HARMONIC ANALYSIS OF LOCAL TIMES AND SAMPLE FUNCTIONS OF GAUSSIAN PROCESSES⁽¹⁾

BY
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Introduction and summary. Let I be a topological space, $\mathcal{B}$ its class of Borel sets, μ a positive finite measure on $\mathcal{B}$, and $x=x(t)$, $t \in I$, a real valued Borel function. Define the measure

$$\nu(A) = \mu(x^{-1}(A))$$

on the linear Borel sets A . If ν is absolutely continuous with respect to the usual linear Borel measure, then its derivative is called the local time of x . In this paper we deduce some properties of the variation of x from smoothness properties of the local time when μ is fixed and x is a sample function of a Gaussian process. The main results are Theorems 5.1 and 5.2 which imply: If $X(t)$, $0 \leq t \leq 1$, is a Gaussian process with mean 0 satisfying $E(X(s) - X(t))^2 \geq C|t - s|^\beta$, for some β , $0 < \beta < 2$, then the sample functions

- (a) are of unbounded γ -variation for $\gamma < 2/\beta$;
- (b) nowhere satisfy a Hölder condition of order $2/(2m + \varepsilon + 1)$ if $\beta < 2/(2m + \varepsilon + 1)$, where m is a nonnegative integer and $0 \leq \varepsilon < 1$;
- (c) are nowhere differentiable if $\beta < 1$;
- (d) return infinitely often to every neighborhood of almost all the points they visit.

The result (c) was stated without proof in [5] for a stationary process whose covariance function r is not twice differentiable at the origin. A related result was proved by Yeh [7] under the additional assumption that $X(\cdot)$ is continuous. The paper of Marcus [4] contains results on Hölder conditions for processes with stationary increments satisfying the condition that $E(X(t) - X(0))^2$ is concave in some neighborhood of 0, or related conditions.

Using a previous result on analytic local times [2], we prove in §6 that Lévy's Brownian motion over Hilbert space has sample functions unbounded over certain compact ellipsoids (cf. [1]).

1. Square integrable local times. Put

$$f(u) = \int_{-\infty}^{\infty} e^{iux} d\nu([-\infty, x]), \quad -\infty < u < \infty;$$

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then f is representable as

$$(1.1) \quad f(u) = \int_I \exp(iux(t)) d\mu(t)$$

because $x=x(t)$ is a measure-preserving transformation on (I, μ) to (R, ν) . If f is square integrable, then ν is absolutely continuous and the local time is square integrable (cf. [2]). We denote the latter by $\phi(x)$. For every square integrable Borel function g we have

$$(1.2) \quad \int_I g(x(t)) d\mu(t) = \int_{-\infty}^{\infty} g(x)\phi(x) dx.$$

Now we introduce the relative local time ϕ_B , where B is a Borel subset of I : it is the local time of the restriction of x to B . It is a consequence of the definition of the Radon-Nikodym derivative that $\phi_B(x) \leq \phi(x)$ for almost all x ; hence, ϕ_B is square integrable if ϕ is. If ϕ satisfies the equation (1.2), then ϕ_B satisfies a similar equation with B in place of I as the domain of integration. As a derivative with respect to Borel measure, the local time is Borel measurable; thus, from the definition of the relative local time it follows that if ϕ is square integrable, then, by (1.2)

$$(1.3) \quad \int_C \phi_B(x(t)) d\mu(t) = \int_{-\infty}^{\infty} \phi_B(x)\phi_C(x) dx = \int_B \phi_C(x(t)) d\mu(t), \quad B, C, \varepsilon, \mathcal{B}.$$

If $\mu(B \cap C) = 0$, then

$$(1.4) \quad \phi_{B \cup C} = \phi_B + \phi_C.$$

LEMMA 1.1. *If ϕ is square integrable, then $\phi(x(t)) > 0$, for almost all t .*

Proof. Let B be the set of t upon which $\phi(x(t)) = 0$; then, from (1.3) with $C = I$, we have

$$0 = \int_B \phi(x(t)) d\mu(t) = \int_{-\infty}^{\infty} \phi(x)\phi_B(x) dx.$$

Since ϕ dominates ϕ_B , the last integral dominates $\int_{-\infty}^{\infty} \phi_B^2(x) dx$; thus ϕ_B vanishes almost everywhere, and so

$$\mu(B) = \int_{-\infty}^{\infty} \phi_B(x) dx = 0.$$

2. Recurrence properties of a class of functions with square integrable local times.

In this section we take I to be the closed unit interval and μ the usual Borel measure (Lebesgue measure on the Borel sets). Let $x(t)$, $0 \leq t \leq 1$, be a Borel function with a square integrable local time. For each positive integer n , let I_{nk} be the closed interval $[2^{-n}(k-1), 2^{-n}k]$, $k=1, \dots, 2^n$. Let ϕ_{nk} be the local time relative to I_{nk} , and f_{nk} the corresponding Fourier transform.

