REARRANGEMENTS OF SERIES OF FUNCTIONS

BY
VLADIMIR DROBOT

1. Introduction and the statement of the main result. It is well known [3, p. 301] that a conditionally convergent series of real numbers can be rearranged in such a way so as to converge to any preassigned value. Suppose now we have a series of functions

$$(1) \quad \sum_{n=1}^{\infty} x_n(t), \quad 0 \leq t \leq 1.$$

In this paper we shall be concerned with the studies of rearrangements of such series, the convergence being that of $L_2(0, 1)$. The work is motivated by the paper of E. Steinitz [4], who considered the rearrangements of conditionally convergent series of vectors in the finite dimensional Euclidean spaces. We are going to prove the following result:

THEOREM 1. *Suppose $x_n(t)$ is a sequence of real valued functions, belonging to the real space $L_2(0, 1)$. Suppose also that:*

- (a) *the series (1) converges in norm to some $x \in L_2$,*
- (b) $\sum \|x_n\| = +\infty$,
- (c) $\sum \|x_n\|^2 < \infty$,
- (d) *the linear subspace $M = \{y \in L_2 : \sum |(x_n, y)| < \infty\}$ is closed ((a, b) is the real inner product of a and b).*

Then there exists a closed linear subspace N and a function $x_0 \in L_2$ such that:

I. *any rearrangement of (1), which converges in norm, must have the limit of the form $x_0 + z$, where $z \in N$;*

II. *for any $z \in N$, there exists a rearrangement of (1), which converges in norm to $x_0 + z$.*

In fact $N = M^\perp$ i.e., $N \oplus M = L_2$.

2. Proof of Theorem.1. First we need several lemmas.

LEMMA 1. *Let $x_1, \dots, x_{n+1}$ be $n+1$ linearly dependent elements in L_2 and let*

$$x = \alpha_1 x_1 + \dots + \alpha_{n+1} x_{n+1}, \quad 0 \leq \alpha_i \leq 1.$$

Then we can express x as

$$x = \gamma_1 x_1 + \dots + \gamma_{n+1} x_{n+1}, \quad 0 \leq \gamma_i \leq 1$$

and at least one $\gamma_i = 0$ or 1.

Proof. This Lemma is proved in [4, p. 167] for the case when $x_i \in R^n$ and the proof carries verbatim to the present situation.

LEMMA 2. Let $x_1, x_2, \dots, x_n \in L_2$ and let

$$x = \lambda_1 x_1 + \dots + \lambda_n x_n, \quad 0 \leq \lambda_i \leq 1.$$

Then there exists a vector $x' \in L_2$ of the form

$$x' = \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n, \quad \delta_i = 0 \text{ or } 1$$

such that $\|x - x'\|^2 \leq \|x_1\|^2 + \dots + \|x_n\|^2$.

Proof. We proceed by induction on n . The case $n=1$ is clear since $|\lambda_1| \leq 1$. Suppose then the lemma is true for any n vectors in L_2 and let

$$(2) \quad x = \lambda_1 x_1 + \dots + \lambda_n x_n + \lambda_{n+1} x_{n+1}, \quad 0 \leq \lambda_i \leq 1.$$

We may write $x_{n+1} = y_{n+1} + z_{n+1}$, where $y_{n+1} \in \text{sp}\{x_1, \dots, x_n\}$ and $(z_{n+1}, x_i) = 0$, $i=1, 2, \dots, n$. Clearly

$$(3) \quad \|x_{n+1}\|^2 = \|y_{n+1}\|^2 + \|z_{n+1}\|^2,$$

$$(4) \quad x - \lambda_{n+1} z_{n+1} = \lambda_1 x_1 + \dots + \lambda_n x_n + \lambda_{n+1} y_{n+1}, \quad 0 \leq \lambda_i \leq 1$$

and the vectors $x_1, \dots, x_n, y_{n+1}$ are linearly dependent. Using Lemma 1 we can write (4) as

$$(5) \quad x - \lambda_{n+1} z_{n+1} = \gamma_1 x_1 + \dots + \gamma_n x_n + \gamma_{n+1} y_{n+1}, \quad 0 \leq \gamma_i \leq 1$$

and at least one $\gamma_{i_0} = 0$ or 1.

