ON THE HAUPTVERMUTUNG FOR A CLASS OF OPEN MANIFOLDS[1]

BY

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1. **Introduction.** The Hauptvermutung for PL manifolds is the conjecture that homeomorphic PL manifolds are PL homeomorphic. No counterexample to this conjecture is known. It is known to be true for manifolds of dimension three or less ([13], [14]) and for many high dimensional compact manifolds ([20], [21]), but the only high dimensional result for open (i.e. noncompact with empty boundary) manifolds is that it is true in case the manifolds are topologically E^n ($n \geqq 5$) [19].

The main result of this paper is that the conjecture is true whenever the manifolds are topologically S^n minus a nonempty tame [8] compact 0-dimensional subset and $n>5$. The analogous result in the differentiable category holds if $n=6, 7$ (Theorem 5.2), but there exist many counterexamples to the complete transfer of the theorem, there is one in dimension 8. We also obtain a connectivity characterization of S^n ($n>5$) minus a nonempty tame compact 0-dimensional subset (Theorem 4.3).

2. **Definitions and basic facts.** An *end* of a manifold M is a function

$$\varepsilon\colon \{\text{compact subsets of } M\} \to \{\text{open subsets of } M\}$$

such that

(1) $\varepsilon(C)$ is a nonempty component of $M-C$ and

(2) $\varepsilon(C_1) \supset \varepsilon(C_2)$ whenever $C_1 \subset C_2$.

This definition is equivalent to that given by Siebenmann in [16].

Throughout this paper $\mathscr{E}X$, $\mathscr{C}X$, $\mathscr{C}_u(X-A)$ and σX will denote the set of ends of X, the set of components of X, the set of unbounded components of $X-A$ (i.e. components with noncompact closure in X), and the cardinality of X respectively. The following elementary lemma is given without proof.

LEMMA 2.1. *Let M^n ($n \geqq 2$) be a compact manifold and let K be a 0-dimensional closed subset of M. Then $\sigma\mathscr{E}(M-K)=\sigma K$.*

A manifold M is said to be *q-connected at infinity* if and only if given any compact subset C of M, there is a compact subset D (depending upon C) of M such that $C \subset D$ and each component of $M-D$ is q-connected. M is *(p, q)-connected* if and only if M is p-connected and q-connected at infinity.

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Throughout this paper E^n, B^n, and S^n will denote the PL manifolds, Euclidean n-space, a standard n-simplex, and the boundary of B^{n+1} respectively. A *cored n-ball* CB_k^n *of index* k (k a positive integer) is a PL n-manifold which is PL homeomorphic to S^n minus the interiors of k mutually disjoint PL n-balls. M is the *monotone union of the submanifolds* $M_1, M_2, \ldots$ if and only if $M = \bigcup M_i$ and $M_i \subset \operatorname{Int} M_{i+1}$ for all i.

THEOREM 2.2. *If U is an open* PL *n-manifold* ($n>5$) *which is the monotone union of cored n-balls, then U is* PL *homeomorphic to S^n minus a nonempty tame compact* 0-*dimensional subset.*

REMARK. We will not give all the details of the proof of Theorem 2.2. The theorem actually can be proven in every dimension except four (in question here because we use the Hauptvermutung for balls). Edwards [6] gives a proof of the 3-dimensional case. One proves Theorem 2.2 in a straightforward manner using the following three lemmas and a theorem of McMillan [8] which tells us that the subset is tame.

LEMMA 2.3. *Let $h: M \to S^n$ be a* PL *embedding where M is a compact connected* PL *n-manifold* ($n>5$) *with* Bd M *being a disjoint union of k* PL $(n-1)$-*spheres*, $S_1, S_2, \ldots, S_k$. *Then* Cl $(S^n - h(M))$ *is the disjoint union of k* PL *n-balls*, $B_1, \ldots, B_k$ *with* $h(S_i) = \operatorname{Bd} B_i$; *and hence M is a cored n-ball.*

Proof. By duality $h(S_i)$ separates S^n into two components. One of these, say Q_i, does not intersect $h(M)$. Since $h(S_i)$ is a PL sphere, Cl $Q_i = B_i$ is a topological ball by applying a theorem of Brown [3]. Obviously B_i is a component of Cl $(S^n - h(M))$ and hence is a PL manifold by [1]. Since for $n>5$ the PL Hauptvermutung for balls is true [18], B_i is a PL n-ball. The lemma now follows easily.

LEMMA 2.4. *Let* $CB_r^n \subset \operatorname{Int} CB_s^n$ ($n>5$). *Let $S_1, S_2, \ldots, S_r$ be the boundary spheres of* CB_r^n. *Then* Cl $(CB_s^n - CB_r^n)$ *has r components*, $Q_1, \ldots, Q_r$ *with*

$$Q_i \cap CB_r^n = \operatorname{Bd} Q_i \cap \operatorname{Bd} CB_r^n = S_i$$

and each Q_i is a cored n-ball.

LEMMA 2.5. *Let M be a cored n-ball* ($n>5$) *with boundary spheres* $S_1, S_2, \ldots, S_k$. *Let $h: S_1 \to S^n$ be a* PL *embedding such that* Cl Q *is a* PL *n-ball where Q is a component of* $S^n - h(S_1)$. *Let* $M' \subset$ Cl Q *be a cored n-ball with boundary spheres*, $S_1', S_2', \ldots, S_k'$ *and* $S_1' = \operatorname{Bd} \operatorname{Cl} Q$. *Then h extends to a* PL *homeomorphism* $h': M \twoheadrightarrow M'$ *such that* $h'(S_i) = S_i'$ *for* $i = 1, \ldots, k$.

The PL n-manifold M_2 is said to be *obtained* from the PL n-manifold M_1 by *surgery of index* k ($0 \leq k < n$) if and only if there are PL embeddings $f_1: S^k \times B^{n-k} \to M_1$ and $f_2: B^{k+1} \times S^{n-k-1} \to M_2$ such that

$$\begin{aligned} M_1 \cap M_2 &= M_1 - f_1(S^k \times \operatorname{Int} B^{n-k}) \\ &= M_2 - f_2(\operatorname{Int} B^{k+1} \times S^{n-k-1}), \end{aligned} \tag{1}$$

and

$$f_1|S^k \times S^{n-k-1} = f_2|S^k \times S^{n-k-1}. \tag{2}$$

PROPOSITION 2.6. *Let f: $B^{k+1} \to M^n$ be a proper (i.e. $f^{-1}(\text{Bd } M) = S^k$) PL embedding. Then there is a regular neighborhood N of $f(B^{k+1})$ in M such that*

(1) *$M' = \text{Cl}(M-N)$ is a PL n-manifold.*

(2) *If $j < n-k-2$ and M is j-connected, so is M'.*

(3) *$N \cap \text{Bd } M \approx S^k \times B^{n-k-1}$.*

(4) *$N \cap \text{Bd } M' \approx B^{k+1} \times S^{n-k-2}$.*

(5) *Bd M' is obtained from Bd M by surgery of index k.*

(6) *If $2k+2 < n$ and λ is the homotopy class of $f|S^k$ in $\pi_k(\text{Bd } M)$, then $\pi_i(\text{Bd } M') \approx \pi_i(\text{Bd } M)$ for $i<k$ and $\pi_k(\text{Bd } M') \approx \pi_k(\text{Bd } M)/(\lambda)$ where (λ) is a subgroup containing λ.*

Proof. Let N be a regular neighborhood of $f(B^{k+1})$ in M such that $N \cap \text{Bd } M$ is a regular neighborhood of $f(S^k)$ in Bd M. By [15], there is a block bundle ξ over $f(B^{k+1})$ whose total space is N and whose restriction to $f(S^k)$ has total space $N \cap \text{Bd } M$. Also by [15], since $f(B^{k+1})$ is collapsible, ξ and $\xi|f(S^k)$ are product bundles. Hence, there is a PL homeomorphism h: $B^{k+1} \times B^{n-k-1} \twoheadrightarrow N$ such that $h(x, 0) = f(x)$ for all $x \in B^{k+1}$, and $h(S^k \times B^{n-k-1}) = N \cap \text{Bd } M$. Hence (3) is satisfied, and by [1], it follows that (1) is satisfied. Now M' and $M - f(B^{k+1})$ have the same homotopy type and hence (2) follows easily using standard general position techniques [23]. An elementary point-set argument yields (4), and (5) follows. (6) is an immediate consequence of a theorem of Milnor [10].

