DECOMPOSITIONS INTO COMPACT SETS WITH UV PROPERTIES(1)

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1. **Introduction.** In the study of cellular decompositions of manifolds, homotopy properties have proved to be of considerable importance. One of the first results along these lines was established by Price [10], [11]. He proved that if G is a cellular decomposition of E^n and U is a simply connected open set in the associated decomposition space, then $P^{-1}[U]$ is simply connected. Here P denotes the projection mapping. In [8], Martin pointed out that a similar result holds for upper semicontinuous decompositions of S^3 into compact absolute retracts. Martin's result also follows from results of Smale [12].

In the cases mentioned, the crucial lemmas assert that certain types of maps into the decomposition space can be "lifted" to yield maps into the space being decomposed. The hypotheses of [11] and [8] are used to show that if g is any element of the decomposition and U is an open set containing g, there is an open set V such that $g \subset V \subset U$ and each singular 1-sphere in V is homotopic to 0 in U. This is a type of homotopy triviality in dimension 1, and in [1], is called "property 1-UV".

The construction of [8] and [11] can be extended to higher dimensions. In [1], a family of properties, the UV properties for compact sets, are studied. For a given positive integer n, a compact set M in a space X has property n-UV if pairs of neighborhoods of M satisfy a condition of homotopy triviality in dimension n analogous to that stated above for dimension 1.

The purpose of this paper is to extend the results of [8] and [11] mentioned above to upper semicontinuous decompositions of metric spaces into compact sets with suitable UV properties. We shall establish stronger results than those of [8] and [11]. We shall obtain results for decompositions of the type mentioned analogous to those of Smale [12]. Closely related results have also been obtained by Lacher [6].

The main lemmas of the paper are established in §§3 and 4. In these results, we consider a map f from a finite complex K into a decomposition space X/G. Under suitable hypotheses, such maps can be "lifted" into the space X being decomposed. In §3, it is supposed that there is a map g from a subcomplex L of K such that f is an extension of Pg. We show that (under appropriate hypotheses), g has an extension. In §4, we show that, in the cases considered, there is a map g from K into X such that Pg is "near" f; thus, g "almost covers" f.

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The lemmas of §§3 and 4 are applied in §5. Other applications of these lemmas may be found in [2].

The first major result of §5 concerns the relation between the homotopy groups of an open set U in a decomposition space and the homotopy groups of $P^{-1}[U]$. We show that under suitable hypotheses, P induces isomorphisms from the groups of $P^{-1}[U]$ onto the corresponding groups of U. The second major result, which generalizes a theorem due to Price [11], gives conditions under which, if U is an open cell in a decomposition space, $P^{-1}[U]$ is an open cell.

There is a close connection between upper semicontinuous decompositions of metric spaces and compact mappings from a metric space onto a metric space. In §6, we have formulated some of our results in terms of compact mappings.

2. **Notation and terminology.** If X is a topological space and G is an upper semicontinuous decomposition of X, then X/G denotes the associated decomposition space and P denotes the projection map from X onto X/G. A number of basic results concerning upper semicontinuous decompositions are given in [9, Chapter V] and [15, Chapter 7].

Suppose X is a topological space, M is a subset of X, and n is a nonnegative integer. M has *property n-UV* if and only if for each open set U containing M, there is an open set V containing M such that (1) $V \subset U$ and (2) each singular n-sphere in V is homotopic to 0 in U. M has *property* UV^n if and only if for each nonnegative integer i such that $i \leq n$, M has property i-UV.

M has *property* UV^ω if and only if for each nonnegative integer k, M has property k-UV. M has *property* UV^∞ if and only if for each open set U containing M, there is an open set V containing M such that (1) $V \subset U$ and (2) V is contractible in U.

If X is a topological space and n is a nonnegative integer, the statement that G is a UV^n *decomposition* of X means that G is an upper semicontinuous decomposition of X into compact sets, each with property UV^n.

Suppose n is a positive integer. The statement that M is an *n-manifold* means that M is a separable metric space such that each point of M has an open neighborhood which is an open n-cell. The statement that M is an n-manifold-with-boundary means that M is a separable metric space such that each point of M has a neighborhood which is an n-cell. If M is an n-manifold-with-boundary, then a point p of M is an *interior point* of M if and only if p has an open neighborhood which is an open n-cell. The set of all interior points of M is the *interior* of M, denoted by Int M. The *boundary* of M, Bd M, is $M - \text{Int } M$.

Suppose M is an n-manifold. If A is a subset of M, then A is *cellular* in M if and only if there exists a sequence $C_1, C_2, C_3, \ldots$ of n-cells in M such that (1) for each positive integer i, $C_{i+1} \subset \text{Int } C_i$, and (2) $\bigcap_{i=1}^{\infty} C_i = A$. If M is a n-manifold, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M and each element of G is a cellular subset of M.

