ON BAZILEVIČ AND CONVEX FUNCTIONS

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1. Introduction. Let

$$(1) \quad f(z) = \left\{ \frac{\beta}{1+\alpha^2} \int_0^z (h(\zeta) - \alpha i) \zeta^{[-\alpha\beta/(1+\alpha^2)]-1} g(\zeta)^{\beta/(1+\alpha^2)} d\zeta \right\}^{(1+\alpha^2)/\beta}$$

where $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ satisfies $\operatorname{Re} h(z) > 0$ in $|z| < 1$, $g(z)$ is starlike in $|z| < 1$, α is any real number and $\beta > 0$.

Bazilevič [1], [8] introduced the above class of functions and showed that each such function is univalent in $|z| < 1$.

Let $\alpha = 0$ in (1). On differentiating we get

$$(2) \quad zf'(z) = f(z)^{1-\beta} g(z)^{\beta} h(z)$$

and

$$(3) \quad \operatorname{Re} h(z) = \operatorname{Re} (zf'(z)/f(z)^{1-\beta} g(z)^{\beta}) > 0 \quad \text{in } |z| < 1.$$

Thomas [12] called a function satisfying the condition (3) a Bazilevič function of type β .

Let $C(r)$ denote the curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$, and let $L(r)$ denote the length of $C(r)$. Let $M(r) = \max_{|z|=r} |f(z)|$.

Then Thomas [12] showed that if $f(z)$ is a Bazilevič function of type β , $0 < \beta \leq 1$, then

$$L(r) \leq K(\beta) M(r) \log (1/(1-r)),$$

and if $M(r) \leq (1-r)^{-\alpha}$ then

$$L(r) \leq K(\alpha, \beta) (1-r)^{-\alpha} \quad \text{for } 0 < \alpha \leq 2$$

where $K(\beta)$ and $K(\alpha, \beta)$ are constants depending only upon α and β .

2. On Bazilevič functions.

LEMMA 1. Let $f(z)$ be mean p -valent in $|z| < 1$. Then

$$\int_0^r \int_0^{2\pi} \frac{|f'(\rho e^{i\theta})|^2}{|f(\rho e^{i\theta})|} \rho d\theta d\rho \leq 2p\pi M(r) \quad \text{for } 0 < r < 1.$$

We owe Lemma 1 to Hayman [3, p. 45].

THEOREM 1. Let $f(z)$ be a Bazilevič function of type β and $\arg f(z)$ be a function of bounded variation on $|z| = \rho < 1$.

Then we have

$$L(r) \leq K(\beta)M(r) \log(1/(1-r)),$$

and also if $M(r) \leq (1-r)^{-\alpha}$ then $L(r) \leq K(\alpha, \beta)(1-r)^{-\alpha}$ for $0 < \alpha \leq 2$ where $K(\beta)$ and $K(\alpha, \beta)$ are constants depending only upon α and β .

Proof. We have

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)| d\theta = \int_0^{2\pi} |f(z)^{1-\beta} g(z)^\beta h(z)| d\theta \\ &\leq \int_0^r \int_0^{2\pi} |(1-\beta)f'(z)f(z)^{-\beta} g(z)^\beta h(z)| d\theta d\rho \\ &\quad + \int_0^r \int_0^{2\pi} |f(z)^{1-\beta} \beta g'(z)g(z)^{\beta-1} h(z)| d\theta d\rho \\ &\quad + \int_0^r \int_0^{2\pi} |f(z)^{1-\beta} g(z)^\beta h'(z)| d\theta d\rho \\ &= J_1 + J_2 + J_3, \quad \text{say.} \end{aligned}$$

Applying (2) and Lemma 1 we obtain

$$(4) \quad J_1 = |1-\beta| \int_0^r \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| d\theta d\rho \leq 2\pi|1-\beta|M(r).$$

Since $g(z)$ is a starlike function we may write $zg'(z) = g(z)\phi(z)$, where $\phi(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ and $\operatorname{Re} \phi(z) > 0$ in $|z| < 1$.

