FREE SURFACES IN S^3

BY

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1. **Introduction.** Suppose M is a subcontinuum of the n-sphere S^n, and U is a component of $S^n - M$. We say that M is *free relative to* U if for each $\varepsilon > 0$ there is a map $f: M \to U$ such that $d(x, f(x)) < \varepsilon$ for each $x \in M$. If M is free relative to each component of $S^n - M$ we say M is *free in* S^n.

Free continua in S^n have been studied by R. L. Wilder with regard to determining those intrinsic properties of M which can be deduced from the existence of an embedding of M as a free subcontinuum of S^n. In [17] it is shown, under appropriate assumptions, that if M is free in S^n then M is a generalized $(n-1)$-manifold. In particular, when $n=3$, it is known [16, Theorem 3] that if M separates S^3, is locally connected, has finitely generated homology, and is free in S^3, then M is a closed 2-manifold.

In this paper we pursue this topic by investigating those positional properties which are satisfied by a given embedding of M as a free subset of S^3. From now on M will denote a closed, connected 2-manifold in S^3.

Bing has shown [2] that M is tame in S^3 if it can be homeomorphically approximated in each complementary domain—that is, if it can be *freed* to each side by homeomorphisms. Hempel has shown [10] that M is tame in S^3 if for each component U of $S^3 - M$ there is a homotopy $h: M \times I \to \overline{U}$ such that $h(x, 0) = x$ and for $t > 0$ $h(x, t) \in U$. Thus M is tame if it can be freed to each side by a continuous family of maps.

In light of these results it is natural to conjecture that M is tame if it is free. The purpose of this paper is to prove this conjecture in the presence of an additional assumption on the placement of M in S^3. This assumption is discussed in §3. We note the contrast between this result and the fact [8] that any $M \subset S^3$ is *partially free* in the sense that, given a component U of $S^3 - M$ and $\varepsilon > 0$, there is a compact zero-dimensional set $T \subset M$ and a map $f: M \to U \cup T$ satisfying $f(M - T) \subset U$ and $d(x, f(x)) < \varepsilon$.

We say M is *tame* in S^3 if there is a homeomorphism of S^3 onto itself which takes M onto a polyhedron. If U is a component of $S^3 - M$ we say M is *tame from* U if $\overline{U} = M \cup U$ is a 3-manifold with boundary. It follows from [4], [5], [13] that M is tame if and only if it is tame from each complementary domain. For other definitions we refer the reader to [3].

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In [3] it is shown that M is tame if S^3-M is *uniformly locally simply connected* (1-ULC). This result is extended in [9] to show that M is tame from U if U is 1-ULC. In §2 we describe a condition which is weaker than being 1-ULC but which suffices to prove the above results. In §4 we show that if M satisfies our special assumption (§3) and is free relative to U then U satisfies our modified form of 1-ULC, and hence that M is tame from U. We note that it is always the case that U is uniformly locally 1-connected in homology (1-u.l.c.) (cf. Theorem 3.2, p. 295 of [18]). In our context the assumption of freeness imposes on U the stronger property of uniform local 1-connectedness in homotopy. In §5 it is shown that if M is assumed to be locally tame off a zero-dimensional set then freeness implies tameness without the special assumptions of §3.

2. **A modification of** 1-ULC. We say that an open subset U of S^3 is 1-ULC *with respect to unknotted simple closed curves* if for each $\varepsilon>0$ there corresponds $\delta>0$ such that if J is a polyhedral simple closed curve in U which is unknotted (i.e. J bounds a (nonsingular) disk in S^3) and has diameter less than δ then J can be shrunk to a point on a subset of U of diameter less than ε.

We note that for an arbitrary open set $U\subset S^3$, the condition of being 1-ULC with respect to unknotted curves is strictly weaker than being 1-ULC. For an example of this, let C be obtained by boring out of a 3-cell a disjoint sequence of knotted "tunnels" which converges to a point of the boundary of the 3-cell. The construction can be made so that $U=S^3-C$ is 1-ULC with respect to unknotted curves, but U is not 1-ULC. We will see, by an indirect route, that if U is a complementary domain of a closed 2-manifold then the two conditions are equivalent. This is a consequence of the following two lemmas.