LEMMA 2.1. *The sequence of sums*

$$(2.1) \quad \sum_{k=1}^{2^n} \int_{-\infty}^{\infty} |f_{nk}(u)|^2 du, \quad n = 1, 2, \dots,$$

is nonincreasing; consequently, it converges.

Proof. Each interval I_{nk} is the union of two intervals $I_{n+1,j}$ and $I_{n+1,j+1}$. Their intersection contains one point so that, by (1.4):

$$(2.2) \quad \phi_{nk} = \phi_{n+1,j} + \phi_{n+1,j+1};$$

moreover $(\phi_{n+1,j} + \phi_{n+1,j+1})^2 \geq \phi_{n+1,j}^2 + \phi_{n+1,j+1}^2$ because the local time is non-negative; therefore, $\phi_{nk}^2 \geq \phi_{n+1,j}^2 + \phi_{n+1,j+1}^2$. This inequality is preserved under integration; therefore, by Parseval's theorem, the same inequality also holds for the integrals of the squared moduli of the corresponding Fourier transforms. This completes the proof.

For each real number y , let $N(y)$ be the number (finite or infinite) of elements in the set

$$\{t : 0 \leq t \leq 1, y \text{ is in the closure of the image of every neighborhood of } t\}.$$

(If the function is continuous $N(y)$ is the number of times that it assumes the value y .) Let $N_n(y)$ be the number of positive elements in the set $\phi_{nk}(y)$, $k=1, \dots, 2^n$, $n=1, 2, \dots$

LEMMA 2.2. *For each y , the sequence $\{N_n(y)\}$ is nondecreasing; thus, it has a limit which may be finite or infinite.*

Proof. The equation (2.2) implies that for every positive element of the set $\{\phi_{nk}(y), k=1, \dots, 2^n\}$ there is at least one positive element in the set $\{\phi_{n+1,k}(y), k=1, \dots, 2^{n+1}\}$.

LEMMA 2.3. *If $N_n(y) \rightarrow \infty$ for $n \rightarrow \infty$, then $N(y) = \infty$.*

Proof. Without losing generality, we may suppose that the local time vanishes outside the closure of the range of x ; the reasoning is as follows. Let y be a point outside the closure of the range; then some neighborhood of y is never visited by the function x ; thus, the integral of the local time over the neighborhood is 0; therefore, the local time vanishes almost everywhere in this neighborhood. Recall that the Radon-Nikodym derivative is unique in its definition except for a Borel set of measure 0; thus, the local time may, if necessary, be redefined so as to vanish *everywhere* in this neighborhood of y . Since the complement of the closure of the range may be covered by countably many such neighborhoods, the derivative may be so redefined on each of these, each time on a set of measure 0; thus, the local time, by an alteration on a set of measure 0, may be taken to vanish throughout the complement of the closure of the range.

Under the hypothesis of the lemma, for every positive integer M , there is an index n sufficiently large so that $N_n(y) > 2M$. The assertion of the previous paragraph implies that y belongs to the closure of the range of $x(t)$, $t \in I_{nk}$, for more than $2M$ indices k ; thus, $N(y) > M$. Since M is arbitrary, the lemma is proved.

LEMMA 2.4. *The set $\{y : N(y) < \infty\}$ is Borel measurable.*

Proof. For ε , $0 < \varepsilon < 1$, form a finite covering of $[0, 1]$ by open intervals J of length less than ε such that each point is covered by at most two intervals J . For each y , let $N_\varepsilon(y)$ be the number of intervals J whose images have closures containing y ; $N_\varepsilon(y)$ is a Borel function because x is. $N(y)$ is infinite if and only if $N_\varepsilon(y)$ becomes infinite for $\varepsilon \rightarrow 0$; indeed, if t is a point whose every neighborhood has an image with closure containing y , then there are *one* or *two* intervals J whose images also have closures containing y . This proves:

$$\{y : N(y) < \infty\} = \left\{ y : \liminf_{\varepsilon \rightarrow 0} N_\varepsilon(y) < \infty \right\};$$

hence the set is Borel measurable.

THEOREM 2.1. *If the limit of the sequence (2.1) is 0, then $N(x(t)) = \infty$ for almost all t (in the sense of Borel measure).*

Proof. Formula (1.4) and Lemma 1.1 imply that

$$(2.3) \quad \sum_{k=1}^{2^n} \phi_{nk}(x(t)) = \phi(x(t)) > 0$$

for almost all t , for every n .