Suppose X is a topological space. If $\mathscr{U}$ is a collection of subsets of X and $A \subset X$, then the *star of A with respect to $\mathscr{U}$*, denoted by st $(A, \mathscr{U})$, is $\bigcup \{U : U \in \mathscr{U}$ *and* U *intersects* $A\}$. Suppose $\mathscr{U}$ and $\mathscr{V}$ are collections of open subsets of X. Then $\mathscr{V}$ *star refines* $\mathscr{U}$ if and only if for each set V of $\mathscr{V}$, there is a set U of $\mathscr{U}$ such that st $(V, \mathscr{V}) \subset U$. If n is any nonnegative integer, then $\mathscr{V}$ *star n-homotopy refines* $\mathscr{U}$ if and only if for each set V of $\mathscr{V}$, there is a set U of $\mathscr{U}$ such that (1) st $(V, \mathscr{V}) \subset U$ and (2) if $0 \leqq k \leqq n$, each singular k-sphere in st $(V, \mathscr{V})$ is homotopic to 0 in U.

I denotes the closed interval [0, 1]. If n is a positive integer, D^n denotes the closed unit ball (n-cell) with center at the origin in E^n, and S^n denotes the unit sphere in E^{n+1} with center at the origin. By a *complex* is meant a finite simplicial complex. By a *mapping* is meant a continuous function. If X is a compact metric space and f and g are mappings from X into a metric space Y, then

$$D(f, g) = \text{l.u.b. } \{d(f(x), g(x)) : x \in X\}.$$

3. **Lifting extensions of maps.** Lemma 3.2 below is one of the two main lemmas of this paper. It deals essentially with lifting extensions of maps. It may also be viewed in the following way: From the existence of homotopies in a decomposition space, we may infer the existence of certain homotopies in the space being decomposed. We first establish a preliminary result.

LEMMA 3.1. *Suppose X is a metric space, n is a nonnegative integer, G is a UV^n decomposition of X, and A is a subset of X/G. If $\mathscr{U}$ is an open covering of A, there exists an open covering $\mathscr{V}$ of A such that $\{P^{-1}[V] : V \in \mathscr{V}\}$ star n-homotopy refines $\{P^{-1}[U] : U \in \mathscr{U}\}$.*

Proof. If $y \in A$, there is an open set U_y in $\mathscr{U}$ such that $y \in U_y$. Then $P^{-1}[U_y]$ is an open set in X containing $P^{-1}[y]$. Since each element of G has property UV^n, there is an open set W_y in X such that (1) $P^{-1}[y] \subset W_y \subset P^{-1}[U_y]$ and (2) if $0 \leqq k \leqq n$, each singular k-sphere in W_y is homotopic to 0 in $P^{-1}[U_y]$. Since G is upper semicontinuous and $P^{-1}[y] \subset W_y$, we may assume that (3) W_y is a union of elements of G; thus $P[W_y]$ is an open set in X/G containing y. Let $\mathscr{W}$ denote $\{P[W_y] : y \in A\}$; $\mathscr{W}$ is an open covering of A.

By [13], X/G is a metric space. By [4, p. 167], there is an open covering $\mathscr{V}$ of A such that $\mathscr{V}$ star refines $\mathscr{W}$. Then $\{P^{-1}[V] : V \in \mathscr{W}\}$ star refines $\{W_y : y \in A\}$, and $\{W_y : y \in A\}$ refines $\{P^{-1}[U_y] : y \in A\}$. It follows that $\{P^{-1}[V] : V \in \mathscr{V}\}$ star n-homotopy refines $\{P^{-1}[U] : U \in \mathscr{U}\}$.

Lemma 3.2 was established for pointlike mappings of S^n by Černavskiĭ and Kompaniec [3, Lemma 1]; see Theorem 2.1 of [6] for a closely related result.

LEMMA 3.2. *Suppose X is a metric space, n is a nonnegative integer, and G is a UV^{n-1} decomposition of X. Suppose k is a nonnegative integer such that $k \leqq n$, K is a finite simplicial k-complex, and L is a subcomplex of K. Suppose f is a map from L into X and g is a map from K into X/G such that $g|L = Pf$. Suppose ε is a positive number. Then there exists an extension F of f sending K into X such that $D(g, PF) < \varepsilon$*