The following elementary lemmas are given without proof.

LEMMA 2.7. *The complement of a compact subpolyhedron in a connected polyhedron has only finitely many components.*

LEMMA 2.8. *Let C be a compact subset of a connected PL n-manifold M. Then $C' = C \cup \bigcup \{Q : Q$ is a bounded component of $M - C$ (i.e. Cl Q is compact)$\}$ is compact and $M - C'$ has only finitely many components.*

3. Approximating open manifolds by compact submanifolds.

PROPOSITION 3.1. *Let U^n ($n > 5$) be an open (0, 1)-connected PL manifold. Let C be a compact subset of U such that each component of $U - C$ is 1-connected and $i: C \subset U$ induces onto homomorphisms i_*: $H_r(C) \twoheadrightarrow H_r(U)$ for $r \leqq n-2$.*

Then there is a compact connected PL n-submanifold N of U such that

(1) *$C \subset \text{Int } N$,*

(2) *$\sigma\mathscr{C}$ Bd $N = \sigma\mathscr{C}(U-N) = \sigma\mathscr{C}_u(U-C)$,*

(3) *each component of Bd N and of $U-N$ is 1-connected, and*

(4) *$i: N \subset U$ induces isomorphisms i_*: $H_r(N) \twoheadrightarrow H_r(U)$ for $r \leqq n-3$.*

The proof of this proposition parallels closely that of Proposition 4 of [2].

For $1 \leqq k \leqq n-3$ let $\mathscr{S}_k$ be the statement that *there is a compact* PL *n-submanifold* N_k *of* U *such that* (1), (2), *and* (3) *of* 3.1 *hold if* N_k *replaces* N, *and* $(4)_k$ $i: N_k \subset U$ *induces isomorphisms* $i_*: H_r(N_k) \twoheadrightarrow H_r(U)$ *for* $r \leqq k$ *and is onto if* $k < r \leqq n-2$.

Let $\mathscr{S}'_k$ be the statement that *if* P *is a compact subpolyhedron of* U *with* $\dim P \leqq n-2$ *and* N_k *is as above then there is a compact connected* PL *n-submanifold* N'_k *of* U *such that* (1), (2), (3) *and* $(4)_k$ *hold if* N'_k *replaces* N_k, $N_k \cup P \subset \operatorname{Int} N'_k$, *and* $(N'_k - N_k)$ *intersected with any component of* $U - N_k$ *is* 1*-connected.*

The proof will proceed through the following steps.

Step 1. Verify $\mathscr{S}_1$.

Step 2. Verify $\mathscr{S}_2$.

Step 3. Show that $\mathscr{S}_k$ implies $\mathscr{S}'_k$.

Step 4. Show that $\mathscr{S}_k$ implies $\mathscr{S}_{k+1}$ if $2 \leqq k \leqq n-5$.

Step 5. Show that $\mathscr{S}_{n-4}$ implies $\mathscr{S}_{n-3}$.

It is clear that $\mathscr{S}_{n-3}$ implies the proposition.

Step 1. *The validity of* $\mathscr{S}_1$.

Proof. Let $C' = C \cup \bigcup \{Q : Q$ is a bounded component of $U - C\}$. By Lemma 2.8, C' is compact and $U - C'$ is composed of finitely many components, $Q_1, \ldots, Q_r$, which are precisely the unbounded components of $U - C$. Since C' is compact, let N'_0 be a compact PL n-submanifold of U such that $C' \subset \operatorname{Int} N'_0$. Without loss of generality we assume that N'_0 is connected; for if not, we can join components of N'_0 by arcs, take regular neighborhoods, and let the new N'_0 be the old N'_0 union these neighborhoods.

Now let a and a' be the number of components of $\operatorname{Cl}(U - N'_0)$ and $U - C'$ respectively. By Lemma 2.7, a is finite; and since $C' \subset \operatorname{Int} N'_0$, $a \geqq a'$. We must alter N'_0 so that $a = a'$. But if $a > a'$, then two components V_1 and V_2 of $\operatorname{Cl}(U - N'_0)$ lie in the same component Q_j of $U - C'$. Let α be a polygonal arc in Q_j such that $\alpha \cap V_i = \operatorname{Bd} \alpha \cap \operatorname{Bd} V_i = \{x_i\}$ $(i = 1, 2)$ and $\operatorname{Int} \alpha \subset \operatorname{Int} N'_0$. Let P be a regular neighborhood of α in N'_0 such that $P \cap C' = \varnothing$. Let $N''_0 = \operatorname{Cl}(N'_0 - P)$. Since an arc cannot separate N'_0, it now follows that N''_0 is a compact connected PL n-submanifold of U such that $C' \subset \operatorname{Int} N''_0$ and $\operatorname{Cl}(U - N''_0)$ has one less component than $\operatorname{Cl}(U - N'_0)$. By finite induction we now assume that $\sigma\mathscr{C} \operatorname{Cl}(U - N''_0) = \sigma\mathscr{C}_u(U - C)$. But $\sigma\mathscr{C} \operatorname{Cl}(U - N''_0) = \sigma\mathscr{C}(U - N''_0)$, and hence N''_0 almost satisfies (2). Now let b be the number of components of $\operatorname{Bd} N''_0$ and a' be as above. It follows that $b \geqq a'$. If $b > a'$, then since $\operatorname{Bd} N''_0 = \operatorname{Bd} \operatorname{Cl}(U - N''_0)$, two components B_1 and B_2 of $\operatorname{Bd} N''_0$ are components of $\operatorname{Bd} V$ for some component V of $\operatorname{Cl}(U - N''_0)$. Let β be a polygonal arc in V such that $\beta \cap \operatorname{Bd} V = \operatorname{Bd} \beta \cap (B_1 \cup B_2) = \{y_1, y_2\}$ with $y_i \in B_i$. Let W be a regular neighborhood of β in V. Let $N_0 = N''_0 \cup P$. Since β cannot separate V, it is clear, by finite induction, that we may now assume that N_0 satisfies (1) and (2). Now let M be a component of $\operatorname{Bd} N_0$. Let Q be the component of $U - C$ which contains M. Let λ be a generator of $\pi_1(M)$. By general position let $f: S^1 \to M$ be a PL embedding such that $[f] = \lambda$($[f]$ is the homotopy class of f). Since Q is 1-connected extend f to $\bar{f}: B^2 \to Q$. By Irwin's embedding theorem we assume that $\bar{f}$ is a PL embedding, and putting $\bar{f}$ into general position with respect to M keeping $\bar{f}|S^1$ fixed, we may assume that $\bar{f}(B^2) \cap M$ is a finite number of PL 1-spheres $S_1, S_2, \ldots, S_r$ with $S_r = \bar{f}(S^1)$. Now let S_i be an innermost embedded 1-sphere. It follows that $\bar{f}(\bar{f}^{-1}(S_i) \cup \operatorname{Int} \bar{f}^{-1}(S_i))$ is a PL properly embedded 2-ball B_i^2 in N_0 or in a component of $\mathrm{Cl}\,(U-N_0)$. By taking a regular neighborhood of B_i^2, it follows from Proposition 2.6 that we can replace N_0 by a compact connected PL n-submanifold N_1' such that the component M' of Bd N_1' which replaces M by surgery satisfies the condition

$$\pi_1(M') \approx \pi_1(M)/([f|S_i]).$$

Continuing in this fashion we may assume that

$$\pi_1(M') \approx \pi_1(M)/([f|S_1], \ldots, [f|S_r]).$$

Hence we can kill the generator λ. Since $\pi_1(M)$ is finitely generated, we may assume that M' is 1-connected. By finite induction, we can replace N_1' by a compact connected PL n-submanifold N_1 such that each component of Bd N_1 is 1-connected. Since all alterations occured in $U-C$, N_1 satisfies (1). The dimension of the surgery is low so that N_1 satisfies (2). The fact that $U-N_1$ is 1-connected follows easily by general position or the van Kampen theorem depending upon whether in the inductive stage the new compact submanifold was formed by removing or adding on a regular neighborhood to the old compact submanifold. To verify $(4)_1$ consider the exact Mayer-Vietoris sequence for the triad $(U; N_1, \mathrm{Cl}\,(U-N_1))$.

$$H_1(\mathrm{Bd}\, N_1) \to H_1(N_1) \oplus H_1(\mathrm{Cl}\,(U-N_1)) \to H_1(U).$$

Since each component of Bd N_1 is 1-connected, it follows that $i_*: H_1(N_1) \to H_1(U)$ is an injection. Since $C \subset \operatorname{Int} N_1$, using the hypothesis, it follows easily that i_* is a surjection.