Observe

$$\begin{aligned} \int_{-\infty}^{\infty} \liminf_{n \rightarrow \infty} \max_k \phi_{nk}^2(x) dx &\leq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \max_k \phi_{nk}^2(x) dx && \text{(Fatou's lemma)} \\ &\leq \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \sum_{k=1}^{2^n} \phi_{nk}^2(x) dx = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \sum_{k=1}^{2^n} \int_{-\infty}^{\infty} |f_{nk}(u)|^2 du && \text{(Parseval theorem)} \\ &= 0 && \text{(by hypothesis);} \end{aligned}$$

therefore, for almost all x , $\liminf_{n \rightarrow \infty} \max_k \phi_{nk}(x) = 0$. This implies

$$(2.4) \quad \liminf_{n \rightarrow \infty} \max_k \phi_{nk}(x(t)) = 0$$

for almost all t because a function with a local time maps sets of positive measure into similar sets. The relations (2.3) and (2.4) together imply that

$$\lim_{n \rightarrow \infty} N_n(x(t)) = \infty, \quad \text{for almost all } t.$$

This and Lemmas 2.3 and 2.4 imply the assertion of the theorem.

COROLLARY 2.1. *Under the hypothesis of Theorem 2.1, the set $\{y : N(y) < \infty\}$ is of ν -measure 0; in other words, $x(\cdot)$ spends no time in this set.*

Proof. Lemmas 2.3 and 2.4 imply

$$\nu\{y : N(y) < \infty\} \leq \nu\left\{y : \lim_{n \rightarrow \infty} N_n(y) < \infty\right\}$$

and the definition of ν , and Theorem 2.1 imply

$$\nu\left\{y : \lim_{n \rightarrow \infty} N_n(y) < \infty\right\} = \mu\left\{t : \lim_{n \rightarrow \infty} N_n(x(t)) < \infty\right\} = 0.$$

3. An inequality for the range of values of a function.

LEMMA 3.1. *Let x be a Borel function on $(I, \mathcal{B}, \mu)$; put*

$$b = \operatorname{ess\,inf}_I x, \quad c = \operatorname{ess\,sup}_I x, \quad |x(I)| = c - b;$$

then, for every ε , $0 \leq \varepsilon < 1$, and nonnegative integer m ,

$$(3.1) \quad |x(I)|^{2m+1+\varepsilon} \geq \mu^2(I) \cdot K \int_{-\infty}^{\infty} |u|^{2m+\varepsilon} |f(u)|^2 du,$$

where K is a numerical constant. If the integral in the denominator is infinite, the fraction is understood to be 0; in this case, the inequality is trivial.

Proof. If either $|b|$ or c is infinite, (3.1) is obvious; thus, we now assume both to be finite. We have

$$(3.2) \quad \mu(I) = \int_{-\infty}^{\infty} \frac{e^{-iuc} - e^{-iub}}{-2\pi iu} f(u) du;$$

for, on one hand, $\mu(I)$ is equal to

$$(3.3) \quad \int_b^c \phi(x) dx$$

because the integral of ϕ over the complement of the range of $x(t)$, $t \in I$, vanishes; on the other hand, by Parseval's theorem the integrals in (3.2) and (3.3) are equal.

First we prove (3.1) for $m=0$. The integral in (3.2) is dominated by

$$\int_{-\infty}^{\infty} |u|^{-\varepsilon/2} \left| \frac{e^{-iuc} - e^{-iub}}{2\pi iu} \right| |u|^{\varepsilon/2} |f(u)| du,$$

whose square, by the Cauchy-Schwarz inequality, is dominated by

$$\begin{aligned} \int_{-\infty}^{\infty} |u|^{-\varepsilon} \left| \frac{1 - e^{-iu(c-b)}}{2\pi iu} \right|^2 du \cdot \int_{-\infty}^{\infty} |u|^{\varepsilon} |f(u)|^2 du \\ = \text{constant} \cdot (c-b)^{1+\varepsilon} \int_{-\infty}^{\infty} |u|^{\varepsilon} |f(u)|^2 du. \end{aligned}$$

Now suppose $m \geq 1$. We assume that the integral in (3.1) is finite; otherwise, there is nothing to prove. The finiteness of this integral implies that ϕ and its first m derivatives $\phi^{(1)}, \dots, \phi^{(m)}$ exist and are square integrable [6, p. 92]; in particular, ϕ and its first $m-1$ derivatives are continuous; thus, they necessarily vanish at b

and c because they vanish everywhere outside $[b, c]$. Using this fact we find, by successive integration by parts:

$$(3.4) \quad \int_b^c \phi(x) dx = \frac{(-1)^m}{m!} \int_b^c (x-b)^m \phi^{(m)}(x) dx.$$