We divide the remaining proof into several cases.

Case 1. $i_0 = n+1$, $\gamma_{n+1} = 0$. By the inductive hypothesis there exists a vector $x' = \delta_1 x_1 + \dots + \delta_n x_n$, $\delta_i = 0$ or 1, such that

$$(6) \quad \|x - \lambda_{n+1} z_{n+1} - x'\|^2 \leq \|x_1\|^2 + \dots + \|x_n\|^2.$$

Since $x = (x - \lambda_{n+1} z_{n+1}) + \lambda_{n+1} z_{n+1}$ and z_{n+1} is orthogonal to $x - \lambda_{n+1} z_{n+1}$ (equation 5 and the conditions of this case) and x' , we get from (6) and (3)

$$\|x - x'\|^2 = \|x - \lambda_{n+1} z_{n+1} - x'\|^2 + |\lambda_{n+1}|^2 \|z_{n+1}\|^2 \leq \|x_1\|^2 + \dots + \|x_n\|^2.$$

Case 2. $i_0 < n+1$ and $\gamma_{i_0} = 0$. We may assume without the loss of generality that $i_0 = 1$. By the inductive hypothesis there is a vector $x'' = \delta_2 x_2 + \dots + \delta_{n+1} x_{n+1}$, $\delta_i = 0$ or 1, such that

$$(7) \quad \|x - \lambda_{n+1} z_{n+1} - x''\| \leq \|x_2\|^2 + \dots + \|x_n\|^2 + \|y_{n+1}\|^2.$$

Let $x' = x'' + \delta_{n+1} z_{n+1} = \delta_2 x_2 + \dots + \delta_{n+1} x_{n+1}$. Since z_{n+1} is orthogonal to $x_1, x_2, \dots, x_n, y_{n+1}$, and $|\lambda_{n+1} - \delta_{n+1}| \leq 1$ we obtain from (7) and (3)

$$\begin{aligned} \|x - x'\|^2 &= \|x - \lambda_{n+1} z_{n+1} - x'' + (\lambda_{n+1} - \delta_{n+1}) z_{n+1}\|^2 \\ &= \|x - \lambda_{n+1} z_{n+1} - x''\|^2 + |\lambda_{n+1} - \delta_{n+1}|^2 \|z_{n+1}\|^2 \\ &\leq \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2 + \|y_{n+1}\|^2 + \|z_{n+1}\|^2. \end{aligned}$$

Since the last 2 terms add up to $\|x_{n+1}\|^2$ we get the result.

Case 3. $i_0 = n+1$, $\gamma_{n+1} = 1$. Here

$$(8) \quad x - \lambda_{n+1}z_{n+1} = \gamma_1x_1 + \cdots + \gamma_{n+1}x_n + y_{n+1}.$$

Let $x'' = \delta_1x_1 + \cdots + \delta_nx_n$, $\delta_i = 0$ or 1, be such that

$$(9) \quad \|x - \lambda_{n+1}z_{n+1} - y_n - x''\|^2 \leq \|x_1\|^2 + \cdots + \|x_n\|^2$$

and put $x' = x'' + x_{n+1} = x_{n+1} + y_{n+1} + z_{n+1}$. Since $|\lambda_{n+1} - 1| \leq 1$ and $z_i \perp x_i, \dots, x_n, y_{n+1}$, we get from (9) and (8)

$$\begin{aligned} \|x - x'\|^2 &= \|x - \lambda_{n+1}z_{n+1} - y_{n+1} - x'' + (\lambda_{n+1} - 1)z_{n+1}\|^2 \\ &= \|x - \lambda_{n+1}z_{n+1} - y_{n+1} - x''\|^2 + |\lambda_{n+1} - 1|^2 \|z_{n+1}\|^2 \\ &\leq \|x_1\|^2 + \cdots + \|x_{n+1}\|^2. \end{aligned}$$

Case 4. $i < n+1$. Here again we assume that $i_0 = 1$ and consequently $\gamma_1 = 1$. We write now

$$x - \lambda_{n+1}z_{n+1} - x_1 = \gamma_2x_2 + \cdots + \gamma_nx_n + \gamma_{n+1}y_{n+1}.$$