Step 2. *The validity of* $\mathscr{S}_2$.

Proof. Let N_1 satisfy $\mathscr{S}_1$. We will obtain N_2 from N_1. Let $Q_1, Q_2, \ldots, Q_s$ be the components of $\mathrm{Cl}\,(U-N_1)$. Consider the excision map $\varepsilon: (\bigcup Q_t, \bigcup \mathrm{Bd}\, Q_t) \to (U, N_1)$ and the following commutative diagram with exact rows.

$$\begin{tikzcd}
H_{k+1}(U, N_1) \arrow[r, "\partial_k"] & H_k(N_1) \arrow[r, "i_k"] & H_k(U) \arrow[r, "j_k"] & H_k(U, N_1) \\
H_{k+1}(\bigcup Q_t, \bigcup \mathrm{Bd}\, Q_t) \arrow[u, "\varepsilon", "\approx"'] \arrow[r, "\partial_k'"] & H_k(\bigcup \mathrm{Bd}\, Q_t) \arrow[u, "i"'] \arrow[r, "i_k'"] & H_k(\bigcup Q_t) \arrow[u, "i"'] \arrow[r, "j_k'"] & H_k(\bigcup Q_t, \bigcup \mathrm{Bd} Q_t) \arrow[u, "\varepsilon", "\approx"']
\end{tikzcd}$$

i_2 is surjective, so we will alter N_1 to N_2 so that N_2 satisfies $\mathscr{S}_2$ by killing ker i_2. Let $0 \neq x \in \ker i_2$. Choose $y \in H_3(U, N_1)$ such that $\partial_2(y) = x$. Hence $z = \partial_2' \varepsilon^{-1}(y)$ is a nonzero element of $H_2(\bigcup \mathrm{Bd}\, Q_t)$ such that $i(z) = x$. But $H_2(\bigcup \mathrm{Bd}\, Q_t) \approx$

$\sum H_2(\text{Bd } Q_t)$ and hence z can be thought of as $(z_1, \ldots, z_s)$ where $z_t \in H_2(\text{Bd } Q_t)$. Since each Bd Q_t is 1-connected, we represent each z_t by a PL embedding $f_t\colon S^2 \to \text{Bd } Q_t$. But $z \in \text{im } \partial_2' = \ker i_2'$ and hence each $[f_t]$ is homologically trivial in Q_t. Since each Q_t is 1-connected, by the Hurewicz theorem and Irwin's embedding theorem, we may extend each f_t to a PL embedding $\bar{f}_t\colon B^3 \to Q_t$. Take regular neighborhoods P_t of $\bar{f}_t(B^3)$ in Q_t. By the proof of Proposition 2.6, these are PL balls and $P_t \cap \text{Bd } Q_t \approx \bar{f}_t(S^2) \times B^{n-3}$. Let $N_2' = N_1 \cup (\bigcup_{t=1}^s P_t)$. Consider the proper triad $(N_1 \cup P_1; N_1, P_1)$ and the resulting exact Mayer-Vietoris sequence

$$H_2(\bar{f}_1(S^2) \times B^{n-3}) \xrightarrow{\alpha} H_2(N_1) \oplus H_2(P_1) \longrightarrow H_2(N_1 \cup P_1) \longrightarrow 0.$$

It follows that $H_2(N_1)/(\text{im } \alpha) \approx H_2(N_1 \cup P_1)$, i.e.

$$H_2(N_1)/([f_1]) \approx H_2(N_1 \cup P_1).$$

Hence $H_2(N_2') \approx H_2(N_1)/([f_1], \ldots, [f_s]) \approx H_2(N_1)/(x)$. Since $\ker i_2$ is finitely generated, by finite induction, there is a compact PL n-submanifold N_2 of U such that $i\colon N_2 \subset U$ induces an isomorphism $i_*\colon H_2(N_2) \twoheadrightarrow H_2(U)$. It can easily be checked that N_2 satisfies (1), (2), (3), and $(4)_2$.

Step 3. $\mathscr{S}_k$ *implies* $\mathscr{S}_k'$.

Proof. Let N_k satisfy $\mathscr{S}_k$. Since U is 1-connected at infinity, let C' be a compact subset of U such that $C' \supset N_k \cup P$ and each component of $U - C'$ is 1-connected. Let N' be a compact connected PL n-submanifold of U which satisfies $\mathscr{S}_1$ for the compact subset C'. Every component of $\text{Cl}(U - N')$ is a subset of a component of $U - N_k$ and hence by an argument like that given in Step 1, we may assume that, in addition, $\mathscr{o}\mathscr{C}\, \text{Cl}(U - N') = \mathscr{o}\mathscr{C}(U - N_k)$. (Note: It is here that we use that fact that $\dim P \leqq n-2$, for recall in Step 1 we join components by arcs; and here we must be sure that the arcs do not intersect P, for otherwise it may be that $P \not\subset N'$.) Now let N_k' be a PL n-submanifold of U which satisfies $\mathscr{S}_k$ for the compact set N'. It is easy to check that N_k' satisfies $\mathscr{S}_k'$.

Step 4. $\mathscr{S}_k$ *implies* $\mathscr{S}_{k+1}$, $2 \leqq k \leqq n-5$.

Proof. Let N_k satisfy $\mathscr{S}_k$. Roughly, we wish to alter N_k so as to kill i_{k+1*} where $i\colon N_k \subset U$ and $i_{k+1*}\colon H_{k+1}(N_k) \to H_{k+1}(U)$ without disturbing the other nice properties of N_k. Let $0 \neq x \in \ker i_{k+1*}$. Choose $y \in H_{k+2}(U, N_k)$ such that $\partial_{k+2}(y) = x$. Since $\dim(\text{carrier } y) < n-2$, by Step 3, choose N_k' which satisfies $\mathscr{S}_k'$ for $N_k \cup \text{carrier } y$.

Let $Q_1, Q_2, \ldots, Q_s$ and $Q_1', Q_2', \ldots, Q_s'$ be the components of $\text{Cl}(U - N_k)$ and $\text{Cl}(U - N_k')$ respectively. Let $X_i = Q_i \cap \text{Cl}(N_k' - N_k)$. Hence, each X_i is 1-connected, $\text{Bd } X_i = \text{Bd } Q_i \cup \text{Bd } Q_i'$, and Bd Q_i, Bd Q_i' are each 1-connected. Consider the excisions

$$\varepsilon\colon \left(\bigcup Q_i, \bigcup \text{Bd } Q_i\right) \subset (U, N_k), \qquad \varepsilon'\colon \left(\bigcup Q_i, \bigcup X_i\right) \subset (U, N_k')$$

and the following commutative diagram

$$\begin{tikzcd}
H_{k+2}(\bigcup X_i, \bigcup \mathrm{Bd}\, Q_i) \arrow[d, "\partial"'] & & & \\
H_{k+1}(\bigcup \mathrm{Bd}\, Q_i) \arrow[d, "h_*"] \arrow[rrr, bend left, "i_*"] & H_{k+2}(\bigcup Q_i, \bigcup \mathrm{Bd}\, Q_i) \arrow[l, "\bar{\partial}"'] \arrow[r, "\varepsilon_*", "\approx"'] \arrow[d, "k_*"] & H_{k+2}(U, N_k) \arrow[r, "\partial_{k+2}"] \arrow[d, "l_*"] & H_{k+1}(N_k) \arrow[d, "j_{k+1*}"'] \\
H_{k+1}(\bigcup X_i) & H_{k+2}(\bigcup Q_i, \bigcup X_i) \arrow[l, "\bar{\partial}'"'] \arrow[r, "\varepsilon'_*", "\approx"'] & H_{k+2}(U, N'_k) \arrow[r, "\partial'"] & H_{k+1}(N'_k)
\end{tikzcd}$$