Apply the Parseval relation to the integral on the right-hand side. The Fourier transform of $\phi^{(m)}$ is $(iu)^m f(u)$, and that of the function

$$\begin{aligned} g(x) &= (x-b)^m, & b \leq x \leq c, \\ &= 0, & \text{elsewhere} \end{aligned}$$

is

$$\int_b^c e^{iux} (x-b)^m dx;$$

hence, the integral on the right-hand side of (3.4) is equal to

$$\frac{(-i)^m}{2\pi m!} \int_{-\infty}^{\infty} \left(\int_b^c e^{iux} (x-b)^m dx \right) (u^m f(u)) du.$$

Multiply the first function in the integrand by $|u|^{-\varepsilon/2}$, the second by $|u|^{\varepsilon/2}$, and apply the Cauchy-Schwarz inequality: the square of the integral is dominated by the product of the two integrals

$$\int_{-\infty}^{\infty} |u|^{2m+\varepsilon} |f(u)|^2 du$$

and

$$\frac{1}{(2\pi m!)^2} \int_{-\infty}^{\infty} |u|^{-\varepsilon} \left| \int_b^c e^{iux} (x-b)^m dx \right|^2 du.$$

The latter integral is dominated by a constant multiple of $(c-b)^{2m+1+\varepsilon}$; this can be verified by successive substitution of variables in the inner and outer integrals: $y = (x-b)/(c-b)$ and $v = u(c-b)$. The proof of the inequality (3.1) is now complete: compare the estimate of the right-hand side of (3.4) to the left-hand side; the latter, as shown in the remark leading to (3.3), is equal to $\mu(I)$.

4. Absence of Hölder conditions and infinite γ -variation for functions with a smooth local time. We apply the inequality (3.1) in the particular case in which I is the closed unit interval and μ the usual Borel measure. We continue to use the notation of §2; here $\mu(I_{nk}) = 2^{-n}$, $k = 1, \dots, 2^n$. Put I_{nk} and f_{nk} in the places of I and f , respectively, in (3.1) for $k = 1, \dots, 2^n$; and let x be a Borel function. Two inequalities follow:

$$(4.1) \quad \min_k |x(I_{nk})|^{2m+1+\varepsilon} \geq K/2^{2n} \sum_k \int_{-\infty}^{\infty} |u|^{2m+\varepsilon} |f_{nk}(u)|^2 du$$

$$(4.2) \quad \sum_k |x(I_{nk})|^{2m+1+\varepsilon} \geq K / \sum_k \int_{-\infty}^{\infty} |u|^{2m+\varepsilon} |f_{nk}(u)|^2 du.$$

The inequality (4.1) is an elementary algebraic consequence of the 2^n inequalities (3.1); and (4.2) is obtained by averaging over the 2^n inequalities (3.1), and noting that the arithmetic mean is at least equal to the harmonic mean.

For $\gamma \geq 1$, the γ -variation of the function $x(t)$, $0 \leq t \leq 1$, is defined as

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t_1 \leq \dots \leq t_n \leq 1} \sum_{j=1}^{n-1} |x(t_{j+1}) - x(t_j)|^\gamma.$$

An immediate implication of the inequality (4.2) is

LEMMA 4.1. *If the $\liminf (n \rightarrow \infty)$ of the denominator in (4.2) is equal to 0, then the γ -variation of x is infinite for $\gamma = 2m + 1 + \varepsilon$.*

The function x is said to satisfy a Hölder condition of order p at the point t_0 in $[0, 1]$ if there are positive constants h and δ such that $|x(t) - x(t_0)| \leq h|t - t_0|^p$ whenever $|t - t_0| \leq \delta$.

LEMMA 4.2. *Let x satisfy a Hölder condition of order p at the point t_0 for some constants h and δ . If I is a subinterval of $[0, 1]$ of positive length $|I| \leq \delta$, and I contains t_0 , then*

$$(4.3) \quad |x(I)| \leq 2|I|^p h.$$

Proof. In order to prove the lemma we shall show that the inequality

$$(4.4) \quad |x(t) - x(s)| \leq 2|I|^p h$$

is satisfied for all points s and t in I . For $s = t$ the inequality is trivial. If $s \neq t$ and $t = t_0$ (or $s = t_0$), then (4.4) is a consequence of the assumed Hölder condition. If s , t and t_0 are distinct, we deduce (4.4) by noting that (triangle inequality)

$$\frac{|x(t) - x(s)|}{|I|^p} \leq \frac{|x(t) - x(t_0)|}{|t - t_0|^p} \frac{|t - t_0|^p}{|I|^p} + \frac{|x(s) - x(t_0)|}{|s - t_0|^p} \frac{|s - t_0|^p}{|I|^p},$$

and that $|t - t_0| \leq |I|$, $|s - t_0| \leq |I|$, and then invoking the assumed Hölder condition.