Proof. If $y \in g[K]$, there is an open set U_y in X such that (1) $P[U_y]$ is open in X/G, (2) $y \in P[U_y]$, and (3) $P[U_y]$ has diameter less than $\varepsilon/2$. Since K is a finite complex, there is a finite subset $\mathscr{U}_k$ of $\{U_y : y \in g[K]\}$ such that $\mathscr{U}_k$ covers $P^{-1}g[K]$. By repeated application of Lemma 3.1, we obtain collections $\mathscr{U}_{k-1}, \mathscr{U}_{k-2}, \ldots,$ and $\mathscr{U}_0$ such that if $0 \leqq i \leqq k-1$, (1) each set of $\mathscr{U}_i$ is an open inverse set, (2) $\mathscr{U}_i$ is a finite open covering of $P^{-1}g[K]$, and (3) $\mathscr{U}_i$ star $(n-1)$-homotopy refines $\mathscr{U}_{i+1}$. If $0 \leqq i \leqq k$, let $U_{i1}, U_{i2}, \ldots,$ and U_{im_i} denote the sets of $\mathscr{U}_i$.

We are now prepared to construct an extension F of f to K. There exists a subdivision T of K such that if $\sigma \in T$, then for some j, $g[\sigma] \subset P[U_{0j}]$; equivalently, $P^{-1}g[\sigma] \subset U_{0j}$. We shall define F first on the 0-skeleton of T, then on the 1-skeleton of T, and so on. We shall denote these maps by $F_0, F_1, \ldots,$ and F_k.

Suppose v is a vertex of T. If $v \in L$, define $F_0(v)$ to be $f(v)$. If $v \notin L$, let $F_0(v)$ be a point of $P^{-1}[g(v)]$. Hence F_0 is defined on the 0-skeleton of T. It follows from the construction of T that if σ' is a 1-simplex of T, then for some j, $F_0[\mathrm{Bd}\ \sigma'] \subset U_{0j}$. This is true because for some j, $P^{-1}g[\sigma'] \subset U_{0j}$, and by construction, $F_0[\mathrm{Bd}\ \sigma'] \subset P^{-1}g[\sigma']$.

Suppose $0 \leqq i < k$ and we have defined F_i on the i-skeleton of T. Suppose further that (1) if $i > 0$, F_i is an extension of F_{i-1}, (2) if x is in both L and the i-skeleton of T, then $F_i(x) = f(x)$, and (3) if σ^i is an i-simplex of T, then for some j, $F_i[\sigma^i] \subset U_{ij}$. We shall now construct an extension F_{i+1} of F_i to the $(i+1)$-skeleton of T.

If σ is an $(i+1)$-simplex of T in L, define $F_{i+1}|\sigma$ to be $f|\sigma$. Suppose σ is an $(i+1)$-simplex of T not in L. Let τ be an i-face of σ. Then for some j, $F_i[\tau] \subset U_{ij}$. If $i > 0$, each i-face of Bd σ intersects τ, and it follows that $F_i[\mathrm{Bd}\ \sigma] \subset \mathrm{st}\,(U_{ij}, \mathscr{U}_i)$. If $i = 0$, then it was mentioned above that for some j, $F_i[\mathrm{Bd}\ \sigma] \subset U_{ij}$. Since $\mathscr{U}_i$ star $(n-1)$-homotopy refines $\mathscr{U}_{i+1}$, it follows that for some r, $F_i[\mathrm{Bd}\ \sigma] \subset U_{(i+1)r}$ and $F_i|\mathrm{Bd}\ \sigma \sim 0$ in $U_{(i+1)r}$. Thus we define an extension F_{i+1} of F_i such that (1) F_{i+1} takes the $(i+1)$-skeleton of T into X, (2) if $x \in L$ and x lies in the $(i+1)$-skeleton of T, $F_{i+1}(x) = f(x)$, and (3) if σ^{i+1} is an $(i+1)$-simplex of T, then for some j, $F_{i+1}[\sigma^{i+1}] \subset U_{(i+1)j}$.

Hence in finitely many steps we construct an extension F of f that takes K into X and has the property that if σ is a simplex of K, then for some j, $F[\sigma] \subset U_{kj}$.

Now we shall show that $D(g, PF) < \varepsilon$. Suppose $x \in K$ and let σ be a simplex of K containing x. Then for some j, $F[\sigma] \subset U_{kj}$ and so $F(x) \in U_{kj}$. Further, diam $P[U_{kj}] < \varepsilon/2$, and since for some r and s, $g[\sigma] \subset P[U_{r0}] \subset P[U_{ks}]$, it follows that diam $g[\sigma] < \varepsilon/2$. Let v be a vertex of σ and recall that $g(v) = PF(v)$. Then

$$\rho(g(x), PF(x)) \leqq \rho(g(x), g(v)) + \rho(g(v), PF(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It then follows that $D(g, PF) < \varepsilon$.

We may state Lemma 3.2 in the following way.