Let $x'' = \delta_2x_2 + \cdots + \delta_nx_n + \delta_{n+1}y_{n+1}$, $\delta_i = 0$ or 1, be such that

$$(10) \quad \|x - \lambda_{n+1}z_{n+1} - x_1 - x''\|^2 \leq \|x_2\|^2 + \cdots + \|y_{n+1}\|^2.$$

Put

$$x' = x_1 + x'' + \delta_{n+1}z_{n+1} = x_1 + \delta_2x_2 + \cdots + \delta_{n+1}x_{n+1}.$$

Noting again that $z_{n+1} \perp x_1, \dots, y_{n+1}$, $|\lambda_{n+1} - \delta_{n+1}| \leq 1$ we get

$$\begin{aligned} \|x - x'\|^2 &= \|x - \lambda_{n+1}z_{n+1} - x_1 - x'' + (\lambda_{n+1} - \delta_{n+1})z_{n+1}\|^2 \\ &\leq \|x_2\|^2 + \cdots + \|y_{n+1}\|^2 + |\lambda_{n+1} - \delta_{n+1}|^2 \|z_{n+1}\|^2 \leq \|x_1\|^2 + \cdots + \|x_{n+1}\|^2. \end{aligned}$$

This proves Lemma 2.

LEMMA 3. Let $X = \{x_n\}$ be a sequence of elements of L_2 . Let

$$P(X) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k\},$$

$$Q(X) = \text{co } P(X) \quad (\text{convex hull of } P(X)).$$

$$R(X) = \{\gamma_1x_{i_1} + \cdots + \gamma_kx_{i_k} : 0 \leq \gamma_i \leq 1, i_1 < i_2 < \cdots < i_k\}.$$

Then $Q(X) \subset R(X)$.

Proof. It is enough to show that $R(X)$ is convex, since $R(X) \supset P(X)$. Let now $y, z \in R(X)$. We may assume

$$y = \gamma_1x_{i_1} + \gamma_2x_{i_2} + \cdots + \gamma_kx_{i_k}, \quad z = \delta_1x_{i_1} + \delta_2x_{i_2} + \cdots + \delta_kx_{i_k}$$

by inserting the terms $0 \cdot x_j$ if necessary.

Let now $0 \leq \lambda \leq 1$. We have

$$\lambda y + (1 - \lambda)z = \sum_j [\lambda\gamma_j + (1 - \lambda)\delta_j]x_{i_j}$$

and since $0 \leq \lambda\gamma_j + (1 - \lambda)\delta_j \leq \lambda + (1 - \lambda) = 1$ we get the result.

LEMMA 4. Let N be a closed linear subspace of L_2 and let B be a convex subset of N . Suppose that for any $x \in N$ and any $T > 0$, there exist elements b_1 and b_2 in B so that $(x, b_1) \leq -T$ and $(x, b_2) \geq T$. Then B is dense in N . (We recall that our L_2 is a real space.)

Proof. Consider N as a Hilbert space and suppose the closure of B is different from N . Let $K = \{x \in N : \|x - x_0\| < r\} \subset N \setminus \bar{B}$, for some $x_0 \in N$ and $r > 0$. Since K and $\bar{B}$ are convex and K has an interior point (in the relative topology of N), there exists a continuous linear functional x' such that $x'(x) \leq c \leq x'(y)$, for all $x \in \bar{B}$, $y \in K$ for some constant c . (See [2, p. 412].) By the Riesz representation theorem for the Hilbert spaces, $x'(x) = (x, x')$ for some $x' \in N$. But this implies that $(x, x') \leq c$ for all $x \in B$, which contradicts the hypothesis of the theorem.