LEMMA 1. *Suppose S is a 2-sphere in S^3 and U is a component of S^3-S which is 1-ULC with respect to unknotted simple closed curves. Then S is tame from U.*

Proof. If we leave out the statement about unknottedness this is a direct consequence of [3, Theorem 1] which shows that S can be homeomorphically approximated in U, together with [2, Theorem 2.1] which then shows that $U\cup S$ is a 3-cell. We merely note that the proof given in [3] works just as well with the statement about unknottedness left in. Specifically the only "small" simple closed curves in U that need to be shrunk to a point on a "small" subset of U are ones which lie on a polyhedral 2-sphere $S'\subset S^3$ which is homeomorphically close to S and lies "almost" in U.

LEMMA 2. *Suppose M is a closed, connected 2-manifold in S^3 and U is a component of S^3-M which is 1-ULC with respect to unknotted simple closed curves. Then M is tame from U.*

Proof. Without the statement about unknottedness, Lemma 2 becomes [9, Theorem 5]. It seems likely that we could dismiss this proof just as we did the proof of Lemma 1. However the proof given in [9] does not proceed directly to show that

M can be homeomorphically approximated from U, but rather shows that M is locally tame from U. Since it is not evident that the proof by localization given in [9] can be adapted to our weaker hypothesis, we give a proof.

Take $p \in M$. It follows from [9, Theorem 1] that there is a disk $D \subset M$ and a 2-sphere $S \subset S^3$ such that

(i) $p \in \text{Int } D \subset S$,

(ii) S is locally polyhedral off D, and

(iii) Bd D is tame.

Now one component of $S^3 - S$, which we denote by V, has the property that D lies in the (topological) boundary of $U \cap V$. We will show that V is 1-ULC with respect to unknotted curves. It then follows by Lemma 1 that S is tame from V. This in turn implies that M is locally tame from U at p. Since this holds for each $p \in M$, the conclusion follows.

Let $E = S - \text{Int } D$. Since E is locally tame at each point of its interior and has a tame boundary it follows (cf. [13, Lemma 2.1]) that E is tame.

Now let $\varepsilon > 0$ be given. There exists $\eta > 0$ such that any subset of E of diameter less than η lies in a disk in E of diameter less than $\varepsilon/2$. Now choose $\theta > 0$ such that any unknotted simple closed curve in U of diameter less than θ can be shrunk to a point on a subset of U of diameter less than $\eta/3$. Finally choose $\delta > 0$ so that any subset of E of diameter less than δ lies in a disk $F \subset E$ of diameter less than $\theta/2$ with $F \cap \text{Bd } E$ either empty or an arc.

Suppose J is an unknotted polyhedral simple closed curve in V of diameter less than δ. Since J bounds a disk in S^3, it is an easy argument to show that J bounds a polyhedral disk $K \subset S^3$ with diameter $K < \delta$; for one can modify an arbitrary polyhedral disk bounded by J by standard "cut and paste" techniques to obtain one which lies in a small regular neighborhood of the convex hull of J. We can obtain a polyhedral disk K_1 such that Bd $K_1 = J$, $K_1 \cap E = \varnothing$ and diameter $K_1 < \theta$; for $K \cap E$ (if not already empty) lies in a disk $F \subset E$ of diameter less than $\theta/2$. If $K \cap \text{Bd } E \neq \varnothing$, then $F \cap \text{Bd } E$ is an arc (in Bd F), and K_1 is obtained by stretching K across the boundary of the tame disk E by a motion that is the identity outside a small neighborhood of F. If $K \cap \text{Bd } E = \varnothing$, we obtain K_1 by systematically chopping off K where it pokes through F (we assume K is in general position with respect to F).