Then we have

$$\begin{aligned} J_2 &= |\beta| \int_0^r \int_0^{2\pi} |f'(z)\phi(z)| d\theta d\rho \\ &\leq |\beta| \left\{ \int_0^r \int_0^{2\pi} |f'(z)|^2 d\theta d\rho \right\}^{1/2} \left\{ \int_0^r \int_0^{2\pi} |\phi(z)|^2 d\theta d\rho \right\}^{1/2}. \end{aligned}$$

It is well known [6], [10] that

$$\int_0^r \int_0^{2\pi} |f'(z)|^2 d\theta d\rho \leq \frac{2\pi M(r)^2}{r}$$

and

$$\int_0^r \int_0^{2\pi} |\phi(z)|^2 d\theta d\rho \leq 4\pi \log \frac{1+r}{1-r}.$$

Hence we have

$$(5) \quad J_2 \leq 2|\beta| \sqrt{2\pi} \left(\frac{1}{r} \log \frac{1+r}{1-r} \right)^{1/2} M(r).$$

On the other hand it is well known [2] that

$$|h'(z)| \leq 2 \operatorname{Re} h(z)/(1-\rho^2) \quad \text{for } |z| \leq \rho < 1.$$

Hence we have

$$\begin{aligned} J_3 &\leq \int_0^r \int_0^{2\pi} |f(z)^{1-\beta} g(z)^\beta| \frac{2 \operatorname{Re} h(z)}{1-\rho^2} d\theta d\rho \\ &= 2 \operatorname{Re} \int_0^r \int_0^{2\pi} |f(z)^{1-\beta} g(z)^\beta| \frac{zf'(z)}{f(z)^{1-\beta} g(z)^\beta} \cdot \frac{1}{1-\rho^2} d\theta d\rho \\ &= 2 \operatorname{Re} \int_0^r \int_0^{2\pi} zf'(z) \exp(-i\{(1-\beta) \arg f(z) + \beta \arg g(z)\}) \frac{1}{1-\rho^2} d\theta d\rho. \end{aligned}$$

Now let us put $\Theta = (1-\beta) \arg f(z) + \beta \arg g(z)$. Then we have

$$\begin{aligned} &\operatorname{Re} \int_0^{2\pi} zf'(z) \exp(-i\{(1-\beta) \arg f(z) + \beta \arg g(z)\}) d\theta \\ &= \operatorname{Re} \int_0^{2\pi} \frac{1}{i} \cdot \frac{df(z)}{d\theta} \cdot e^{-i\Theta} d\theta = \operatorname{Re} \frac{1}{i} \int_{|z|=\rho} e^{-i\Theta} \frac{df(z)}{d\Theta} d\Theta \\ &= \operatorname{Re} \frac{1}{i} [e^{-i\Theta} f(z)]_{|z|=\rho} + \operatorname{Re} \int_{|z|=\rho} e^{-i\Theta} f(z) d\Theta \\ &= \operatorname{Re} \int_{|z|=\rho} e^{-i\Theta} f(z) d\Theta \\ &= \operatorname{Re} (1-\beta) \int_{|z|=\rho} e^{-i\Theta} f(z) d \arg f(z) + \operatorname{Re} \beta \int_{|z|=\rho} e^{-i\Theta} f(z) d \arg g(z) \\ &\leq \operatorname{Re} (1-\beta) \int_{|z|=\rho} e^{-i\Theta} f(z) d \arg f(z) + 2\pi\beta M(r). \end{aligned}$$

Since $\arg f(z)$ is a function of bounded variation on $|z|=\rho < 1$, there exists a bounded constant K and $\int_{|z|=\rho} |d \arg f(z)| \leq 2\pi K$. Hence we have

$$\begin{aligned} J_3 &\leq 2\pi\{|1-\beta|K + |\beta|\}M(r) \int_0^r \frac{1}{1-\rho^2} d\rho \\ (6) \quad &= \pi\{|1-\beta|K + |\beta|\}M(r) \log \frac{1+r}{1-r}. \end{aligned}$$

From (4), (5) and (6) we obtain

$$L(r) \leq K(\beta)M(r) \log(1/(1-r)).$$

Applying the same method as the above we have also that if $M(r) \leq (1-r)^{-\alpha}$, then $L(r) \leq K(\alpha, \beta)(1-r)^{-\alpha}$ for $0 < \alpha \leq 2$. This completes our proof.

COROLLARY 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and starlike in $|z| < 1$. Then we have $L(r) \leq KM(r) \log(1/(1-r))$, and also if $M(r) \leq (1-r)^{-\alpha}$ then $L(r) \leq K(\alpha)(1-r)^{-\alpha}$ for $0 < \alpha \leq 2$, where K is a bounded constant and $K(\alpha)$ is a constant depending only upon α .*

These results are due to Keogh and Pommerenke [4], [5].

LEMMA 2. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be regular and univalent in $|z| < 1$.
Then we have

$$M(r) \leq 4(\pi^{-1}A(r) \log(3/(1-r)))^{1/2}$$

where $A(r)$ is the area enclosed by the curve $C(r)$.