LEMMA 5. Suppose $x_i \in L_2$, $\|x_i\| \leq M$, $i = 1, 2, \dots, n$. Let

$$(11) \quad x_1 + x_2 + \dots + x_n = a.$$

Then we can rearrange the order of x_i 's, say into $\{x'_1, x'_2, \dots, x'_n\}$, such that

$$(12) \quad \|x'_1 + x'_2 + \dots + x'_p\|^2 \leq \|x_1\|^2 + \dots + \|x_n\|^2 + \|a\|(\|a\| + 2M), \quad p = 1, 2, \dots, n.$$

Proof. First assume that $a = 0$ and call the right hand side of (12) K . We proceed to rearrange $x_1, \dots, x_n$ as follows. Let $x'_1 = x_1$. Clearly $\|x'_1\|^2 \leq K$. On account of (11) we have $\sum (x'_1, x_1) = (x'_1, 0) = 0$ and the first term of the sum is equal to $\|x'_1\|^2 \geq 0$. Hence for some x'_2 among $x_2, \dots, x_n$ we must have $(x'_1, x'_2) \leq 0$. From this it follows that

$$\|x'_1 + x'_2\|^2 = \|x'_1\|^2 + 2(x'_1, x'_2) + \|x'_2\|^2 \leq \|x'_1\|^2 + \|x'_2\|^2 \leq K.$$

Consider next $\sum (x'_1 + x'_2, x_i) = (x'_1 + x'_2, 0) = 0$. The first 2 terms add up to $\|x'_1 + x'_2\|^2 \geq 0$; hence for some x'_3 among x 's different from x'_1, x'_2 we must have $(x'_1 + x'_2, x'_3) \leq 0$. So

$$\|x'_1 + x'_2 + x'_3\|^2 = \|x'_1 + x'_2\|^2 + 2(x'_1 + x'_2, x'_3) + \|x'_3\|^2 \leq K.$$

Continuing in this fashion we get the result.

Suppose now $a \neq 0$. Then $\sum (x_i - n^{-1}a) = 0$; so we can order x_i 's in such a way that

$$\begin{aligned} \left\| \sum_1^p \left(x'_i - \frac{1}{n} a \right) \right\|^2 &\leq \sum_1^n \left\| x_i - \frac{1}{n} a \right\|^2 \leq \sum_1^n \left(\|x_i\|^2 + 2 \frac{1}{n} \|x_i\| \|a\| + \frac{1}{n^2} \|a\|^2 \right) \\ &\leq (2M + \|a\|)\|a\| + \sum \|x_i\|^2. \end{aligned} \quad \text{Q.E.D.}$$

We are now ready to prove Theorem 1. Let $X = \{x_n : n = 1, 2, \dots\}$ be the sequence of functions satisfying the conditions a-d. Each x_n can be written as $x_n = y_n + z_n$, $y_n \in M$, $z_n \in M^\perp = N$. It is clear that if $\{x'_n\}$ is a rearrangement of X , then $\sum x'_n$

converges in L_2 if and only if $\sum y'_n$ and $\sum z'_n$ converge in L_2 . We shall show first that $\sum y'_n$ converges for any rearrangements of y 's. For any $y \in M$ we have

$$(13) \quad \sum |(y_n, y)| = \sum |(x_n, y)| < \infty.$$

Hence $\sum (y_n, y)$ converges absolutely, so every rearrangement of $\sum (y_n, y)$ will converge to the same limit, say $W(y)$. This shows that $\sum y_n$ converges weakly, and for every $y \in M$, every subseries of $\sum (y_n, y)$ will also converge. This implies [1, p. 60] that every subseries of $\sum y_n$ will converge, which in turn implies [1, p. 59 (1-b)] that every rearrangement of $\sum y_n$ converges in norm. The strong limit of $\sum y_n$ must be equal to the weak limit of $\sum y_n$. The weak limit of $\sum y_n$ is the same for every rearrangement, hence the strong limit must be independent of the rearrangement. This shows that $\sum y_n$ converges unconditionally in norm, i.e. for every rearrangement it converges in norm to the same limit, say x_0 . This proves the first part of the theorem, since if a rearrangement of $\sum x_n$ converges in norm then the limit must be of the form $x_0 + \sum z'_n$ where $\sum z'_n \in N = M^\perp$.

What remains to be shown is that for every $w \in N$ there exists a rearrangement of $\sum z_n$ which converges in norm to w . We introduce the following notation.