Now there is a finite collection $\{J_1, \ldots, J_k\}$ of mutually exclusive polyhedral simple closed curves in $K_1 \cap U \cap V$ such that $\bigcup_{i=1}^k J_i$ separates J from $K_1 \cap D$ on K_1. Let C be the component of $K_1 - \bigcup_{i=1}^k J_i$ which contains J. Each J_i is unknotted, lies in U and has diameter less than θ. Thus J_i bounds a singular disk C_i in U with diameter $C_i < \eta/3$. Thus $K_2 = C \cup \bigcup_{i=1}^k C_i$ is a singular disk bounded by J, having diameter less than $\theta + 2\eta/3 < \eta$, and missing D. Thus there is a piecewise linear map $f: \Delta \to U$ (Δ a disk) such that $f|\text{Bd } \Delta$ is a homeomorphism onto J, $f(\Delta) \cap D = \varnothing$, diameter $f(\Delta) < \eta$, and f is in general position with respect to E. By our choice of η, $f(\Delta) \cap E$ lies in a disk $G \subset \text{Int } E$ of diameter less than $\varepsilon/2$. Each component of

$f^{-1}(f(\Delta) \cap E)$ is a simple closed curve in Int Δ; hence f can be modified to obtain a map $g: \Delta \to V \cup G$ so that $g|\text{Bd}\,\Delta = f|\text{Bd}\,\Delta$ and $g(\Delta) \subset G \cup (f(\Delta) \cap V)$. Since G is tame, g can be further modified (on a neighborhood of $g^{-1}(G)$) to obtain a map $h: \Delta \to V$ with $h|\text{Bd}\,\Delta = g|\text{Bd}\,\Delta$ and with diameter $h(\Delta) \leqq$ diameter $f(\Delta)$ + diameter $G < \eta + \varepsilon/2 < \varepsilon$. This establishes that V is 1-ULC with respect to unknotted simple closed curves and therefore completes our proof.

3. **Condition A.** We say that a 2-manifold $M \subset S^3$ satisfies *condition* (A) provided that whenever D is a polyhedral disk in S^3 with Bd $D \subset S^3 - M$ and V is any neighborhood of D there is a disk D' (not necessarily tame) such that

(i) Bd $D' =$ Bd D,

(ii) $D' \subset V$, and

(iii) if C is the component of $D' - M$ which contains Bd D', then $D' - C$ has finitely many components.

If M is a closed manifold and U is a component of $S^3 - M$ we say that *M satisfies condition* (A) *relative to U* if the above holds for those disks D with Bd $D \subset U$.

Let W denote the set of wild points of M, i.e. those points at which M fails to be locally tame. If W has the property that any disk can be moved slightly so as to miss W, then it is an easy general position argument to show that M satisfies (A). Thus for example if W is a zero-dimensional set which is tame in the sense that $S^3 - W$ is 1-ULC (see [7]) then M satisfies (A). The familiar "horned sphere" described by Alexander [1] is such an example.

4. **Main results.** We show here that a surface in S^3 which satisfies (A) is tame if and only if it is free.

THEOREM 1. *Suppose M is a closed, connected 2-manifold in S^3 and U is a component of $S^3 - M$ such that M satisfies condition* (A) *relative to U. If M is free relative to U, then M is tame from U.*

Theorem 1 follows from Theorem 2, below, in which the hypothesis of freeness is localized. Before proceeding we note that by applying Theorem 1 to each component of $S^3 - M$ we obtain

COROLLARY. *Suppose M is a closed, connected 2-manifold in S^3 which satisfies condition* (A). *If M is free in S^3 then M is tame.*

THEOREM 2. *Suppose M is a closed, connected 2-manifold in S^3 and U is a component of $S^3 - M$ such that M satisfies condition* (A) *relative to U. Suppose further that for each disk $G \subset M$ and for each $\eta > 0$ there is a map $f: G \to U$ such that $d(x, f(x)) < \eta$ for each $x \in G$. Then M is tame from U.*

Proof. By Lemma 2 it suffices to show that U is 1-ULC with respect to unknotted simple closed curves. Thus let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that any subset of M of diameter less than δ lies in a disk on M of diameter less than $\varepsilon/9$.