LEMMA 3.3. *Suppose X is a metric space, n is a nonnegative integer, K is a finite simplicial n-complex, L is a subcomplex of K, and G is a UV^n decomposition of X. Suppose f and h are continuous functions from K into X such that* (1) $f|L = h|L$ *and* (2) *there is a homotopy H from $K \times I$ into X/G such that* (a) $H_0 = Pf$, (b) $H_1 = Ph$, *and*

(c) *if* $t \in I$, $H_t|L = Pf|L$. *Suppose* ε *is a positive number. Then there is a homotopy* F *from* $K \times I$ *into* X *such that* (1) $F_0 = f$, (2) $F_1 = h$, (3) *if* $t \in I$, $F_t|L = f|L$, *and* (4) $D(H, PF) < \varepsilon$.

If A and B are sets in a topological space such that $B \subset A$ and C and D are sets in a topological space such that $D \subset C$, then a function f is *from* (A, B) *into* (C, D) if and only if f has domain A, $f[A] \subset C$, and $f[B] \subset D$. Suppose X is a topological space, U and V are open sets in X, $V \subset U$, and n is a nonnegative integer. Then the pair (U, V) is *n-connected* if and only if for each nonnegative integer i such that $i \leqq n$ and each continuous function from $(\Delta^i, \mathrm{Bd}\,\Delta^i)$ into (U, V), there is a homotopy H from $\Delta^i \times I$ into U such that (1) if $t \in I$, H_t is from $(\Delta^i, \mathrm{Bd}\,\Delta^i)$ into (U, V), (2) $H_0 = f$, and (3) $H_1[\Delta^i] \subset V$. Here Δ^i denotes an i-simplex; Δ^0 denotes a point and $\mathrm{Bd}\,\Delta^0$ is void.

LEMMA 3.4. *Suppose* X *is a metric space,* n *is a nonnegative integer, and* G *is a* UV^n *decomposition of* X. *Suppose* U *and* V *are open sets in* X/G *such that* $V \subset U$. *If* (U, V) *is n-connected, then* $(P^{-1}[U], P^{-1}[V])$ *is n-connected.*

Proof. Suppose i is an integer such that $0 \leqq i \leqq n$ and f is a continuous function from $(\Delta^i, \mathrm{Bd}\,\Delta^i)$ into $(P^{-1}[U], P^{-1}[V])$. Then Pf is a continuous function from $(\Delta^i, \mathrm{Bd}\,\Delta^i)$ into (U, V). Since (U, V) is n-connected, there is a homotopy H from $\Delta^i \times I$ into U such that (1) if $t \in I$, H_t is from $(\Delta^i, \mathrm{Bd}\,\Delta^i)$ into (U, V), (2) $H_0 = Pf$ and (3) $H_1[\Delta^i] \subset V$.

Let ε be the distance in X/G from $H[(\Delta^i \times \{i\}) \cup ([\mathrm{Bd}\,\Delta^i] \times I)]$ to $U - V$. Now by Lemma 3.3, there is a homotopy F from $\Delta^i \times I$ into $P^{-1}[U]$ such that (1) $F_0 = f$ and (2) $D(H, PF) < \varepsilon$. It then follows that $F[\Delta^i \times \{i\}] \subset P^{-1}[V]$ and that if $t \in I$, $F_t[\mathrm{Bd}\,\Delta] \subset P^{-1}[V]$.

4. **Construction of continuous functions.** The objective of this section is to establish the second main lemma, Lemma 4.2, of this paper. In Lemma 4.2, we consider the following situation. We have a metric space X and an upper semi-continuous decomposition G of X. Suppose that f is a map from a finite simplicial complex K into X/G. There may not exist a map g from K into X such that $Pg = f$. However, under suitable hypotheses, we can show that there exists a map h from K into X such that $Ph \sim f$ in X/G. We may require, in addition, that Ph and f be homotopic via a homotopy with short paths.

LEMMA 4.1. *Suppose* X *is a metric space,* n *is a nonnegative integer,* G *is a* UV^n *decomposition of* X, $n \geqq 1$, K *is a finite simplicial k-complex, and* L *is a subcomplex of* K. *Suppose* f *is a map from* L *into* X *and* g *is a map from* K *into* X/G *such that* $g|L = Pf$. *If* ε *is any positive number, there exists a positive number* δ *such that if* h *and* l *are continuous extensions of* f *sending* K *into* X *such that, in* X/G, $D(g, Ph) < \delta$ *and* $D(g, Pl) < \delta$, *then there is a homotopy* H *from* $K \times I$ *into* X *such that* (1) $H_0 = h$, (2) $H_1 = l$, (3) *if* $t \in I$, $H_t|L = f$, *and* (4) *if* $t \in I$, $D(g, PH_t) < \varepsilon$ *in* X/G.