where i_*, h_*, k_*, l_*, and j_{k+1*} are all induced by inclusions. Since $N'_k \supset$ carrier y, $l_*(y)=0$. Let $z=\bar{\partial}\varepsilon_*^{-1}(y)$. Hence $i_*(z)=x$ and $h_*(z)=0$. Now choose $w \in H_{k+2}(\bigcup X_i, \bigcup \mathrm{Bd}\, Q_i)$ such that $\partial(w)=z$. But $H_{k+1}(\bigcup \mathrm{Bd}\, Q_i) \approx \sum H_{k+1}(\mathrm{Bd}\, Q_i)$ and $H_{k+2}(\bigcup X_i, \bigcup \mathrm{Bd}\, Q_i) \approx \sum H_{k+2}(X_i, \mathrm{Bd}\, Q_i)$. Hence we may think of z and w as $z=(z_1, z_2, \ldots, z_s)$ where $z_i \in H_{k+1}(\mathrm{Bd}\, Q_i)$, $w=(w_1, w_2, \ldots, w_s)$ where

$$w_i \in H_{k+2}(X_i, \mathrm{Bd}\, Q_i).$$

Since each X_i and $\mathrm{Bd}\, Q_i$ is 1-connected, $\pi_1(X_i, \mathrm{Bd}\, Q_i)=0$. Since $i_*\colon H_r(N_k) \twoheadrightarrow H_r(U)$ and $i_*\colon H_r(N'_k) \twoheadrightarrow H_r(U)$ are isomorphisms for $r \leqq k$, so is $j_*\colon H_r(N_k) \twoheadrightarrow H_r(N'_k)$. Hence $H_r(N'_k, N_k)=0$ for $r \leqq k$. Using excision it follows that $H_r(X_i, \mathrm{Bd}\, Q_i)=0$ for $1 \leqq i \leqq s$ and $r \leqq k$. By the relative Hurewicz theorem, $\pi_r(X_i, \mathrm{Bd}\, Q_i)=0$ for $r \leqq k$; and hence by Lemma 8 of [2], each w_i can be represented by an embedded handle $B_i^{k+2} \times B_i^{n-k-2}$ meeting $\mathrm{Bd}\, X_i$ in $S_i^{k+1} \times B_i^{n-k-2} \subset \mathrm{Bd}\, Q_i$.

Let $N_{k+1}=N_k \cup \bigcup_{i=1}^s (B_i^{k+2} \times B_i^{n-k-2})$. It follows that $x=i_*(z)=i_*(\partial w)$ includes trivially in $H_{k+1}(N_{k+1})$, where we recall that x is an arbitrary generator of ker i_{k+1*}. The desired result will follow by finite induction if we do not change any of the nice properties of N_k. By finite induction, we may assume that $N_{k+1}=N_k \cup B^n$ with $B^n \cap N_k=S^{k+1} \times B^{n-k-2}$. Considering the Mayer-Vietoris sequence for the triad (N_{k+1}, N_k, B^n), one concludes that $N_k \subset N_{k+1}$ induces homology isomorphisms through dimension k, and since $N_k \subset U$ induces homology isomorphisms through dimension k; it follows that $N_{k+1} \subset U$ induces homology isomorphisms through dimension k. Furthermore, it is easy to see that $H_{k+1}(N_{k+1})$ is being reduced so that eventually we will kill ker i_{k+1*}. It now follows easily that N_{k+1} satisfies (1), (2) and $(4)_{k+1}$ of $\mathscr{S}_{k+1}$. We will check (3). Let $\dot{Q}'$ be a component of $\mathrm{Bd}\, N_{k+1}$. It follows that $\dot{Q}'$ is obtained from a component $\dot{Q}$ of $\mathrm{Bd}\, N_k$ by $(k+1)$-dimensional surgery. Hence

$$\dot{Q}' = (\dot{Q}-(S^{k+1} \times \mathrm{Int}\, B^{n-k-2})) \cup (B^{k+2} \times S^{n-k-3}) = C \cup D$$

where $C \cap D=S^{k+1} \times S^{n-k-3}$. But C has the homotopy type of $\dot{Q}-S^{k+1}$ and $k+1 \leqq n-4$. Hence, by general position, since $\dot{Q}$ is 1-connected, so is C. Since $\min\{k+1, n-k-3\} \geqq 2$, by the van Kampen theorem it now follows that $\dot{Q}'$ is

1-connected. Let Q' be a component of $\mathrm{Cl}\,(U-N_{k+1})$. Let Q be the component of $\mathrm{Cl}\,(U-N_k)$ such that $Q'=\mathrm{Cl}\,(Q-(B^{k+2}\times B^{n-k-2}))$. Since Q is 1-connected and Q' has the homotopy type of $Q-B^{k+2}$, by general position Q' is 1-connected.

Step 5. *$\mathscr{S}_{n-4}$ implies $\mathscr{S}_{n-3}$.*

Proof. Let N satisfy $\mathscr{S}_{n-4}$ and let $V=\mathrm{Cl}\,(U-N)$. Consider the excision map $\varepsilon:(V, \mathrm{Bd}\, N)\subset(U, N)$ and the following commutative diagram

$$\begin{tikzcd}
H_r(N) \arrow[r, "i_r"] & H_r(U) \arrow[r, "j_r"] & H_r(U, N) \arrow[r, "\partial_r"] & H_{r-1}(N) \arrow[r, "i_{r-1}"] & H_{r-1}(U) \\
H_r(\mathrm{Bd}\, N) \arrow[r, "i'_r"] \arrow[u, "k_r"] & H_r(V) \arrow[r, "j'_r"] \arrow[u, "l_r"] & H_r(V, \mathrm{Bd}\, N) \arrow[r, "\partial'_r"] \arrow[u, "\approx"', "\varepsilon_r"] & H_{r-1}(\mathrm{Bd}\, N) \arrow[r, "i'_{r-1}"] \arrow[u, "k_{r-1}"] & H_{r-1}(V). \arrow[u, "l_{r-1}"']
\end{tikzcd}$$

For $r\leqq n-2$, i_r is onto, and hence $j_r=0$ and ∂_r is injective. For $r-1\leqq n-4$, i_{r-1} is an isomorphism and hence $\partial_r=0$ for $r\leqq n-3$. Hence $H_r(U, N)=0=H_r(V, \mathrm{Bd}\, N)$ for $r\leqq n-3$. Since each component of $\mathrm{Bd}\, N$ is 1-connected, by duality and the Universal Coefficient theorem, it follows that $H_{n-3}(\mathrm{Bd}\, N)$ is free. But $j_{n-2}=0$ implies that $j'_{n-2}=0$, and hence ∂'_{n-2} is injective. So $H_{n-2}(V, \mathrm{Bd}\, N)$ and $H_{n-2}(U, N)$ are both free. Now consider the following portion of the above diagram

$$\begin{tikzcd}
{} \arrow[r, "0"] & H_{n-2}(U, N) \arrow[r, "\partial"] & H_{n-3}(N) \arrow[r, "i_*"] & H_{n-3}(U) \arrow[r] & 0 \\
{} \arrow[r, "0"] & H_{n-2}(V, \mathrm{Bd}\, N) \arrow[r] \arrow[u] & H_{n-3}(\mathrm{Bd}\, N) \arrow[r] \arrow[u] & H_{n-3}(V) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

We wish to kill $\ker i_*$. Since N is compact, let $\lambda_1, \lambda_2, \dots, \lambda_s$ be a set of generators of $\ker i_*$. Choose $(n-2)$-chains $c_1, \dots, c_s$ in U such that $\partial(c_i)=\lambda_i$. Let $P=\bigcup$ carrier c_i. By Step 3, let N' satisfy the conclusion of $\mathscr{S}'_{n-4}$ relative to N and P. Since N' satisfies the same homology conditions as that of N, the conclusions we made above for N and V are valid for N' and $V'=\mathrm{Cl}\,(U-N')$. Now consider the commutative diagram with exact rows