COROLLARY 4.1. *If*

$$\limsup_{n \rightarrow \infty} 2^{np} \min_k |x(I_{nk})| = \infty,$$

then x does not satisfy a Hölder condition of order p at any point of $[0, 1]$.

Combining Corollary 4.1 and the inequality (4.1) we conclude:

LEMMA 4.3. *If the $\liminf (n \rightarrow \infty)$ of the denominator in (4.1) is equal to 0, then x nowhere satisfies a Hölder condition of order $p = 2/(2m + 1 + \varepsilon)$; in particular, if the hypothesis holds for $m = 1$, then x is nowhere differentiable.*

5. Local times and sample functions of Gaussian processes. Let $(I, \mathcal{B}, \mu)$ be as defined in the introduction; $(\Omega, \mathcal{F}, P)$ a probability space; and $X = X(t, \omega)$ a real

valued stochastic process on Ω , measurable with respect to $\mathcal{B} \times \mathcal{F}$. For each sample function $X(\cdot, \omega)$ we define the occupation time distribution: $\nu(A, \omega) = \mu(X^{-1}(A))$, for every linear Borel set A . The Fourier-Stieltjes transform of the corresponding distribution function is, by formula (1.2), given by

$$f(u, \omega) = \int_I e^{i u X(t, \omega)} d\mu(t), \quad -\infty < u < \infty;$$

it is a measurable function of (u, ω) .

Let E be the expectation operator corresponding to the probability measure P . When a function of (t, ω) or (u, ω) appears under the expectation operator, the argument ω will be suppressed; for example, $Ef(u, \omega)$ is written $Ef(u)$.

In this section X is a Gaussian process. It will be assumed that $EX(t) \equiv 0$ ($t \in I$). The application of the above theory of local times to Gaussian processes depends almost entirely on this formula:

$$\begin{aligned} (5.1) \quad & E \int_{-\infty}^{\infty} |u|^p |f(u)|^2 du \\ &= 2^{(p+1)/2} \Gamma\left(\frac{p+1}{2}\right) (2\pi)^{-1} \int_I \int_I [E(X(s) - X(t))^2]^{-(p+1)/2} d\mu(s) d\mu(t), \quad p \geq 0. \end{aligned}$$

The proof is: $X(t, \omega)$ and $f(u, \omega)$ are measurable functions of their respective pairs of arguments; thus, application of Fubini's theorem permits the interchange of order of integration:

$$\begin{aligned} E \int_{-\infty}^{\infty} |u|^p |f(u)|^2 du &= \int_{-\infty}^{\infty} |u|^p \int_I \int_I E e^{i u (X(t) - X(s))} d\mu(s) d\mu(t) du \\ &= \int_{-\infty}^{\infty} |u|^p \int_I \int_I \exp \left[-\frac{1}{2} u^2 E(X(t) - X(s))^2 \right] d\mu(s) d\mu(t) du. \end{aligned}$$

Another interchange of order of integration, and the change of variable $v^2 = u^2 E(X(t) - X(s))^2$ show that the last integral is equal to that on the right-hand side of (5.1).

For use in §6 we record a second formula:

$$\begin{aligned} (5.2) \quad & E \int_{-\infty}^{\infty} e^{ub} |f(u)|^2 du \\ &= \int_I \int_I (2\pi)^{1/2} [E(X(t) - X(s))^2]^{-1/2} \exp [b^2/2 E(X(t) - X(s))^2] d\mu(s) d\mu(t), \end{aligned}$$

for any real b . (The proof is similar to that for (5.1) with the modification that the formula for the moment generating function of the Gaussian distribution is used.)

LEMMA 5.1. *If the integral on the right-hand side of (5.1) is finite for some $p \geq 0$, then, for almost all ω , $\nu(\cdot, \omega)$ is absolutely continuous with respect to linear Borel measure. Its derivative, the local time of $X(\cdot, \omega)$, is square integrable. The first $[p/2]$ derivatives of the local time exist and are square integrable ($[u]$ = integral part of u).*

Proof. The hypothesis implies: for almost all ω , $|u|^{[p/2]}|f(u, \omega)|$ is square integrable; thus, the conclusion follows the application of [6, p. 92].

We now take I to be unit interval with the usual Borel measure, and return to the notation of the previous section.