Proof. The proof of this lemma is similar to that of Lemma 3.2. If $y \in g[K]$, there is an open set U_y in X such that (1) $P[U_y]$ is open in X/G and (2) $P[U_y]$ has diameter less than $\varepsilon/2$. Since K is a finite complex, there is a finite subset $\mathscr{U}_{k+1}$ of $\{U_y : y \in g[K]\}$ covering $P^{-1}g[K]$. By repeated application of Lemma 3.1, we obtain collections $\mathscr{U}_k, \mathscr{U}_{k-1}, \ldots$ and $\mathscr{U}_0$ such that if $0 \leqq i \leqq k$, (1) each set of $\mathscr{U}_i$ is an open inverse set, (2) $\mathscr{U}_i$ is a finite open covering of $P^{-1}g[K]$, and (3) $\mathscr{U}_i$ star n-homotopy refines $\mathscr{U}_{i+1}$. If $0 \leqq i \leqq k+1$, let $U_{i1}, U_{i2}, \ldots$, and U_{im_i} denote the sets of $\mathscr{U}_i$.

There exists a subdivision T of K such that if σ is a simplex of T, then for some j, $g[\sigma] \subset P[U_{0j}]$. Let δ be a positive number such that (1) $\delta < \varepsilon/2$ and (2) if σ is any simplex of T, $1 \leqq j \leqq m_0$, and $g[\sigma]$ is contained in $P[U_{0j}]$, then the open δ-neighborhood of $g[\sigma]$ lies in $P[U_{0j}]$.

Suppose h and l are continuous extensions of f sending K into X such that $D(g, Ph) < \delta$ and $D(g, Pl) < \delta$. Note that if τ is any simplex of T, $1 \leqq j \leqq m_0$, and $g[\tau] \subset P[U_{0j}]$, then both $h[\tau] \subset P[U_{0j}]$ and $l[\tau] \subset P[U_{0j}]$. This follows from the choice of δ.

Let K' denote $(K \times \{0\}) \cup (K \times \{1\}) \cup (L \times I)$. Define H' on K' as follows: (1) If $x \in K$, then $H'(x, 0) = h(x)$ and $H'(x, 1) = l(x)$. (2) If $x \in L$ and $t \in I$, $H'(x, t) = f(x)$. We shall now construct an extension H of H' to $K \times I$. We extend by skeletons of T, and shall denote the maps constructed by $H^0, H^1, H^2, \ldots$, and H^k.

Let v be a vertex of T not in L. As noted above, for some j, U_{0j} contains both $h(v)$ and $k(v)$. Since $\mathscr{U}_0$ star n-homotopy refines $\mathscr{U}_1$, it follows that $H' | (\{v\} \times \{0, 1\})$ can be extended to take $\{v\} \times I$ into some element of $\mathscr{U}_1$. Note that if v is a vertex of T in L, $\{v\} \times I \subset K'$. In this manner, we construct an extension H^0 of H' to $(T^0 \times I) \cup K'$, where T^0 is the 0-skeleton of T. Further, H^0 has the property that if σ is a simplex of T in $T^0 \cup L$, then $H^0[\sigma \times I]$ lies in some set of $\mathscr{U}_1$.

Suppose that $0 \leqq i < k$ and that we have constructed an extension H^i of H' to $(T^i \times I) \cup K'$, where T^i denotes the i-skeleton of T. Suppose further that if τ is a simplex of T in $T^i \cup L$, then $H^i[\tau \times I]$ lies in some element of $\mathscr{U}_{i+1}$.

Let σ be an $(i+1)$-simplex of T not in L. Then H^i is defined on $(\sigma \times \{0\}) \cup (\sigma \times \{1\}) \cup ([\text{Bd } \sigma] \times I)$. Further, if j has the property that $g[\sigma] \subset P[U_{(i+1)j}]$, then

$$H^i[(\sigma \times \{0\}) \cup (\sigma \times \{1\}) \cup ([\text{Bd } \sigma] \times I)] \subset \text{St}\,(U_{(i+1)j}, \mathscr{U}_{i+1}).$$

Since $\mathscr{U}_{i+1}$ star n-homotopy refines $\mathscr{U}_{i+2}$, it follows that

$$H^i | [(\sigma \times \{0, 1\}) \cup ([\text{Bd } \sigma] \times I)]$$

can be extended to take $\sigma \times I$ into some element of $\mathscr{U}_{i+2}$. Recall that if σ is an $(i+1)$-simplex of T in L, $\sigma \times I \subset K'$. It follows that H^i has an extension H^{i+1} that takes $(T^{i+1} + I) \subset K'$ into X such that if τ is a simplex of T in $T^{i+1} \cup L$, then $H^{i+1}[\tau \times I]$ lies in some element of $\mathscr{U}_{i+2}$.