$$\begin{tikzcd}
{} \arrow[r, "0"] & H_{n-2}(U, N) \arrow[r, "\partial"] \arrow[d, "h'_*"] & H_{n-3}(N) \arrow[r, "i_*"] \arrow[d, "h_*"] & H_{n-3}(U) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
{} \arrow[r, "0"] & H_{n-2}(U, N') \arrow[r, "\partial'"] & H_{n-3}(N') \arrow[r, "i'_*"] & H_{n-3}(U) \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{Int}\, N'\supset\bigcup$ carrier λ_i, $h_*(\ker i_*)=0$. Hence $h'_*=0$. Let $X=\mathrm{Cl}\,(N'-N)$. Since $(V, \mathrm{Bd}\, N)\subset(U, N)$ and $(V, X)\subset(U, N')$ are excisions; it follows that $\bar{h}:(V, \mathrm{Bd}\, N)\subset(V, X)$ induces the 0 map $\bar{h}_*: H_{n-2}(V, \mathrm{Bd}\, N)\to H_{n-2}(V, X)$. Also $H_r(V, X)=0$ for $r\leqq n-3$, and $H_{n-2}(V, X)$ is free. Now since $H_{n-2}(V, X)$ and $H_{n-2}(V, \mathrm{Bd}\, N)$ are both free, using the Universal Coefficient theorem, it follows that $\bar{h}^*=0$. Consider the following commutative diagram with exact rows where all vertical homomorphisms are induced by inclusions

$$\begin{tikzcd}
 & H^{n-2}(V, \mathrm{Bd}\, N) & H^{n-3}(\mathrm{Bd}\, N) \arrow[l] & H^{n-3}(V) \arrow[l] & 0 \arrow[l] \\
0 & H^{n-2}(V, X) \arrow[l] \arrow[u, "0"', "\bar{h}^*"] \arrow[d, "\varepsilon'"] \arrow[r, dashed, shift right, "\alpha"'] & H^{n-3}(X) \arrow[l, shift right, "\delta"'] \arrow[u, "\nu"] \arrow[d, "\nu'"] & H^{n-3}(V) \arrow[l] \arrow[u] \arrow[d] & 0 \arrow[l] \\
 & H^{n-2}(V', \mathrm{Bd}\, N') \arrow[r, dashed, shift right, "\alpha'"'] & H^{n-3}(\mathrm{Bd}\, N') \arrow[l, shift right, "\delta'"'] & H^{n-3}(V') \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

The zeros on the right are valid by the Universal Coefficient theorem since $H_r(V, \mathrm{Bd}\, N)=H_r(V, X)=H_r(V', \mathrm{Bd}\, N')=0$ for $r\leqq n-3$. To show that δ is onto, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\,(H_{n-2}(V, X), \boldsymbol{Z}) & \mathrm{Hom}\,(H_{n-3}(X), \boldsymbol{Z}) \arrow[l, "\mathrm{Hom}\,(\partial, 1)"'] \\
H^{n-2}(V, X) \arrow[u, "\rho"] & H^{n-3}(X) \arrow[l, "\delta"'] \arrow[u, "\rho'"]
\end{tikzcd}$$

where ρ and ρ' come from the Universal Coefficient theorem. It is not difficult to show that $0\to H_{n-2}(V, X)\xrightarrow{\partial} H_{n-3}(X)\to H_{n-3}(V)\to 0$ is split exact and hence $\mathrm{Hom}\,(\partial, 1)$ is onto. But ρ' is onto and hence $\rho\delta$ is onto. But $H^{n-2}(V, X)$ is free, and hence ρ is an isomorphism, and δ is onto. Using the facts that $H^{n-2}(V, X)$ is free and $\bar{h}^*=0$, it is not difficult to define $\alpha: H^{n-2}(V, X)\to H^{n-3}(X)$ such that $\delta\alpha=1$ and $\nu\alpha=0$. Now define $\alpha': H^{n-2}(V', \mathrm{Bd}\, N')\to H^{n-3}(\mathrm{Bd}\, N')$ by $\alpha'=\nu'\alpha(\varepsilon')^{-1}$. (Note that ε' is induced by an excision, so that α' is well defined.) One easily checks that $\delta'\alpha'=1$ and $\mathrm{im}\,\alpha'\subset\nu'(\ker\nu)$. It follows that $\mathrm{im}\,\alpha'$ is a free direct summand of $H^{n-3}(\mathrm{Bd}\, N')$. Now let $V_1,\ldots, V_t$ and $V_1',\ldots, V_t'$ be the components of V and V' respectively such that $V_i'\subset V_i$. (Recall, $V=\mathrm{Cl}\,(U-N)$, $V'=\mathrm{Cl}\,(U-N')$, and N' satisfies $\mathscr{S}_{n-4}'$ with respect to N.) Let $X_i=\mathrm{Cl}\,(V_i-V_i')$. Hence $V=\bigcup_{\mathrm{disjt}} V_i$, $V'=\bigcup_{\mathrm{disjt}} V_i'$, $\mathrm{Bd}\, N=\mathrm{Bd}\, V=\bigcup_{\mathrm{disjt}} \mathrm{Bd}\, V_i$, $\mathrm{Bd}\, N'=\mathrm{Bd}\, V'=\bigcup_{\mathrm{disjt}} \mathrm{Bd}\, V_i'$, $X=\bigcup_{\mathrm{disjt}} X_i$, $\mathrm{Bd}\, X_i=\mathrm{Bd}\, V_i \bigcup_{\mathrm{disjt}} \mathrm{Bd}\, V_i'$. Let $\nu_i:\mathrm{Bd}\, V_i\subset X_i$ and $\nu_i':\mathrm{Bd}\, V_i'\subset X_i$. It follows that, for $1\leqq i\leqq t$, there is a commutative diagram

$$\begin{tikzcd}
H^{n-2}(V_i, \mathrm{Bd}\, V_i) & H^{n-3}(\mathrm{Bd}\, V_i) \arrow[l] & H^{n-3}(V_i) \arrow[l] \\
H^{n-2}(V_i, X_i) \arrow[u, "0="', "\bar{h}_i^*"] \arrow[d] \arrow[r, shift right, "\alpha_i"'] & H^{n-3}(X_i) \arrow[l, shift right, "\delta_i"'] \arrow[u, "\nu_i^*"] \arrow[d, "\nu_i'^*"] & H^{n-3}(V_i) \arrow[l] \arrow[u] \arrow[d] \\
H^{n-2}(V_i', \mathrm{Bd}\, V_i') \arrow[r, shift right, "\alpha_i'"'] & H^{n-3}(\mathrm{Bd}\, V_i') \arrow[l, shift right, "\delta_i'"'] & H^{n-3}(V_i') \arrow[l]
\end{tikzcd}$$

such that $\delta_i\alpha_i=1$, $\nu_i^*\alpha_i=0$, $\delta_i'\alpha_i'=1$, $\operatorname{im}\alpha_i'\subset\nu_i'^*(\ker\nu_i^*)$, and $\operatorname{im}\alpha_i'$ is a free direct summand of $H^{n-3}(\mathrm{Bd}\,V_i')$. By Lemma 11 of [2], the duality isomorphism

$$\psi_i\colon H^{n-3}(\mathrm{Bd}\,V_i')\twoheadrightarrow H_2(\mathrm{Bd}\,V_i')$$

sends $\nu_i'^*(\ker\nu_i^*)$ onto $(\ker\nu_{i*}')_2$. Hence $A_i=\psi_i(\operatorname{im}\alpha_i')$ is a free direct summand of $H_2(\mathrm{Bd}\,V_i')$. Let $\lambda_{1i},\lambda_{2i},\dots,\lambda_{k_ii}$ be a basis for $A_i\subset H_2(\mathrm{Bd}\,V_i')$. Since each $\mathrm{Bd}\,V_i'$ is 1-connected and $\dim\mathrm{Bd}\,V_i'>4$, it follows that each λ_{ji} can be represented by a PL embedded 2-sphere S_{ji} such that $S_{ji}\cap S_{kl}=\varnothing$ unless $j=k$ and $i=l$. Since $A_i\subset(\ker\nu_{i*}')$ we can, using Irwin's embedding theorem, bound each S_{ji} with a PL 3-ball B_{ji} with $\operatorname{Int}B_{ji}\subset\operatorname{Int}X_i$ such that $B_{ji}\cap B_{kl}=\varnothing$ unless $j=k$ and $i=l$. For each B_{ji}, take a regular neighborhood M_{ji} such that $M_{ji}\cap M_{kl}=\varnothing$ unless $j=k$ and $i=l$.

Let $N_{n-3}=\mathrm{Cl}\,(N'-\bigcup M_{ji})$. We want to show that N_{n-3} satisfies $\mathscr{S}_{n-3}$. Clearly (1) is satisfied since $N_{n-3}\supset N$ and $\operatorname{Int}N\supset C$. One easily checks that the deletion of the handles and the surgery is such that, $\sigma\mathscr{C}\,\mathrm{Cl}\,(U-N_{n-3})=\sigma\mathscr{C}\,\mathrm{Cl}\,(U-N')$ and $\sigma\mathscr{C}\,\mathrm{Bd}\,N_{n-3}=\sigma\mathscr{C}\,\mathrm{Bd}\,N'$, and so (2) follows.