Let $W = \{w_i : i = 1, 2, \dots\}$ be an arbitrary sequence of elements in N . Put

$$P(W) = \{w_{i_1} + w_{i_2} + \dots + w_{i_k} : i_1 < i_2 < \dots < i_k\},$$

$$Q(W) = \text{co } P(W) \quad (\text{convex hull of } P(W)),$$

$$R(W) = \{\gamma_1 w_{i_1} + \dots + \gamma_k w_{i_k} : 0 \leq \gamma_i \leq 1, i_1 < i_2 < \dots < i_k\}.$$

Denote the elements of $P(W)$ by p . If $p = w_{i_1} + w_{i_2} + \dots + w_{i_k}$, denote by $W - p$ the sequence $\{w_i : i \neq i_1, i_2, \dots, i_k\}$. Put now W to be the sequence $\{z_i : i = 1, 2, \dots\}$.

Let $z \in N$, $z \neq 0$. Then

$$(14) \quad \left| \sum (z, z_n) \right| = \left| \sum (z, x_n) \right| < \infty \quad \text{since } \sum x_n \text{ converges}$$

$$(15) \quad \sum |(z, z_n)| = \sum |(z, x_n)| = +\infty \quad \text{since } z \notin M.$$

Hence for any $z \neq 0$, $z \in N$, and any $T > 0$, there exist p_1, p_2 in $P(W)$ such that $(z, p_1) \geq T$ and $(z, p_2) \leq -T$. Lemma 4 shows then that $Q(W)$ is dense in N . Since $R(W) \supset Q(W)$ (Lemma 3), $R(W)$ is also dense in N . It follows then that the set $z + R(W) = \{z + r : r \in R(W)\}$ is also dense for every $z \in N$. The equations (14) and (15) also show that the sets $Q(W - p)$ and $z + R(W - p)$ are dense for every $p \in P(W)$ and $z \in N$. Hence we have shown

(A) for every $w \in N$, $z \in N$, $p \in P(W)$ and every $\varepsilon > 0$ there exists $r \in z + R(W - p)$ so that

$$(16) \quad \|w - r\| < \varepsilon.$$

Choose $w \in N$. We shall now construct a rearrangement of $\sum z_n$ which converges to w . Let

$$(17) \quad \varepsilon_k \downarrow 0, \quad A_k^2 = \sum_k \|z_n\|^2 \quad \text{and} \quad B_k = \sup \{\|z_n\| : n \geq k\}.$$

By the hypothesis of the theorem $A_k \downarrow 0$, $B_k \downarrow 0$. We shall define a sequence of elements $w_k \in N$ as follows. Let $p_1 = z_1$. Choose $q_1 \in p_1 + R(z - p_1)$ such that $q_1 = z_1 + u_1$, $u_1 \in R(W - p_1)$ and so that

$$(18) \quad \|w - p_1 - u_1\| < \varepsilon_1.$$

This is possible by (A). We have

$$u_1 = \gamma_1 z_{i_1} + \cdots + \gamma_j z_{i_j}, \quad 0 \leq \gamma_i \leq 1, \quad 1 < i_1 < \cdots < i_j.$$

Using Lemma 2 we can find $t_1 = \delta_1 z_{i_1} + \cdots + \delta_j z_{i_j}$, $\delta_i = 0$ or 1 , such that

$$(19) \quad \|u_1 - t_1\|^2 \leq \sum_1^j \|z_{i_n}\|^2 \leq A_1^2.$$

Let $w_1 = z_1 + t_1 \in P(W)$. Clearly from (18), (19) we have

$$(20) \quad \|w - w_1\| \leq \|w - z_1 - u_1\| + \|u_1 - t_1\| \leq \varepsilon_1 + A_1.$$

We now put p_2 as the first z not used as a summand in w_1 . Choose $q_2 \in p_2 + R(W - w_1 - p_2)$ such that $q_2 = p_2 + u_2$, $u_2 \in R(W - (p_1 + w_1))$ and

$$\|w - w_1 - p_2 - u_2\| \leq \varepsilon_2.$$

Using Lemma 2 we choose $t_2 \in P(W - (p_2 + w_1))$ so that

$$\|u_2 - t_2\|^2 \leq \sum_2^\infty \|z_i\|^2 = A_2^2.$$

Let $w_2 = p_2 + t_2 \in P(W - w_1)$. We get again

$$\|w - w_1 - w_2\| \leq \|w - w_1 - p_2 - u_2\| + \|u_2 - t_2\| \leq \varepsilon_2 + A_2.$$

