fourier series of g^* . We claim S' and T are the same series.

Suppose not, and let k_0 be the least integer for which $\alpha_{k_0} \neq c_{k_0}$. Clearly $T_{k_0+1}(x) - S'_{k_0+1}(x) \equiv d$ in $\Delta_{k_0}^{(i_0)}$ where d is a nonzero constant and $i_0 = 1$ or 2 .

Let $E \cup \text{D.R.} = \{Z_1, Z_2, \dots\}$; we use Lemma 6 countably many times to obtain sequences $\{n_{j_k}\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ such that

$$(21) \quad Z_k \notin \Delta_{p_k}^{(i_{p_k})^-} \subset \Delta_{p_k-1}^{(i_{p_k-1})} \quad \text{for } k = 2, 3, \dots,$$

$$(22) \quad |T_{2^{n_{j_k}}}(x)| > k \quad \text{for } x \in \Delta_{p_k}^{(i_{p_k})}, \quad k = 1, 2, \dots$$

Since the dyadic rationals are excluded from the nested sequence $\Delta_{p_k}^{(i_{p_k})}$, $m(\Delta_{p_k}^{(i_{p_k})})$ tends to zero as $k \rightarrow \infty$. By (21) we let $\xi \in \bigcap_{k=1}^{\infty} \Delta_{p_k}^{(i_{p_k})}$. Then by (22),

$$\limsup_{k \rightarrow \infty} |S_{2^{n_{j_k}}}(\xi)| = \infty$$

which implies by hypothesis (ii) that $\xi \in E \subseteq \{Z, Z_2, \dots\}$ which contradicts (21). Thus the assumption was false and $S' \equiv T$.

Thus by (19), if $n > j_0 + 1$ and $x \in \Delta_{j_0}^{(i)}$,

$$(23) \quad S_n(x) = T_n(x) + S_{j_0+1}(x).$$

Since T is a Fourier series and S_{j_0+1} is a polynomial we use (17) and hypothesis (i) to see S_n satisfies the hypotheses of Lemma 7 inside $\Delta_{j_0}^{(i)}$. Choose $\rho \in \text{D.R.} \cap \Delta_{j_0}^{(i)}$ to the left of x_0 , and use Lemma 7 to obtain

$$\begin{aligned} L(S, x_0, n_j) - L(S, \rho) &= \lim_{j \rightarrow \infty} \int_{\rho}^{x_0} S_{2^{n_j}}(u) du \\ &= \int_{\rho}^{x_0} g(u) du < \infty \end{aligned}$$

since $(\rho, x_0) \subseteq \Delta_{j_0}^{(i)}$. But by Lemma 1, $L(S, \rho)$ is finite and thus $L(S, x_0, n_j)$ exists and is finite. The claim is thus true.

Let $N = \{x \in (0, 1) \mid L(S, x, n_j) \text{ does not exist}\}$. By our claim $N \subseteq A_g$. If we can show that N has no isolated points, then $\bar{N} \subseteq (A_g)^- = A_g$ would be a perfect countable set, which would force N to be empty. But by (4) and orthogonality of the Haar functions $L(S, 0) = 0$, $L(S, 1) = a_0$, and so we will have shown that $L(S, x, n_j)$ exists and is finite everywhere in $[0, 1]$.

Suppose indeed that N has an isolated point x_0 . Then for numbers $0 < a < c < d < b < 1$, $L(S, x, n_j)$ exists and is finite in $[c, x_0) \cup (x_0, d]$. Choose f_1 by the Vitali-Carathéodory Theorem as in Lemma 4, and define $F_1(x) = \int_0^x f_1(u) du$. Using (15), Lemma 2, hypothesis (ii), and continuity of F_1 ,

$$(24) \quad \liminf_{j \rightarrow \infty} 2^{n_j} \{F_1(\beta_{n_j}) - L(S, \beta_{n_j}) - [F(\alpha_{n_j}) - L(S, \alpha_{n_j})]\} \leq 0 \quad \text{for } x \notin E,$$

$$(25) \quad \begin{aligned} F_1(\alpha'_{n_j}) - L(S, \alpha'_{n_j}) &\rightarrow F_1(x) - L(S, x, n_j), \\ F_1(\beta_{n_j}) - L(S, \beta_{n_j}) &\rightarrow F_1(x) - L(S, x, n_j). \end{aligned}$$

Thus by Lemma 3, $F_1(x) - L(S, x, n_j)$ is monotone nonincreasing in each of the intervals (c, x_0) , (x_0, d) . But F_1 is continuous so $L(S, x, n_j)$ has a right and left limit (in the extended real plane) as $x \rightarrow x_0$. By (6) these limits must be equal, and by monotonicity they are finite. Thus by using (4), and the fact that the first 2^{n_j} Haar functions are constant in $(\alpha_{n_j}, \beta_{n_j})$

$$\begin{aligned} |L_{2^{n_j}}(S, x_0) - L_{2^{n_j}}(S, \alpha_{n_j})| &= \left| \int_{\alpha_{n_j}}^{x_0} S_{2^{n_j}}(u) du \right| \\ &= |(x_0 - \alpha_{n_j}) \cdot S_{2^{n_j}}(x_0)| \\ &\leq \frac{|S_{2^{n_j}}(x_0)|}{2^{n_j}} = o(1) \end{aligned}$$

by (9). Thus $L_{2^{n_j}}(S, x_0) = L(S, \alpha_{n_j}) + o(1)$ which implies $L(S, x_0, n_j)$ exists and is finite, and we have proved N has no isolated points.

By periodicity of the Haar functions we conclude that $L(S, x, n_j)$ exists and is finite for all real x . Hence using Lemma 4 we conclude $g \in L^1[-1/2, 3/2]$ and $S_{2^{n_j}}(x) \rightarrow g(x)$ almost everywhere.

We now proceed as we did locally for g^* . If the Haar Fourier series of the integrable function $g(x)$ is not identically equal to the series S , then using Lemma 6 countably many times we conclude the E is uncountable contrary to hypothesis. Thus S is the Haar Fourier series of the function g and Theorem 1 is established.

4. Proof of Theorem 2. Using (1) and defining

$$(26) \quad \alpha_k = \alpha_n^{(k')} = \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} a_i,$$

we see that the Haar series

$$T(x) = \sum_{k=0}^{\infty} \alpha_k \chi_k(x)$$

satisfies

$$T_{2^n}(x) = S_{2^n}(x) \quad \text{for } x \notin \text{D.R.}$$

Thus the Haar series T satisfies hypotheses (i), (ii) and (iii) of Theorem 1, for the countable set $E \cup \text{D.R.}$ We now claim that hypothesis (iv) implies T satisfies *condition G*. To see this use (26) to obtain

$$(27) \quad |\alpha_n^{(k)}| \leq (2^n)^{1/2} \max_{2^n \leq i < 2^{n+1}} |a_i|.$$

But by definition of the Haar functions, $\max_x |\chi_n^{(k)}(x)| = (2^n)^{1/2}$, so we use the argument in Lemma 5 to conclude T satisfies *condition G*.

Thus by Theorem 1, g is integrable, and T is its Haar Fourier series. We now observe that (26) implies

$$(28) \quad a_k = \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} \alpha_i.$$

Hence by (1), (28) and Theorem 1,

$$\begin{aligned} a_k &= \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} \int_0^1 g(x) \chi_i(x) dx \\ &= \int_0^1 g(x) \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik} \chi_i(x)}{(2^n)^{1/2}} dx \\ &= \int_0^1 g(x) \Psi_k(x) dx \end{aligned}$$

which means that S is the Walsh Fourier series of the integrable function g .

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