Now suppose J is an unknotted, polyhedral simple closed curve in U with diameter $J<\delta$. Since J bounds a disk in S^3 we can show, as we did in the proof of Lemma 2, that J bounds a disk D of diameter less than δ. We further assume that D satisfies (iii) of condition (A). We also assume, using [6, Theorem 7], that D is locally polyhedral off $D \cap M$; so that each compact subset of $D-M$ lies in a finite polyhedron in $D-M$.

Now $D-C$ has finitely many components (C is the component of $D-M$ which contains Bd D). Each of these components is a continuum in Int D which does not separate D; such a continuum can be represented as the intersection of a decreasing sequence in disks in Int D. Thus there is a finite collection $\{D_1, \ldots, D_k\}$ of mutually exclusive disks in Int D such that

(i) $D-\bigcup_{i=1}^k \operatorname{Int} D_i \subset U$, and

(ii) if A_i is the interior of the component of D_i-M which contains $J_i=\operatorname{Bd} D_i$, then A_i has two boundary components—J_i and a subset of a component of $D \cap S$. It then follows that A_i is homeomorphic to $S^1 \times (0, 1)$.

We assume that each J_i is polyhedral; so D_i is locally polyhedral off $D_i \cap M$.

We will show that each J_i bounds a (possibly singular) disk D_i' in U with diameter $D_i'<\varepsilon/3$.

Once this is done, the proof is finished; for then J can be shrunk to a point on

$$\left(D-\bigcup_{i=1}^{k} D_i\right) \cup \bigcup_{i=1}^{k} D_i'.$$

This set lies in U and has diameter less than $\delta+2\varepsilon/3<\varepsilon$.

We therefore focus our attention on J_1 (we assume there is at least one D_i—otherwise $D \cap M=\varnothing$ and we are already finished).

Now diameter $(D_1 \cap M)<\delta$; so there is a disk $E \subset M$ of diameter less than $\varepsilon/9$ with $D_1 \cap M \subset \operatorname{Int} E$. Choose a disk $G \subset M$ with $E \subset \operatorname{Int} G$. Choose a point $p \in S^3-\overline{U}$ and join p to a point of $D_1 \cap E$ by an arc X which lies except for one end point in $S^3-\overline{U}$. Now M separates J_1 from p in S^3; the same is true of the image of M under any map which is homotopic to the identity in $S^3-(J_1 \cup p)$.

Now choose $\eta>0$ satisfying

(i) $\eta<\delta/2$,

(ii) $\eta<d(D_1 \cup X, M-\operatorname{Int} E)$,

(iii) $\eta<d(J_1 \cup p, M)$, and

(iv) $\eta<d(E, M-\operatorname{Int} G)$.

By hypothesis there is a map $f\colon G \to U$ such that $d(x,f(x))<\eta$ for each $x \in G$. We can extend f to a map $F\colon M \to S^3$ such that $d(x, F(x))<\eta$ for each $x \in M$ (we can make F the identity outside an arbitrary neighborhood of G, but do not require this).

By condition (ii) on η it follows that $F(M-\operatorname{Int} E) \cap D_1=\varnothing$.

By condition (iii) it follows that F is homotopic to the inclusion that $M \to S^3$ in $S^3-(J_1 \cup p)$. Thus $F(M)$ separates J_1 from p in S^3. Furthermore, by (ii), (iv), and the fact that $F(G) \subset U$, it follows that $F(M) \cap (E \cup X) = \varnothing$; so $F(M)$ separates J_1 from E in S^3. Since $F(M) \cap D_1 = F(E) \cap D_1$, $F(E)$ separates J_1 from $D_1 \cap M$ on D_1.