Inductively it goes as follows. Suppose $w_1, w_2, \dots, w_n$ are already defined and satisfy

$$(21) \quad \|w - (w_1 + \cdots + w_k)\| \leq \varepsilon_k + A_k,$$

$$(22) \quad w_k \in P(W - (w_1 + \cdots + w_{k-1})), \quad k = 1, 2, \dots, n,$$

$$(23) \quad z_1, z_2, \dots, z_k \text{ are included as summands in } w_1 + w_2 + \cdots + w_k, \quad k = 1, 2, \dots, n.$$

Choose p_{n+1} to be the z with the smallest subscript not included in the sum $w_1 + \cdots + w_n$.

Let $q_{n+1} \in p_{n+1} + R(W - (p_{n+1} + w_1 + \cdots + w_n))$ be such that $q_{n+1} = p_{n+1} + u_{n+1} \in R(W - (p_{n+1} + \cdots + w_n))$ and

$$(24) \quad \|w - (w_1 + \cdots + w_n + p_{n+1} + u_{n+1})\| \leq \varepsilon_{n+1}.$$

This is possible by (A). Using Lemma 2 choose $t_{n+1} \in P(W - (w_1 + \cdots + w_n + p_{n+1}))$ so that

$$(25) \quad \|u_{n+1} - t_{n+1}\|^2 \leq \sum_{n+1}^\infty \|z_i\|^2 \leq A_{n+1}^2.$$

Put $w_{n+1} = p_{n+1} + t_{n+1} \in P(W - (w_1 + \cdots + w_n))$. We have from (24), (25)

$$(26) \quad \|w - (w_1 + \cdots + w_{n+1})\| \leq \varepsilon_{n+1} + A_{n+1}.$$

Clearly $w_1, w_2, \dots, w_{n+1}$ satisfy (21), (22), (23). Thus the sequence w_n is defined, every z_i is included as a summand and in exactly one w_n and

$$(27) \quad \left\| w - \sum_1^n w_i \right\| \leq \varepsilon_n + A_n.$$

This shows that $\sum w_n$ defines a rearrangement of $\sum z_n$, say $\sum z_n''$ such that

$$(28) \quad w_n = z_{i_n}'' + \cdots + z_{i_{n+1}-1}''$$

and

$$(29) \quad \left\| w - \sum_1^{i_n+1-1} z_i'' \right\| \leq \varepsilon_n + A_n.$$

Now we rearrange each of the sums w_n in such a way that the resulting rearrangement $\sum z_i'$ will converge to w . Let

$$w_n = z_{j_1}'' + \cdots + z_{j_k}'', \quad n \leq i_1 < i_2 < \cdots < i_k.$$

Using Lemma 5 we can rearrange z_{j_i}'' s in such a way that

$$(30) \quad w_n = z_{j_1}' + \cdots + z_{j_k}'$$

$$(31) \quad \begin{aligned} \|z_{j_1}' + \cdots + z_{j_p}'\|^2 &\leq \|z_{j_1}'\|^2 + \cdots + \|z_{j_k}'\|^2 + \|w_n\|(\|w_n\| + 2B_n) \\ &\leq A_n^2 + \|w_n\|(\|w_n\| + 2B_n), \quad p = 1, 2, \dots, k. \end{aligned}$$

Since

$$\begin{aligned} \|w_n\| &\leq \|w - (w_1 + \cdots + w_{n-1})\| + \|w - (w_1 + \cdots + w_n)\| \\ &\leq \varepsilon_{n-1} + A_{n-1} + \varepsilon_n + A_n = d_n \end{aligned}$$

we have

$$(32) \quad \|z_{j_1}' + \cdots + z_{j_p}'\|^2 \leq A_n^2 + d_n(d_n + B_n), \quad p = 1, 2, \dots, k.$$

Let now $\sum z_i'$ be the resulting rearrangement. Let N be a positive integer and choose $n = n(N)$ such that

$$w_{n+1} = z_{i_1}' + z_{i_1+1}' + \cdots + z_{i_k}', \quad i_1 \leq N \leq i_k.$$

Then

$$(33) \quad \begin{aligned} \left\| w - \sum_1^N z_i' \right\| &\leq \left\| w - \sum_1^n w_i \right\| + \|z_{i_1}' + \cdots + z_{i_k}'\| \\ &\leq \varepsilon_n + A_n + [A_n^2 + d_n(d_n + B_n)]^{1/2}. \end{aligned}$$

It is clear that $n(N) \rightarrow \infty$ as $N \rightarrow \infty$; so the right hand side of (33) tends to 0 as $N \rightarrow \infty$. This completes the proof of Theorem 1.