Hence there is an extension H of H' to $K \times I$ and such that (1) H takes $K \times I$ into X and (2) if σ is any simplex of K, $H[\sigma \times I]$ lies in some set of $\mathscr{U}_{k+1}$.

Now we shall show that in X/G, if $t \in I$, $D(g, PH_t) < \varepsilon$. Suppose $x \in K$, and let σ be a simplex of T such that $x \in \sigma$. Then for some j, $H[\sigma \times I] \subset U_{(k+1)j}$, so that $H(x, 0) \in U_{(k+1)j}$ and $H(x, t) \in U_{(k+1)j}$. Thus $PH(x, t) \in P[U_{(k+1)j}]$ and since $H_0 = h$, $PH(x, 0) \in P[U_{(k+1)j}]$. By construction, $\operatorname{diam} P[U_{(k+1)j}] < \varepsilon/2$, and by hypothesis, $D(g, Ph) < \delta$ so that $\rho(g(x), Ph(x)) < \delta < \varepsilon/2$. Thus $\rho(g(x), PH(x, t)) < \varepsilon$. Therefore, if $t \in I$, $D(g, PH_t) < \varepsilon$.

LEMMA 4.2. *Suppose X is a metric space, n is a nonnegative integer, G is a UV^n decomposition of X, $k \leqq n$, K is a finite simplicial k-complex, and L is a subcomplex of K. Suppose f^0 is a map from L into X and g is a map from K into X/G such that $g|L = Pf^0$. If ε is a positive number, there exists an extension f of f^0 sending K into X and an ε-homotopy H from $K \times I$ into X/G such that $H_0 = Pf$, $H_1 = g$, and if $t \in I$, $H_t|L = g|L$. In particular, $Pf \sim g$ in X/G by an ε-homotopy.*

Proof. We shall use Lemma 3.2 to construct a sequence $h_1, h_2, h_3, \ldots$ of continuous extensions of h to K such that (1) for each i, h_i is homotopic to h_{i+1} under a homotopy H^i such that PH^i does not move points far and (2) the sequence $Ph_1, Ph_2, Ph_3, \ldots$ converges to g.

Suppose ε is a positive number. For each i, let ε_i be $\varepsilon/2^{i+1}$. For each i, there exists, by Lemma 4.1, a positive number δ_i such that if α and β are any continuous extensions of f^0 sending K into X such that $D(g, P\alpha) < \delta_i$ and $D(g, P\beta) < \delta_i$, then there is a homotopy F from $K \times I$ into X such that (1) $F_0 = \alpha$, (2) $F_1 = \beta$, (3) if $t \in I$, $F_t|L = f^0$, and (4) if $t \in I$, $D(g, PF_t) < \varepsilon_i$ in X/G. In addition, we assume that for each i, $\delta_i < \delta_{i+1}$.

For each positive integer i, there exists, by Lemma 3.2, a map h_i from K into X such that $h_i|L = f^0$ and $D(g, Ph_i) < \delta_i$. Since $\delta_{i+1} < \delta_i$, it follows that $D(g, Ph_{i+1}) < \delta_i$. Hence there is a homotopy H^i from $K \times I$ into X such that (1) $H_0^i = h_i$, (2) $H_1^i = h_{i+1}$, (3) if $t \in I$, $H_t^i|L = f^0$, and (4) in X/G, if $t \in I$, $D(g, PH_t^i) < \varepsilon_i$.

Define f to be h_1. Define H as follows: (1) If $x \in K$, then $H(x, 1) = g(x)$. (2) If i is any positive integer, $t \in [1 - 2^{-(i-1)}, 1 - 2^{-i}]$, and $x \in K$, let

$$H(x, t) = PH^i(x, 2^{i+1}[t + (1 - 2^{-(i-1)})]).$$

For each i, $H_1^i = h_i$ and $H_0^{i+1} = h_i$. It follows that H is a well-defined function from $K \times I$ into X/G. Since $H_0^1 = h_1 = f$, then $H_0 = Pf$. By definition, $H_1 = g$. Since for each i, if $t \in I$, $H_t^i|L = f^0$, it is clear that if $t \in L$, $H_t|L = g|L$.

It is easy to see that H is continuous at each point of $K \times [0, 1)$. For each i and each t in I, $D(g, PH_t^i) < \varepsilon_i$, $\varepsilon_i = \varepsilon/2^{i+1}$, and it is routine to show that H is continuous at each point of $K \times \{1\}$ as well.

Since for each i, $\varepsilon_i = \varepsilon/2^{i+1}$, it follows that for any point x of K, $\operatorname{diam} PH[\{x\} \times I] < \varepsilon$, and thus that in X/G, H is an ε-homotopy.