We assume that F is locally piecewise linear at each point of Int G (M has a triangulation, but of course not necessarily as a subcomplex of S^3) and that F is in general position with respect to A_1; so that $F^{-1}(A_1) = F^{-1}(A_1 \cap F(E))$ is the union of a finite collection $\{L_1, \ldots, L_q\}$ of mutually exclusive simple closed curves in Int E.

We would like the situation to be such that, for $i = 1, \ldots, q$, $F|L_i$ is an essential map of the simple closed curve L_i into the open annulus A_1. If this is not already the case, we modify F as follows.

Choose i so that $F|L_i$ is homotopic to zero in A_1. Choose a disk H in Int E slightly larger than the one bounded by L_i. We define a map $F' \colon M \to S^3$ so that $F'|M-H = F|M-H$ and $F'(H)$ lies in U and slightly to one side of A_1. Now F' may move some points more than η; however, we do have

$$F'(E) \subset F(E) \cup \text{small neighborhood of } A_1.$$

We wish to assert that $F'(M)$ separates J_1 from E in S^3. Suppose this is not the case. Then F' is not homotopic to F in $S^3 - (J_1 \cup E \cup X)$. In particular the map α: Bd $(H \times I) \to U - J_1$ given by

$$\begin{aligned} \alpha(x, t) &= F(x), && t = 0, \\ &= F'(x), && t = 1, \\ &= F(x) = F'(x), && x \in \text{Bd } H, t \in I, \end{aligned}$$

is an essential map of the 2-sphere Bd $(H \times I)$ into $U - J_1$. By the sphere theorem [14] there is a piecewise linear homeomorphism β: Bd $(H \times I) \to U - J_1$ which is essential. By the proof of the sphere theorem given in [14], it follows that β can be chosen so that $\beta(\text{Bd }(H \times I))$ lies in an arbitrarily given neighborhood of $\alpha(\text{Bd }(H \times I))$. Since β is an essential homeomorphism of a 2-sphere into $U - J_1$, it follows that J_1 lies in the component, W, of $S^3 - \beta(\text{Bd }(H \times I))$ which lies in U.

Now diameter $W \leqq$ diameter $F(E)$ + diameter $A_1 < \varepsilon/3$. Thus we can choose $D_1' \subset W$ and the proof is finished. We therefore assume the alternative—namely that $F'(M)$ separates J_1 from E in S^3 and hence that $F'(E)$ separates J_1 from $D_1 \cap M$ on D_1.

We repeat the above procedure, if necessary, and after a finite number of steps we obtain a map $F_1 \colon M \to S^3$ satisfying

(i) $F_1|M-E = F|M-E$,

(ii) $F_1(E) \subset U \cap (F(E) \cup \text{small neighborhood of } A_1)$,

(iii) $F_1(E)$ separates J_1 from $D_1 \cap M$ on D_1, and

(iv) $F_1^{-1}(A_1)$ is the union of a finite collection $\{L_1, \ldots, L_k\}$ of mutually exclusive simple closed curves in Int E such that for each i $F_1|L_i$ is an essential map of L_i into A_1.

Now choose an innermost L_i, say L_1, so that L_1 bounds a disk $E_1 \subset \text{Int } E$ with $F_1(\text{Int } E_1) \cap A_1 = \varnothing$. (By condition (iii) there is at least one L_i.) We construct a (noncompact) 3-manifold $Q \subset U$ with Bd $Q = A_1$ by thickening A_1 slightly to one side and adding to this a small neighborhood of $F_1(\text{Int } E_1)$ which does not meet A_1.

Now kernel $(\pi_1(A_1) \to \pi_1(Q)) \neq 0$; for $F_1 | L_1$ represents a nontrivial element of this kernel. Thus by the loop theorem [15] there is a simple closed curve K in A_1 which cannot be shrunk to a point in A_1, but which bounds a disk R in Q. If B is the component of $\overline{A}_1 - K$ which contains J_1, then $B \subset U$. Thus $D'_1 = B \cup R \subset U$. Finally $D'_1 \subset B \cup Q$, $Q \subset$ small neighborhood of $A_1 \cup F_1(E)$, and $F_1(E) \subset F(E) \cup$ small neighborhood of A_1. Thus $D'_1 \subset$ small neighborhood of $(A_1 \cup F(E))$. So diameter $D'_1 < \delta + \varepsilon/9 + 2\eta < \varepsilon/3$, and the proof is complete.