By finite induction, to show that each component of $\mathrm{Bd}\,N_{n-3}$ is 1-connected, it suffices to show that $\mathrm{Bd}\,(V_i'\cup M_{ji})$ is 1-connected. But $\mathrm{Bd}\,V_i'$ is 1-connected and $\mathrm{Bd}\,(V_i'\cup M_{ji})$ is obtained from $\mathrm{Bd}\,V_i'$ by surgery of index 2. So, using general position and the van Kampen theorem, it follows that $\mathrm{Bd}\,(V_i'\cup M_{ji})$ is 1-connected. Each component of $U-N_{n-3}$ is 1-connected, for by finite induction we may assume that such a component is $V_i'\cup M_{ji}$ with $M_{ji}\approx B^3\times B^{n-3}$ and $V_i'\cap M_{ji}\approx S^2\times B^{n-3}$, and since V_i' is 1-connected, so is $V_i'\cup M_{ji}$ by the van Kampen theorem.

It remains to show (4); that $i\colon N_{n-3}\subset U$ induces isomorphisms $i_*\colon H_r(N_{n-3})\twoheadrightarrow H_r(U)$ for $r\leqq n-3$.

Let $\bar V_i=V_i'\cup\bigcup_{j=1}^{k_i}M_{ji}$, $i=1,\dots,t$ be the components of $\mathrm{Cl}\,(U-N_{n-3})$. To show (4), it suffices to show that $H_r(\bar V_i,\mathrm{Bd}\,\bar V_i)=0$ for $r\leqq n-2$ and $i=1,\dots,t$; for since $\bar V_i\cap\bar V_j=\varnothing$ if $i\neq j$, this will imply that $H_r(\bigcup\bar V_i,\bigcup\mathrm{Bd}\,\bar V_i)=0$. Hence by excision $H_r(U,N_{n-3})=0$ for $r\leqq n-2$, which implies (4). By finite induction we may assume that $\bar V_i$ is obtained from V_i' by adjoining a single M_{ji}. Let $W_j=\mathrm{Bd}\,V_j'-(S^2\times\operatorname{Int}B^{n-3})$ and consider the following commutative diagram where all maps are inclusions

$$\begin{tikzcd} & \mathrm{Bd}\,V_j' \arrow[r, "i_2"] & V_j' \arrow[dd, "i_3"] \\ W_j \arrow[ur, "i_1"] \arrow[dr, "\bar{i}_1"'] & & \\ & \mathrm{Bd}\,\bar{V}_j \arrow[r, "\bar{i}_2"] & \bar{V}_j . \end{tikzcd}$$

It now suffices to show the following:

(a) $\bar{i}_{2*}\colon H_r(\mathrm{Bd}\,\bar V_j)\to H_r(\bar V_j)$ is an isomorphism for $r\neq 2$ but $r\leqq n-4$.

(b) $\bar{i}_{2*}\colon H_2(\mathrm{Bd}\ \bar{V}_j) \to H_2(\bar{V}_j)$ is an isomorphism.

(c) $H_r(\bar{V}_j, \mathrm{Bd}\ \bar{V}_j)=0$ for $r=n-3$, $n-2$.

To verify (a) it suffices to show that i_1, $\bar{i}_1$, i_2, i_3 induce homology isomorphisms in dimension k where $k\leqq n-4$ but $k\neq 2$. We will only check that i_1 induces isomorphisms; the arguments for the others being similar or easy. Let $T=S^2\times B^{n-3}$ and consider the reduced Mayer-Vietoris sequence for the triad $(\mathrm{Bd}\ V_j'; W_j, T)$.

$$\begin{aligned}\tilde{H}_r(S^2\times S^{n-4}) \longrightarrow \tilde{H}_r(W_j) \oplus \tilde{H}_r(S^2\times B^{n-3})\\ \xrightarrow{\phi'} \tilde{H}_r(\mathrm{Bd}\ V_j') \xrightarrow{\phi} \tilde{H}_{r-1}(S^2\times S^{n-4}).\end{aligned}$$

If $r\neq 2$ but $r<n-4$, this sequence has the appearance $0 \to H_2(W_j) \to H_r(\mathrm{Bd}\ V_j') \xrightarrow{0}$ for we know that in this range ϕ is injective, since $S^2\times 0$ represents a free generator of $H_2(\mathrm{Bd}\ V_j')$, and hence $\phi'=0$. Hence, in this range i_{1*} is an isomorphism. Now if $3\leqq r=n-4$ the sequence appears as

$$Z \xrightarrow{\alpha} H_{n-4}(W_j) \xrightarrow{i_{1*}} H_{n-4}(\mathrm{Bd}\ V_j') \xrightarrow{\phi'} H_{n-5}(S^2\times S^{n-4}) \xrightarrow{\phi}.$$

As above $\phi'=0$. Hence i_{1*} will be an isomorphism if $\alpha=0$. Notice that $\mathrm{Bd}\ W_j=S^2\times S^{n-4}$ and consider the following commutative (up to sign) diagram where the isomorphisms are given by duality.

$$\begin{tikzcd} H^2(W_j) \arrow[r, "i^*"'] \arrow[d, "\approx"'] & H^2(\mathrm{Bd}\ W_j) \arrow[d, "\approx"] & \\ H_{n-3}(W_j, \mathrm{Bd}\ W_j) \arrow[r] & H_{n-4}(\mathrm{Bd}\ W_j) \arrow[r, "\alpha"] & H_{n-4}(W_j). \end{tikzcd}$$

By exactness of the lower row, it now suffices to show that i^* is surjective. But $i_*\colon H_2(\mathrm{Bd}\ W_j) \to H_2(W_j)$ carries the generator of $H_2(\mathrm{Bd}\ W_j)$ onto a free element of $H_2(W_j)$, since $S^2\times 0$ represents a primitive free element of a basis for $H_2(\mathrm{Bd}\ V_j')$. Hence there is a split short exact sequence of the form

$$0 \longrightarrow H_2(\mathrm{Bd}\ W_j) \xrightarrow{i_*} H_2(W_j) \longrightarrow G \longrightarrow 0.$$

Now consider the commutative diagram

$$\begin{tikzcd} H^2(W_j) \arrow[r, "\beta"] \arrow[d, "i^*"] & \mathrm{Hom}\,(H_2(W_j), \boldsymbol{Z}) \arrow[d, "\mathrm{Hom}\,(i_*, 1)"] \\ H^2(\mathrm{Bd}\ W_j) \arrow[r, "\beta'"] & \mathrm{Hom}\,(H_2(\mathrm{Bd}\ W_j), \boldsymbol{Z}) \end{tikzcd}$$

where β and β' come from the Universal Coefficient theorem. It follows that β and $\mathrm{Hom}\,(i_*, 1)$ are surjective. But β' is an isomorphism since $H^2(\mathrm{Bd}\ W_j)$ is free, and hence i^* is surjective.