3. Examples.

1. $M = \{y : \sum |(x_n, y)| < \infty\}$ need not be closed, even if the conditions a–c of Theorem 1 hold.

Let $\{e_n : n = 1, 2, \dots\}$ be an orthonormal basis of L_2 . Put

$$(34) \quad \begin{aligned} l_n &= [(n+2) \log(n+2)]^{-1} & x_n &= (-1)^{n+1} l_n (e_1 + e_2 + \dots + e_n) \\ x &= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} (-1)^{j+1} l_j e_i = \sum_{i=1}^{\infty} \alpha_i e_i. \end{aligned}$$

It follows that

$$(35) \quad \|x_n\|^2 = n l_n^2 < [n(\log n)^2]^{-1}$$

so $\sum \|x_n\|^2 < \infty$, $\sum \|x_n\| = +\infty$. Since $\sum (-1)^{n+1} l_n$ is an alternating series, we have the following estimate

$$(36) \quad \left| \sum_{i=p}^{\infty} (-1)^{n+1} l_i - \sum_{i=p}^{p+q} (-1)^{n+1} l_i \right| < |l_{q+1}|.$$

Hence

$$(37) \quad \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} (-1)^{n+1} l_n \right|^2 = \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} (-1)^{n+1} l_n - \sum_{j=1}^{i-1} l_n \right|^2 \leq \sum_{i=1}^{\infty} |l_i|^2 < \infty$$

so $x \in L_2$. Moreover using (36) we obtain

$$(38) \quad \left\| \sum_1^N x_n - x \right\|^2 = \left\| \sum_1^N x_n - \sum_1^N \alpha_n e_n \right\|^2 + \left\| \sum_{N+1}^{\infty} \alpha_n e_n \right\|^2.$$

The last term clearly goes to 0 as $N \rightarrow \infty$. The first term is equal to

$$\sum_{n=1}^N \left| \sum_{i=n}^N (-1)^{n+1} l_i - \sum_{i=n}^{\infty} (-1)^{n+1} l_i \right|^2 \leq \sum_{n=1}^N |l_{N-1}| = N l_{N+1} \rightarrow 0 \quad (N \rightarrow \infty).$$

Hence $\sum x_n$ converges to x in norm. Put now

$$(39) \quad \begin{aligned} t_i &= [\log(\log(i+2))]^{-1} \\ y_1 &= 0, \quad y_k = t_1 e_1 + \sum_{i=1}^{k-1} (t_{i+1} - t_i) e_i, \quad k \geq 2, \\ y &= t_1 e_1 + \sum_{i=1}^{\infty} (t_{i+1} - t_i) e_i. \end{aligned}$$

Since

$$(40) \quad \begin{aligned} t_{i+1} - t_i &= \int_i^{i+1} \frac{d}{dx} \frac{1}{\log(\log(x+2))} dx \\ &= - \int_i^{i+1} [(x+2) \log(x+2) (\log(\log(x+2)))^2]^{-1} dx \end{aligned}$$

so

$$|t_{i+1} - t_i| \leq \frac{1}{(i+2) \log(i+2) [\log(\log(i+2))]^2}$$

and hence $y \in L_2$. Moreover

$$\|y - y_k\|^2 \leq |t_k|^2 + |t_{k+1} - t_k|^2 + \sum_{i=k}^{\infty} |t_{i+1} - t_i|^2 \rightarrow 0 \quad (k \rightarrow \infty)$$

so y_k converges in norm to y . We also note that $|(x_n, y_k)| = 0$ for $n > k$, hence $\sum |(x_n, y_k)| < \infty$ for all k . However $|(x_n, y)| = l_n t_n$ and since

$$\sum \frac{1}{(i+2) \log(i+2) \log(\log(i+2))} = \infty \quad y \notin M,$$

this shows that M need not be closed.