LEMMA 5.2. *If the integral on the right-hand side of equation (5.1) is finite, then for almost all ω ,*

$$\liminf_{n \rightarrow \infty} \sum_k \int_{-\infty}^{\infty} |u|^p |f_{nk}(u, \omega)|^2 du = 0.$$

Proof. The union of the squares $I_{nk} \times I_{nk}$, $k=1, \dots, 2^n$ forms a decreasing sequence of Borel sets ($n=1, 2, \dots$) of measure tending to 0: therefore,

$$\lim_{n \rightarrow \infty} \sum_k \int_{I_{nk}} \int_{I_{nk}} [E(X(s) - X(t))]^{-(p+1)/2} d\mu(s) d\mu(t) = 0;$$

thus, by application of (5.1),

$$\lim_{n \rightarrow \infty} E \sum_k \int_{-\infty}^{\infty} |u|^p |f_{nk}(u)|^2 du = 0,$$

and Fatou's lemma completes the proof.

THEOREM 5.1. *Let X be a Gaussian process on $[0, 1]$. Let m be a nonnegative integer, ε a real number such that $0 \leq \varepsilon < 1$, and p defined as $p = 2m + \varepsilon$. If the integral on the right-hand side of (5.1) is finite, then, for almost all ω :*

- (a) *The γ -variation of $X(\cdot, \omega)$ is infinite for $\gamma = p + 1$;*
- (b) *$X(\cdot, \omega)$ nowhere satisfies a Hölder condition of order $2/(p+1)$;*
- (c) *$N(X(t, \omega))$ is infinite for almost all t (Theorem 2.1). This is valid as long as the hypothesis is fulfilled for $p=0$.*

Proof. The assertion (a) is based on Lemmas 4.1 and 5.2; (b) on Lemmas 4.3 and 5.2; and (c) on Theorem 2.1 and Lemma 5.2 (for $p=0$).

Suppose that there are constants $C > 0$ and $\beta > 0$ such that

$$(5.3) \quad E(X(s) - X(t))^2 \geq C|t - s|^\beta, \quad 0 \leq s, t \leq 1;$$

then the integral (5.1) is finite if $\beta(p+1) < 2$. If $\beta < 2$, then (5.1) is finite at least for $p=0$; therefore, conclusion (c) of Theorem 5.1 holds. If for some ε , $0 \leq \varepsilon < 1$, we have $\beta < 2/(1+\varepsilon)$, then, for $\gamma = 1 + \varepsilon$ the γ -variation of $X(\cdot, \omega)$ is infinite for almost all ω ; this follows from assertion (a) with $m=0$. Assertion (b) implies: if

$$(5.4) \quad \beta < 2/(2m + \varepsilon + 1)$$

then $X(\cdot, \omega)$ nowhere satisfies a Hölder condition of order $2/(2m + \varepsilon + 1)$.

In order to prove the nowhere differentiability of $X(\cdot, \omega)$, we shall show that the sample function nowhere satisfies a Hölder condition of order 1. The result on

Hölder conditions just given is not completely satisfactory because 1 is not expressible in the form $2/(2m + \varepsilon + 1)$ since $\varepsilon < 1$. The best result available from (5.4) is: if $\beta < 2/3$ ($m = 1, \varepsilon = 0$), there is nowhere a Hölder condition of order $2/3$, so that the sample function is nowhere differentiable. This will now be improved.

THEOREM 5.2. *The sample functions are nowhere differentiable if (5.3) holds for some $\beta < 1$.*

Proof. Let ε be a real number satisfying the double inequality

$$(5.5) \quad \beta/(2 - \beta) < \varepsilon < 1;$$

such a number exists under the assumption $\beta < 1$. Put $m = 0$ in the inequality (4.1) and multiply both sides by $2^{n(1 + \varepsilon)}$:

$$(5.6) \quad 2^{n(1 + \varepsilon)} \min |x(I_{nk})|^{1 + \varepsilon} \geq K/2^{n(1 - \varepsilon)} \sum_k \int_{-\infty}^{\infty} |u|^\varepsilon |f_{nk}(u, \omega)|^2 du.$$

Under the hypothesis (5.3) and by the formula (5.1) the expected value of the denominator in (5.6) is bounded above by a constant multiple of

$$2^{n(2 - \varepsilon)} \int_0^{2^{-n}} \int_0^{2^{-n}} |t - s|^{-\beta(\varepsilon + 1)/2} ds dt = 2^{n[\beta(\varepsilon + 1)/2 - \varepsilon]} \int_0^1 \int_0^1 |t - s|^{-\beta(\varepsilon + 1)/2} ds dt.$$

The latter double integral is finite because ε and β are less than 1; and the factor multiplying the integral converges to 0 as $n \rightarrow \infty$, under the assumption (5.5); thus (Fatou's lemma), the $\liminf (n \rightarrow \infty)$ of the denominator in (5.6) is 0, for almost all ω ; hence, by Corollary 4.1 ($p = 1$), $X(\cdot, \omega)$ nowhere satisfies a Hölder condition of order 1.