(b) follows, for as given in [2],

$$\begin{aligned} H_2(\mathrm{Bd}\ \overline{V}_j) &\approx H_2(\mathrm{Bd}\ V_j')/A_j \\ &\approx H_2(V_j')/\mathrm{inc}_*(A_j) \\ &\approx H_2(\overline{V}_j). \end{aligned}$$

To verify (c), one first easily shows that $\bar{\imath}_{2*}$ is onto for dimensions less than $n-1$. Hence, using (a) and (b) and arguing as we did early in the proof of this step for $(V, \mathrm{Bd}\ N)=(V, \mathrm{Bd}\ V)$, it follows that $H_r(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)=0$ for $r\leqq n-3$ and

$$H_{n-2}(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)$$

is free. From this information, it follows, as it did for $(V_j', \mathrm{Bd}\ V_j')$, that the following is an exact sequence.

$$0\leftarrow H^{n-2}(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)\leftarrow H^{n-3}(\mathrm{Bd}\ \overline{V}_j)\leftarrow H^{n-3}(\overline{V}_j)\leftarrow 0.$$

By Poincaré duality, $H^{n-3}(\mathrm{Bd}\ \overline{V}_j)\approx H^{n-3}(\mathrm{Bd}\ V_j')/A_j'$ where

$$A_j' = \alpha_j'(H^{n-2}(V_j', \mathrm{Bd}\ V_j')).$$

But since α_j' (α_j' of the diagram on page 381) is a splitting homomorphism,

$$H^{n-3}(\mathrm{Bd}\ \overline{V}_j) \approx H^{n-3}(V_j').$$

Now

$$H^{n-3}(V_j') \approx \mathrm{Hom}\,(H_{n-3}(V_j'), \boldsymbol{Z}) \oplus \mathrm{Ext}\,(H_{n-4}(V_j'), \boldsymbol{Z})$$

and

$$H^{n-3}(\overline{V}_j) \approx \mathrm{Hom}\,(H_{n-3}(\overline{V}_j), \boldsymbol{Z}) \oplus \mathrm{Ext}\,(H_{n-4}(\overline{V}_j), \boldsymbol{Z}).$$

But $H_r(V_j')\approx H_r(\overline{V}_j)$ for $r\neq 2$, and since $\mathrm{inc}_*(A_j)$ is a free direct summand of $H_2(V_j')$, from the argument given in (b) above, $\mathrm{Ext}\ H_2(V_j')\approx \mathrm{Ext}\ H_2(\overline{V}_j)$. It follows that $H^{n-3}(\mathrm{Bd}\ \overline{V}_j)\approx H^{n-3}(\overline{V}_j)$. Since these groups are finitely generated and $H^{n-2}(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)$ is free, the above exact sequence tells us that $H^{n-2}(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)=0$. Now by the Universal Coefficient theorem, $H_{n-2}(\overline{V}_j, \mathrm{Bd}\ \overline{V}_j)=0$. This completes the proof of Proposition 3.1.

PROPOSITION 3.2. *If C is a compact subset of the open connected* PL *n-manifold U ($n>5$) such that each component of $U-C$ is q-connected, then there is a compact connected* PL *n-submanifold N of U such that*

(1) $C\subset \mathrm{Int}\ N$,

(2) $\sigma\mathscr{C}\ \mathrm{Bd}\ N=\sigma\mathscr{C}(U-N)=\sigma\mathscr{C}_u(U-C)$,

(3) *each component of* Bd N *is* $\min\{q, [(n-1)/2]-1\}$*-connected, and*

(4) *each component of $U-N$ is* $\min\{q, [n/2]-1\}$*-connected.*

Proof. For $0\leqq i\leqq \min\{q, [n/2]-1\}$ let $\mathscr{H}_i$ be the statement that there is a compact PL n-submanifold N_i of U such that (1), (2), (3), and (4) hold when i and N_i replace q and N respectively. If $q=0$, the techniques used in the proof of $\mathscr{S}_1$ of Proposition

3.1 will prove Proposition 3.2; and if $q \geqq 1$, $\mathscr{H}_1$ is valid by the proof of $\mathscr{S}_1$. Now suppose that $\mathscr{H}_i$ is valid where $1 \leqq i < \min\{q, [n/2]-1\}$. Let $M_1, M_2, \ldots, M_t$ be the components of Bd N_i, and let $Q_1, Q_2, \ldots, Q_t$ be the components of $U-C$ such that $Q_j \supset M_j$. For each j, consider the triad $(Q_j; M_j^+, \mathrm{Cl}\,(Q_j - M_j^+))$ where M_j^+ is the component of $\mathrm{Cl}\,(U-N_i)$ which contains M_j. Now consider the following portion of the resulting Mayer-Vietoris sequence

$$H_{i+1}(M_j) \xrightarrow{\alpha} H_{i+1}(M_j^+) \oplus H_{i+1}(\mathrm{Cl}\,(Q_j - M_j^+)) \longrightarrow H_{i+1}(Q_j).$$

Since Q_j is $(i+1)$-connected, α is onto; and hence it follows that $H_{i+1}(M_j^+)$ is finitely generated. Let λ be a generator of $H_{i+1}(M_j^+)$. Choose $\mu \in H_{i+1}(M_j)$ such that $\alpha(\mu) = (\lambda, 0)$. Since M_j is i-connected, let $f\colon S^{i+1} \to M_j$ be such that, via the Hurewicz isomorphism, $[f] = \mu$. By a general position argument, we may assume that f is a PL embedding. Now $(\lambda, 0) = \alpha(\mu) = \alpha([f]) = ([f], -[f])$ and hence $[f]$ is trivial in $H_{i+1}(\mathrm{Cl}\,(Q - M_j^+))$. But, since M_j and Q_j are both i-connected, so is $\mathrm{Cl}\,(Q_j - M_j^+)$, and hence f is homotopically trivial in $\mathrm{Cl}\,(Q_j - M_j^+)$. Hence, we extend f to $\bar{f}\colon B^{i+2} \to \mathrm{Cl}\,(Q_j - M_j^+)$. One easily verifies the hypothesis of Irwin's embedding theorem [23], and hence we may assume that $\bar{f}$ is a proper PL embedding. Let P be a regular neighborhood of $\bar{f}(B^{i+2})$ in $\mathrm{Cl}\,(Q_j - M_j^+)$, and let $N'_{i+1} = \mathrm{Cl}\,(N_i - P)$. N'_{i+1} will be our first approximation to N_{i+1}. Since $P \subset Q_j \subset U - C$ it is clear that N'_{i+1} satisfies (1). It is easy to see that neither the deletion of handles, nor surgery in this dimension range disconnects N_i or any M_j, and hence N'_{i+1} satisfies (2).

Now let us see what happened to the connectivity conditions. Consider the triad $(M_j^+ \cup P; M_j^+, P)$. By Proposition 2.6 (3), $P \cap M_j^+ \approx S^{i+1} \times B^{n-i-2}$. Now consider the reduced Mayer-Vietoris sequence

$$\begin{aligned}\tilde{H}_k(S^{i+1} \times B^{n-i-2}) \xrightarrow{\alpha_k} \tilde{H}_k(M_j^+) \oplus \tilde{H}_k(P) \xrightarrow{\beta_k} \tilde{H}_k(M_j^+ \cup P) \\ \longrightarrow \tilde{H}_{k-1}(S^{i+1} \times B^{n-i-2}).\end{aligned}$$

Note that $M_j^+ \cup P \in \mathscr{C}\,\mathrm{Cl}\,(U - N'_{i+1})$ replaces $M_j^+ \in \mathscr{C}\,\mathrm{Cl}\,(U-N_i)$, and what we wish to show is that $M_j^+ \cup P$ is as nicely connected as M_j^+, and that in addition we have killed the generator λ of $H_{i+1}(M_j^+)$. It is easy to show that $\tilde{H}_k(M_j^+ \cup P)$ is flanked by zeros in the above sequence if $k \leqq i$, and using the van Kampen and Hurewicz theorems, it does follow that $M_j^+ \cup P$ is i-connected. For $k = i+1$, $H_{k-1}(S^{i+1} \times B^{n-i-2}) = 0$, and hence β_{i+1} is surjective. It follows that

$$H_{i+1}(M_j^+)/\mathrm{im}\,\alpha_{i+1} \approx H_{i+1}(M_j^+ \cup P).$$

But $[\bar{f}|S^{i+1}]$ generates $H_{i+1}(S^{i+1} \times B^{n-i-2})$ and $\alpha_{i+1}[\bar{f}|S^{i+1}] = [f]_{H_{i+1}(M_j^+)} = \lambda$. Hence $H_{i+1}(M_j^+ \cup P) \approx H_{i+1}(M_j^+)/(\lambda)$, and we have reduced $H_{i+1}(M_j^+)$ by one generator. Hence we will be able to alter N_i inductively to N_{i+1} which satisfies (4) of $\mathscr{H}_{i+1}$, if we can show that Bd $(M_j^+ \cup P)$ is $\min\{i, [(n-1)/2]-1\}$-connected. By Proposition 2.6, we obtained Bd $(M_j^+ \cup P)$ from Bd $M_j^+ = M_j$ by surgery of index $i+1$. Let $M_j'=\text{Bd}\,(M_j^+ \cup P)$, $T_1 \approx S^{i+1} \times \text{Int}\, B^{n-i-2}$ and $T_2 \approx B^{i+2} \times S^{n-i-3}$, so that $M_j'=(M_j-T_1) \cup T_2$ with $(M_j-T_1) \cap T_2 \approx S^{i+1} \times S^{n-i-3}$. Consider the triad $(M_j'; M_j-T_1, T_2)$ and the resulting reduced Mayer-Vietoris sequence,

$$\tilde{H}_k(M_j-T_1) \oplus \tilde{H}_k(T_2) \to \tilde{H}_k(M_j') \to \tilde{H}_{k-1}((M_j-T_1) \cap T_2).$$

For $k \leqq \min\{i, [(n-1)/2]-1\}$; $n-i-3>k$ and hence

$$\tilde{H}_k(T_2) = 0 = \tilde{H}_{k-1}((M_j-T_1) \cap T_2).$$

M_j-T_1 has the homotopy type of M_j-S^{i+1} and since M_j is $\min\{i, [(n-1)/2]-1\}$-connected, general position arguments give that M_j-T_1 is also

$$\min\{i, [(n-1)/2]-1\}\text{-connected.}$$

It follows that M_j' is $\min\{i, [(n-1)/2]-1\}$-connected.