2. Next we give an example of a series $\sum x_n$ for which $N = L_2$, or equivalently $M = \{0\}$. Let $\{e_n\}$ be an orthonormal basis of L_2 . Define

$$(41) \quad x_n^{(k)} = (-1)^{n+1} 2^{-k} n^{-1} e_k.$$

Since $\|x_n^{(k)}\|^2 = 2^{-2k} n^{-2}$ then $\sum_n \sum_k \|x_n^{(k)}\|^2 < \infty$. It is also clear that $\sum_{n,k} \|x_n^{(k)}\| = \infty$. We can order $\{x_n^{(k)}\}$ into a single sequence $\{x_n\}$ such that $\sum x_n$ converges. We proceed by induction as follows. Let $x_1 = x_1^{(1)}$, $x_2 = x_1^{(2)}$ and $x_3 = x_2^{(1)}$. Suppose we have ordered all the elements $x_n^{(k)}$, $n+k \leq m$, into a sequence $x_1, x_2, \dots, x_N$ such that if x_i corresponds to $x_n^{(k)}$, x_j corresponds to $x_n^{(k+1)}$ then $i < j$. We put

$$(42) \quad x_{N+1} = x_1^{(m)}, \quad x_{N+2}^{(m)} = x_2^{(m-1)}, \dots, x_{N+j} = x_j^{(m-j+1)}, \quad x_{N+m} = x_m^{(1)}.$$

In this way we have ordered all the elements $\{x_n^{(k)}\}$ in such a way that if x_i corresponds to $x_n^{(k)}$ and x_j corresponds to $x_n^{(k+1)}$ then $i < j$. Hence for each k

$$\sum_i (x_i, e_k) = \frac{1}{2^k} \sum_j \frac{(-1)^{j+1}}{j} = \frac{1}{2^k} \log 2 = y_k.$$

Put

$$(43) \quad x = \sum_k \frac{1}{2^k} \log 2 e_k = \sum_k y_k e_k.$$

We have for any positive integer N

$$(44) \quad \left| \sum_{i=1}^N (-1)^{i+1} \frac{1}{2^k i} - \frac{1}{2^k} \log 2 \right| \leq \frac{1}{2^k N}.$$

Let N be a positive integer and choose p to be the largest integer such that $x_1^{(p)}$ is included among $x_1, x_2, \dots, x_N$. Then by the construction of x_i 's

$$(45) \quad \left\| \sum_{i=1}^N x_i - x \right\| \leq \left\| \sum_1^p x_i - \sum_1^p y_i e_i \right\| + \left\| \sum_{p+1}^N x_i - \sum_{p+1}^N y_i e_i \right\| + \left\| \sum_{N+1}^{\infty} y_i e_i \right\|.$$

The last sum clearly goes to 0 as $N \rightarrow \infty$. By (44) the first and the second sum are each smaller than

$$\frac{1}{p} \frac{1}{1} + \frac{1}{p-1} \frac{1}{2} + \frac{1}{p-2} \frac{1}{2^2} + \cdots + \frac{1}{1} \frac{1}{2^{p-1}} = S_p$$

which goes to 0 as $p \rightarrow \infty$. Since $p \rightarrow \infty$ as $N \rightarrow \infty$ we have shown that $\sum x_i$ converges to x in norm. Let now $y \neq 0$, $y \in L_2$, $y = \sum b_k e_k$. Then

$$(46) \quad \sum |(x_i, y)| \geq \sum_n |(x_n^{(k)}, y)| = \sum_n \frac{b_k}{2^{kn}} = \infty$$

unless $b_k = 0$. Hence $M = \{0\}$ and every function in L_2 can be obtained as a limit of some rearrangement of $\sum x_i$.

REFERENCES

1. M. M. Day, *Normed linear spaces*, Academic Press, New York, 1962.
2. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Wiley, New York, 1958.
3. E. Hille, *Analysis*. Vol. I, Blaisdell, New York, 1964.
4. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme*, J. Reine Angew. Math. **143** (1913).

STATE UNIVERSITY OF NEW YORK AT BUFFALO,
BUFFALO, NEW YORK