6. Analytic local time: an application to Brownian motion over Hilbert space.

We return to the general space I and recall some results given in [2] for the case of the unit interval. If ϕ is analytic, then x spends positive time in every Borel set of positive measure; therefore, x is unbounded. A sufficient condition for the analyticity of ϕ is that there exist $b > 0$ such that

$$(6.1) \quad \int_{-\infty}^{\infty} e^{ub} |f(u)|^2 du < \infty;$$

thus, if there exists a positive Borel measure on $\mathcal{B}$ for which (6.1) holds, then x is unbounded on I .

Now let $X(t, \omega)$, $t \in I$ be a measurable Gaussian process defined as in §5.

LEMMA 6.1. *If there exists a positive Borel measure μ on I , and $b > 0$ such that*

$$(6.2) \quad \int_I \int_I \exp [b^2/E(X(t) - X(s))^2] d\mu(s) d\mu(t) < \infty,$$

then, for almost all ω , $X(\cdot, \omega)$ is unbounded on I .

Proof. The hypothesis implies that

$$\int_I \int_I [E(X(s) - X(t))^2]^{-1} d\mu(s) d\mu(t) < \infty$$

because u^{-1} tends to infinity more slowly than $\exp(b^2/u)$ for $u \downarrow 0$; thus, applying the Cauchy-Schwarz inequality to the integral on the right-hand side of (5.2), we find the left-hand side to be finite; consequently, for almost all ω , (6.1) holds with $f=f(u, \omega)$.

Take I to be the space l_2 of all real square summable sequences $t=(t_n)$ with the usual norm $\|\cdot\|$ defined by $\|t\|^2 = \sum_n t_n^2$. Let X be a measurable version of Levy's Brownian motion, a Gaussian process such that

$$EX(t) \equiv 0, \quad EX^2(0) = 0, \quad E(X(t) - X(s))^2 = \|t - s\|.$$

In [1] it is shown that on each compact subset of l_2 either the sample functions are continuous, or if not, there is a point in the subset at which each sample function is discontinuous. In the particular case of a compact ellipsoid sufficient conditions for continuity were given; now Lemma 6.1 will be used to get sufficient conditions for the unboundedness of the sample functions. The method can also be used for other compact sets in l_2 ; for example, a cube.

An ellipsoid is a set of the form

$$(6.3) \quad \left\{ t : \sum_n t_n^2 / \lambda_n^2 \leq 1 \right\},$$

where $\{\lambda_n\}$ is a sequence of real numbers. The ellipsoid is compact if $\lambda_n \rightarrow 0$ [3, p. 28].

THEOREM 6.1. *Let S be a compact ellipsoid in l_2 , and (λ_n) the sequence in (6.3). If for some p , $1 < p < 2$,*

$$(6.4) \quad \sum_n \frac{1}{n^p \lambda_n^2} < \infty,$$

then the sample functions are unbounded over S .

(The goal of the proof is the construction of a positive Borel measure on S for which (6.2) holds.)

Proof. Let $X_1, X_2, \dots$ be a sequence of independent random variables on some probability space having Gaussian distributions with expectations 0 and $EX_n^2 = n^{-p}$. For each integer $k > 0$ and real numbers $t_1, \dots, t_k$ define the measure of the cylinder set $\{s : s_1 \leq t_1, \dots, s_k \leq t_k\}$ in l_2 as $\Pr(X_1 \leq t_1, \dots, X_k \leq t_k)$. This cylinder set measure is extendable to a countably additive measure (denoted also by μ) on the Borel sets of l_2 ; indeed, the series $\sum_n X_n^2$ converges with probability 1 because the sequence of variances converges (cf. [3, p. 334, Theorem 7]).

The condition (6.4) also implies that the series $\sum_n X_n^2/\lambda_n^2$ converges with probability 1; therefore, for every $n \geq 1$, the series may be represented as the sum of the two independent sub-sums

$$\sum_{j=1}^n X_j^2/\lambda_j^2 \quad \text{and} \quad \sum_{j=n+1}^{\infty} X_j^2/\lambda_j^2.$$

This representation implies

$$\Pr \left(\sum_n X_n^2/\lambda_n^2 \leq 1 \right) > 0;$$

indeed, for every n , the distribution of the first sub-sum certainly assigns positive probability to the interval of values $[0, \frac{1}{2}]$, and, if n is sufficiently large, so does the distribution of the second sub-sum; thus the convolution of the distributions assigns positive probability to $[0, 1]$; hence, $\mu(S) > 0$.