We owe this lemma to Pommerenke [7].

Applying Lemma 2 and the same method as in the proof of Theorem 1 we easily obtain the following theorems.

THEOREM 2. Let $f(z)$ be a Bazilevič function of type β and $\arg f(z)$ be a function of bounded variation on $|z| = \rho < 1$.

Then we have

$$L(r) \leq K(\beta)(A(r))^{1/2}(\log(1/(1-r)))^{3/2}$$

where $K(\beta)$ is a constant depending only upon β .

THEOREM 3. Let $f(z)$ be regular and close-to-convex (a Bazilevič function of type 1) in $|z| < 1$. Then we have

$$L(r) \leq K(A(r))^{1/2}(\log(1/(1-r)))^{3/2}$$

where K is a bounded constant.

REMARK. Thomas [11] has obtained the following:

Let $f(z)$ be regular and starlike in $|z| < 1$. Then

$$L(r) \leq 2(\pi A(r))^{1/2} \left(1 + \log \frac{1+r}{1-r}\right).$$

3. On convex functions.

LEMMA 3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and starlike in $|z| < 1$.
Then we have for any real λ

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |1 + re^{i\theta}|^{-2\lambda} d\theta.$$

We owe this lemma to Robertson [9].

THEOREM 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and convex in $|z| < 1$.
Then we have

$$I(r) = \int_0^{2\pi} |f(z)| d\theta \leq K \log \frac{1}{1-r}$$

where K is a bounded positive constant.

Proof. Since $f(z)$ is convex it follows that $zf'(z)$ is starlike in $|z| < 1$. Then we have the following inequalities for a fixed constant $0 < r_1 < r < 1$,

$$\begin{aligned} I(r) &\leq \int_0^r \int_0^{2\pi} |f'(z)| \, d\theta \, d\rho = \int_0^{r_1} \int_0^{2\pi} |f'(z)| \, d\theta \, d\rho + \int_{r_1}^r \int_0^{2\pi} |f'(z)| \, d\theta \, d\rho \\ &\leq C + \frac{1}{r_1} \int_{r_1}^r \int_0^{2\pi} |zf'(z)| \, d\theta \, d\rho, \end{aligned}$$

where C is a bounded constant.

Applying Lemma 3 to $zf'(z)$ we have

$$\begin{aligned} \int_{r_1}^r \int_0^{2\pi} |zf'(z)| \, d\theta \, d\rho &\leq \int_{r_1}^r \int_0^{2\pi} \frac{1}{|1 - \rho e^{i\theta}|^2} \, d\theta \, d\rho \\ &\leq \int_0^r \frac{1}{1 - \rho^2} \, d\rho = \frac{1}{2} \log \frac{1+r}{1-r}. \end{aligned}$$

Hence we have

$$I(r) \leq C + \frac{1}{2r_1} \log \frac{1+r}{1-r}.$$

Integrating the following convex function we have

$$(7) \quad \int_{|z|=r} \left| \frac{z}{1-z} \right| \, d\theta = \int_0^{2\pi} \frac{r}{(1 - 2r \cos \theta + r^2)^{1/2}} \, d\theta = O(\log(1/(1-r)))$$

where O in (7) cannot be replaced by o .

This shows that the function $f(z) = z/(1-z)$ is an extremal function in Theorem 4.

REFERENCES

1. I. E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Mat. Sb. **37** (1955), 471-476.
2. C. Carathéodory, *Conformal representation*, Cambridge Univ. Press, Cambridge, 1932.
3. W. K. Hayman, *Multivalent functions*, Cambridge Univ. Press, Cambridge, 1958.
4. F. R. Keogh, *Some theorems on conformal mapping of bounded star-shaped domains*, Proc. London. Math. Soc. (3) **9** (1959), 481-491.
5. C. Pommerenke, *On starlike and convex functions*, J. London Math. Soc. **37** (1962), 209-224.
6. ———, *On starlike and close-to-convex functions*, Proc. London Math. Soc. **13** (1963), 290-304.
7. ———, *Über nahezu konvexe analytische Funktionen*, Arch. Math. **16** (1965), 344-347.
8. ———, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159-173.
9. M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. **37** (1936), 374-408.
10. D. K. Thomas, *On starlike and close-to-convex univalent functions*, J. London Math. Soc. **42** (1967), 427-435.
11. ———, *A note on starlike functions*, J. London Math. Soc. **43** (1968), 703-706.
12. ———, *On Bazilevič functions*, Trans. Amer. Math. Soc. **132** (1968), 353-361.

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