5. **Applications.** As our first application, we give a Vietoris mapping theorem for homotopy groups, for decompositions into compact sets with suitable UV properties. Theorem 5.1 below is analogous to results of [12].

THEOREM 5.1. *Suppose X is a metric space, n is a nonnegative integer, and G is a UV^n decomposition of X. Suppose U is an open set in X/G and $x \in P^{-1}[U]$. Then if $0 \leqq k \leqq n$, $P_*: \pi_k(P^{-1}[U], x) \to \pi_k(U, P(x))$ is an isomorphism onto.*

Proof. We need only prove that P_* is one-to-one and onto. Suppose $0 \leqq k \leqq n$ and that $f: (S^k, s) \to (P^{-1}[U], x)$ represents an element, denoted by $[f]$, of $\pi_k(P^{-1}[U], x)$. If $P_*([f]) = 1$, then Pf can be extended to a map $g: D^{k+1} \to W$. By Lemma 3.2, there is an extension F of f that takes D^{k+1} into $P^{-1}[W]$. Hence $f \simeq 0$ in $P^{-1}[W]$ and it follows that P_* is one-to-one.

Suppose $0 \leqq k \leqq n$ and that $g: (S^k, s) \to (U, P(x))$ represents an element of $\pi_k(U, P(x))$. Let f^0 denote the map from $\{s\}$ into $P^{-1}[U]$ such that $f^0(s) = x$. Then by Lemma 4.2, there is an extension f of f^0 sending S^k into $P^{-1}[U]$ such that $Pf \sim g$ in U. Hence $P_*([f]) = [g]$, and it follows that P_* is onto.

COROLLARY 5.2. *Suppose n is a positive integer, M is an n-manifold, and G is a cellular decomposition of M. If U is an open set in M/G and $x \in P^{-1}[U]$, then for each positive integer k, $P_*: \pi_k(P^{-1}[U], x) \to \pi_k(U, P(x))$ is an isomorphism onto. In particular, if $x_0 \in M$, then for each positive integer k,*

$$P_*: \pi_k(M, x_0) \to \pi_k(M/G, P(x_0))$$

is an isomorphism onto.

Proof. Since G is a cellular decomposition, then for each positive integer m, G is a UV^m decomposition. Corollary 5.2 then follows from Theorem 5.1.

For definitions of terms used in the next two corollaries, see [1].

COROLLARY 5.3. *Suppose n is a positive integer, X is a locally compact LC^n metric space, and G is an upper semicontinuous decomposition of X into compact LC^{n-1} n-connected sets. Then if U is any open set X/G, $p \in P^{-1}[U]$, and $q = P(p)$, $\pi_n(U, q)$ and $\pi_n(P^{-1}[U], p)$ are isomorphic.*

Proof. By Lemma 5.5 of [1], each set of G has property UV^n. Hence Corollary 5.3 follows from Theorem 5.1.

COROLLARY 5.4. *Suppose X is a locally compact LC metric space and G is an upper semicontinuous decomposition of X into compact absolute retracts. Then if U is any open set in X/G, $p \in P^{-1}[U]$, $q = P(p)$, and k is any positive integer, then $\pi_k(U, q)$ and $\pi_k(P^{-1}[U], p)$ are isomorphic.*

Proof. By Corollary 5.7 of [1], each set of G has property UV^ω, and hence Theorem 5.1 applies.

In [11], Price proved the following theorem: If n is a positive integer, $n \neq 4$, G is a cellular decomposition of E^n, and U is an open n-cell in E^n/G such that U is open in E^n/G, then $P^{-1}[U]$ is an open n-cell. We shall establish an analogous result for n-manifolds, $n > 4$ and UV^∞ decompositions.

THEOREM 5.5. *Suppose n is a positive integer, $n>4$, M is an n-manifold, G is a UV^{∞} decomposition of M, and U is an open subset of M/G such that U is an open n-cell. Then $P^{-1}[U]$ is an open n-cell.*

Proof. By [7], we need only show that $P^{-1}[U]$ is contractible and 1-connected at infinity. By Theorem 5.1, for each positive integer k, $\pi_k(P^{-1}[U])=0$. It follows from [14] that $P^{-1}[U]$ is contractible.

Now let B be any compact subset of $P^{-1}[U]$. There is a compact subset D of U such that $P[B]\subset D$ and $U-D$ is connected and simply connected. By Lemma 3.2, $P^{-1}[U-D]$, or $P^{-1}[U]-P^{-1}[D]$, is simply connected. Since each set of G is connected, $P^{-1}[U-D]$ is connected [9, Chapter V, Theorem 5]. Since $B\subset P^{-1}[D]$, $P^{-1}[U]$ is 1-connected at infinity. By [7], $P^{-1}[U]$ is an open n-cell.