The next step in the proof is the estimation of the double integral in (6.2) when μ is the measure just constructed and $E(X(t) - X(s))^2 = \|t - s\|$. Let $\{Y_n\}$ be a sequence of random variables on the same space as $\{X_n\}$, distributed identically as the latter but independent of it. By definition of μ , the double integral in (6.2) is equal to

$$E \left\{ \exp \left[b^2 / \left(\sum_n (X_n - Y_n)^2 \right)^{1/2} \right] \right\};$$

thus, since $X_n - Y_n$ is distributed as $\sqrt{2} X_n$:

$$(6.5) \quad \int_{I_2} \int_{I_2} \exp (b^2 / E(X(s) - X(t))^2) d\mu(s) d\mu(t) = E \left\{ \exp \left[b^2 / \sqrt{2} \left(\sum_n X_n^2 \right)^{1/2} \right] \right\}.$$

Define

$$(6.6) \quad \begin{aligned} U_n &= 1 && \text{if } |X_n| > n^{-p/2}, \\ &= 0 && \text{if } |X_n| \leq n^{-p/2}. \end{aligned}$$

The U 's are independent and have a common Bernoulli distribution with mean $\Pr (|X_n| > n^{-p/2}) = 0.3174$ (approximately); and, from (6.6):

$$(6.7) \quad E \left\{ \exp \left[b^2 / \sqrt{2} \left(\sum_n X_n^2 \right)^{1/2} \right] \right\} \leq E \left\{ \exp \left[b^2 / \sqrt{2} \left(\sum_n n^{-p} U_n \right)^{1/2} \right] \right\}.$$

The final step in the proof is showing that the latter expectation is finite. Recall that the expected value of a nonnegative random variable Z is finite if

$$\int_0^\infty \Pr (Z > t) dt < \infty;$$

thus, the expectation in (6.7) is finite if

$$(6.8) \quad \int_1^\infty \Pr \left(\sum n^{-p} U_n < b^4 / 2 |\log t|^2 \right) dt < \infty.$$

For every $\tau > 0$ the inequality $\sum_n n^{-p} U_n < \tau$ holds only if the terms up to index

$n > \tau^{-1/p}$ are all equal to 0; thus, by the independence of the $\{U_n\}$ and common distribution:

$$\begin{aligned} \Pr \left(\sum n^{-p} U_n < \tau \right) &\leq \Pr (U_j = 0, j = 1, \dots, [\tau^{-1/p}]) \\ &\leq (\Pr (U_1 = 0))^{[\tau^{-1/p}]} \leq (0.7)^{[\tau^{-1/p}]}; \end{aligned}$$

therefore, putting $\tau = b^4/2 |\log t|^2$, we see that the integral in (6.8) is at most

$$\int_0^\infty (1/0.7) \exp \{2^{1/p} |\log t|^{2/p} b^{-4/p} \log (0.7)\} dt;$$

it is finite because $p < 2$.

The measure μ assigns positive measure to S , and satisfies (6.2); hence, the sample functions are unbounded on S .

EXAMPLE. Suppose S is the ellipsoid (6.3) and $\lambda_n = n^{-r}$, where $r > 0$. The sample functions are continuous on S if $r > 1$; indeed, in this case $\sum_n |\lambda_n|^p < \infty$ for some $p < 1$ and the continuity condition given in [1] is satisfied. The sample functions are unbounded on S if $r < 1/2$; in fact, there exist number p and ε , $1 < p < 2$, $\varepsilon > 0$, such that $1 < p < 2$ and $2r < (p-1) - \varepsilon$, and the series $\sum_n n^{-p} n^{2r}$ converges.

Added in Proof. S. Varadhan showed that Theorem 6.1 is valid also for $1 < p \leq 3$. From the inequality

$$\prod_{n \geq 1} (1 + \lambda n^{-p})^{-1} \leq \exp (-c \lambda^{1/p}), \quad c > 0 \text{ constant,}$$

the formula for the moment generating function of $\sum_{n \geq 1} X_n^2$, and a Chebyshev-type inequality, we conclude that

$$P \left[\sum X_n^2 \leq t \right] \leq \exp [-\text{constant}/t^{1/(p-1)}], \quad t > 0.$$

This implies the finiteness of (6.5). In the example $\lambda_n = n^{-r}$ the sample functions are unbounded if $r < 1$.

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