5. **Applications.** In this section we give a situation in which condition (A) can be removed from the hypothesis of our results. The author is grateful to D. R. McMillan for pointing out the following.

THEOREM 3. *Suppose M is a closed, connected 2-manifold in S^3 and W is a compact, zero-dimensional subset of M such that M is locally tame at each point of $M - W$. If M is free in S^3, then M is tame.*

Proof. We will show that W is tame. From this it follows (as pointed out in §3) that M does indeed satisfy condition (A); hence by the Corollary to Theorem 1 M is tame.

To show that W is tame it suffices [7, Theorem 3.1] to show the following:

Given $p \in W$ and $\varepsilon > 0$ there is a 2-sphere $S \subset S^3 - W$ having diameter less than ε and such that p lies in the small component of $S^3 - S$.

We proceed to show that the above condition is satisfied by W.

Since M is locally tame at each point of $M - W$ there is a homeomorphism $h\colon (M - W) \times (-1, 1) \to S^3$ such that $h(x, 0) = x$ for each $x \in M - W$.

Let U_1 and U_2 denote the components of $S^3 - M$ with $h((M - W) \times (0, 1)) \subset U_1$ and $h((M - W) \times (-1, 0)) \subset U_2$.

Now let $p \in W$ and $\varepsilon > 0$ be given. Now since M is free relative to U_i $(i = 1, 2)$, it follows from the proof of Theorem 2 that there exist $\delta_i > 0$ $(i = 1, 2)$ such that if J is a simple closed curve in U_i of diameter less than δ_i and if J_i bounds a disk in S^3 whose diameter is less than δ_i and which satisfies condition (A), then J_i can be shrunk to a point on a subset of U_i of diameter less than $\varepsilon/3$.

Let $\delta = \min(\delta_1, \delta_2, \varepsilon/6)$. Choose a simple closed curve $J \subset M - W$ which bounds a disk $E \subset M$ with $p \in \text{int } E$ and diameter $E < \delta$. Choose $t > 0$ small enough that the disk $F = E \cup h(J \times [0, t])$ has diameter less than δ. We note that F satisfies condition (A); thus Bd $F = h(J \times t)$ can be shrunk to a point on a subset of U_1 of diameter less than $\varepsilon/3$. Thus J can be shrunk to a point on a subset of $\overline{U}_1 - W$ of diameter less than $\delta + \varepsilon/3 < \varepsilon/2$. Applying the Dehn's lemma to the 3-manifold-with-boundary $\overline{U}_1 - W$ we obtain a disk D_1 of diameter $< \varepsilon/2$ with Bd $D_1 = J$ and int $D_1 \subset U_1$.

In the same way we obtain a disk D_2 of diameter less than $\varepsilon/2$ with Bd $D_2 = J$ and int $D_2 \subset U_2$. The 2-sphere $D_1 \cup D_2$ satisfies our requirements and the proof is complete.

THEOREM 4. *Suppose M is a 2-sphere in S^3, U is a component of $S^3 - M$ and W is a compact, zero-dimensional subset of M such that M is locally tame from U at each point of $M - W$. If M is free relative to U then M is tame from U.*

Proof. By [11], [12] there is an embedding h of $\overline{U}$ into S^3 in such a way that $S^3 - h(U)$ is a closed 3-cell. Clearly $h(M)$ satisfies the hypothesis of Theorem 3; so $h(M)$ is tame in S^3. This implies that M is tame from U.

REMARK. The restriction in Theorem 4 that M be a 2-sphere seems unnecessary. The general theorem would follow as above from a suitable generalization of the results of [12], but we do not pursue this matter here.

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