THEOREM 5.6. *Suppose G is a UV^{∞} decomposition of E^3 and U is an open subset of E^3/G such that U is an open 3-cell. Then $P^{-1}[U]$ is an open 3-cell.*

Proof. This theorem may be established by an argument like that for Theorem 5.5 above but using [5].

6. **Compact mappings.** There is a close connection between upper semicontinuous decomposition of metric spaces into compact sets and compact mappings from a metric space onto a metric space. In this section, we formulate some of our previous results in terms of such compact mappings. If f is a map from a space X onto a space Y, then f is *compact* if and only if for each compact set A in Y, $f^{-1}[A]$ is compact.

We shall first point out how these two concepts are related. Suppose X is a metric space and G is an upper semicontinuous decomposition of X into compact sets. Then the projection map P from X onto X/G is compact [9, Chapter V, Theorem 6]. It then follows [13] that X/G is metrizable. Hence we have metric spaces X and X/G, and a compact mapping P from X onto X/G.

Now suppose that X and Y are metric spaces and f is a compact mapping from X onto Y. Let G_f be $\{f^{-1}[y];\, y\in Y\}$; G_f is an upper semicontinuous decomposition of X into compact sets. G_f is the decomposition *induced* by f. Let h be the function from X/G_f into Y such that if $x\in X/G_f$, then $h(x)=y$ when $\{y\}=fP^{-1}[x]$. It is easily shown that h is one-to-one, continuous, and onto Y. Using the fact that f is compact, we may prove that h is a homeomorphism. In this case, then, X and Y are homeomorphic under the natural map h. Hence we may, in such a case, work with the induced decomposition into compact sets in place of the compact mapping.

We shall now formulate some of our results in terms of compact mappings. In fact, as the discussion above shows, all the results of this paper could be so formulated. If n is a nonnegative integer, we define a mapping f from a space X into a space Y to be a *UV^n map* if and only if for each point y of Y, $f^{-1}[y]$ has property UV^n.

The following theorem is analogous to a result of [12].

THEOREM 6.1. *Suppose X and Y are metric spaces, n is a nonnegative integer, and f is a compact UV^n map from X onto Y. If U is an open set in Y and $x \in f^{-1}[U]$, then if $1 \leqq k \leqq n$, $f_*: \pi_k(f^{-1}[U], x) \to \pi_k(U, f(x))$ is an isomorphism onto.*

A mapping f from a manifold M into a space Y is a *cellular* mapping if and only if for each point y of Y, $f^{-1}[y]$ is a cellular set in M.

THEOREM 6.2. *Suppose f is a compact cellular mapping from a manifold M onto a metric space Y. Then if $x_0 \in M$ and k is any positive integer,*

$$f_*: \pi_k(M, x_0) \to \pi_k(Y, f(x_0))$$

is an isomorphism onto.

REFERENCES

1. S. Armentrout, *Compact sets with UV properties*, (to appear).

2. ———, *Homotopy properties of decomposition spaces*, (to appear).

3. A. V. Černavskiĭ and V. P. Kompaniec, *Equivalence of two classes of sphere mappings*, Soviet Math. Dokl. **7** (1966), 1083–1085.

4. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

5. C. H. Edwards, Jr., *Open 3-manifolds which are simply connected at infinity*, Proc. Amer. Math. Soc. **14** (1963), 391–395.

6. R. C. Lacher, *Cell-like spaces*, Proc. Amer. Math. Soc. **20** (1969), 598–602.

7. E. Luft, *On contractible open topological manifolds*, Invent. Math. **4** (1967), 192–201.

8. J. Martin, *The sum of two crumpled cubes*, Michigan Math. J. **13** (1966), 147–151.

9. R. L. Moore, *Foundations of point set theory* (rev. ed.), Amer. Math. Soc. Colloq. Publ., Vol. 13, Amer. Math. Soc., Providence, R. I., 1962.

10. T. M. Price, *Upper semi-continuous decompositions of E^3*, Thesis, University of Wisconsin, Madison, 1964.

11. ———, *A necessary condition that a cellular upper semicontinuous decomposition of E^n yield E^n*, Trans. Amer. Math. Soc. **122** (1966), 427–435.

12. S. Smale, *A Vietoris mapping theorem for homotopy*, Proc. Amer. Math. Soc. **8** (1957), 604–610.

13. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. **7** (1956), 690–700.

14. J. H. C. Whitehead, *Combinatorial homotopy*. II, Bull. Amer. Math. Soc. **55** (1949), 453–496.

15. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., Vol. 28, Amer. Math. Soc., Providence, R. I., 1942.

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