It remains to show that (3) can be satisfied on the $i+1$ level. If $i+1 \geqq [(n-1)/2]$, we are done; for any $M_j' \in \mathscr{C}\,\text{Bd}\, N_{i+1}'$ is $\min\{i, [(n-1)/2]-1\}$-connected. Now suppose $i+1<[(n-1)/2]$. Let λ be a generator of $\pi_{i+1}(M_j')$. Since we are in the trivial range, there is a PL embedding $f\colon S^{i+1} \to M_j'$ such that $[f]=\lambda$. Since $i+1 \leqq [n/2]-1$ and $M_j^+ \cup P$ is $\min\{i+1, [n/2]-1\}$-connected, we can extend f to a proper PL embedding $\bar{f}\colon B^{i+2} \to M_j^+ \cup P$. Let P' be a regular neighborhood of $\bar{f}(B^{i+2})$ in $M_j^+ \cup P$ and let $N_{i+1}=N_{i+1}' \cup P'$. By Proposition 2.6, we have obtained $M_j''=\text{Bd Cl}\,((M_j^+ \cup P)-P') \in \mathscr{C}\,\text{Bd}\, N_{i+1}$ from $M_j' \in \mathscr{C}\,\text{Bd}\, N_{i+1}'$ by surgery of index $i+1$. But, we are in the trivial range for surgery, and by Proposition 2.6, $\pi_{i+1}(M_j'') \approx \pi_{i+1}(M_j')/(\lambda)$. Hence the proposition will follow by induction, if we can show that $(M^+ \cup P)-P' \in \mathscr{C}(U-N_{i+1})$ which replaces $M^+ \cup P \in \mathscr{C}(U-N_{i+1}')$ is $\min\{i+1, [n/2]-1\}$-connected, so that we will not destroy (4) at the $i+1$ level. But by Proposition 2.6, $P' \approx \bar{f}(B^{i+2}) \times B^{n-i-2}$ and hence $(M_j^+ \cup P)-P'$ is of the homotopy type of $(M_j^+ \cup P)-\bar{f}(B^{i+2})$. Now using the connectivity of $M_j^+ \cup P$, general position, and the fact that $i+1<[(n-1)/2]$; the desired result follows easily.

Remark. Proposition 3.2 is an important step in proving the PL Hauptvermutung for the open PL manifolds that we consider, for it enables us to take a connectivity (at infinity) condition, which is not a priori related to any PL structure, and relate it to a given PL structure.

4. A connectivity characterization of S^n $(n>5)$ minus a nonempty tame compact 0-dimensional subset.

Lemma 4.1. *Let U be an open (p, q)-connected* PL *n-manifold $(n>5)$ with $p \geqq [n/2]$, $q \geqq 1$ and $p+q \geqq n-2$. Then U is $(n-2)$-connected.*

Proof. Using Proposition 3.2, it follows that U is the monotone union of compact PL n-submanifolds N_i such that

(1) Each component of Bd N_i is 1-connected.
(2) Each component of $U-N_i$ is $\min\{q, [n/2]-1\}$-connected for each i.
(3) $\sigma\mathscr{C}\,\text{Bd}\, N_i=\sigma\mathscr{C}(U-N_i)$.
(4) N_i is 1-connected for each i.

(1), (2), and (3) are immediate. To see (4) consider the decomposition $U=N_i \cup Q_{1i} \cup \cdots \cup Q_{r_i i}$ where the Q_{ji}'s are the closures of the components of $U-N_i$. Since U and each $Q_{ji} \cap N_i$ (a component of Bd N_i) are 1-connected, an application of the van Kampen theorem, a finite number of times, tells us that N_i is 1-connected. Let $Q_i=\mathrm{Cl}\,(U-N_i)$. Now consider the exact reduced homology sequence for the pair (U, Q_i)

$$\tilde{H}_k(U) \to H_k(U, Q_i) \to \tilde{H}_{k-1}(Q_i) \to \tilde{H}_{k-1}(U).$$

Since U is p-connected and each component of Q_i is $\min\{q, [n/2]-1\}$-connected, it follows, using excision, that $H_k(N_i, \mathrm{Bd}\, N_i)=0$ if $2 \leqq k \leqq \min\{q+1, [n/2]\}$ and $H_1(N_i, \mathrm{Bd}\, N_i)$ is free. By duality and the Universal Coefficient theorem,

$$\begin{aligned} H_r(N_i) &\approx H^{n-r}(N_i, \mathrm{Bd}\, N_i) \\ &\approx \mathrm{Hom}\,(H_{n-r}(N_i, \mathrm{Bd}\, N_i), \boldsymbol{Z}) \oplus \mathrm{Ext}\,(H_{n-r-1}(N_i, \mathrm{Bd}\, N_i), \boldsymbol{Z}). \end{aligned}$$

But $p+1 \leqq r \leqq n-2$ implies that $2 \leqq n-r \leqq \min\{q+1, [n/2]\}$, and hence $H_r(N_i)=0$ for $p+1 \leqq r \leqq n-2$. Since U is the monotone union of the N_i's, $H_r(U)=0$ for $p+1 \leqq r \leqq n-2$; and since U is p-connected, U is $(n-2)$-connected.

PROPOSITION 4.2. *Let U be an open (p, q)-connected* PL *n-manifold $(n>5)$ with $p \geqq [n/2]$, $q \geqq 1$ and $p+q \geqq n-2$. Let C be a compact subset of U such that each component of $U-C$ is* 1*-connected. Then there is a cored n-ball CB_k^n such that $C \subset \mathrm{Int}\, CB_k^n$ and $k=\sigma\mathscr{C}_u(U-C)=\sigma\mathscr{C}(U-CB_k^n)$.*

Proof. By Lemma 4.1, U is $(n-2)$-connected. Hence $i: C \subset U$ induces a surjective homomorphism $i_*: H_r(C) \twoheadrightarrow H_r(U)$ for $r \leqq n-2$. Now let N be a compact connected PL n-submanifold of U which satisfies Proposition 3.1 for C. We will show that N is the cored n-ball we are seeking. Note that $C \subset \mathrm{Int}\, N$ and $\sigma\mathscr{C}\, \mathrm{Bd}\, N = \sigma\mathscr{C}_u(U-C)=\sigma\mathscr{C}(U-N)$ by Proposition 3.1. Also, $\tilde{H}_r(N)=0$ for $r \leqq n-3$. Hence N will be $(n-3)$-connected, if N is 1-connected. Let $Q_1, Q_2, \ldots, Q_t$ be the components of $\mathrm{Cl}\,(U-N)$. Hence $U=N \cup Q_1 \cup Q_2 \cup \cdots \cup Q_t$ with each $N \cap Q_i$ being a component of Bd N and hence, 1-connected. Since U is 1-connected, as we remarked in the proof of Lemma 4.1, N is 1-connected. The proposition will follow from the following.

CLAIM. *A compact $(n-3)$-connected* PL *n-manifold N $(n>5)$ with nonempty boundary such that each component of* Bd *N is* 1*-connected, is a cored n-ball.*

Proof. By Lefschetz duality, $H_{n-r}(N, \mathrm{Bd}\, N) \approx H^r(N)$ and since N is $(n-3)$-connected, $H^r(N)=0$ for $1 \leqq r \leqq n-3$. Hence, $H_k(N, \mathrm{Bd}\, N)=0$ for $3 \leqq k \leqq n-1$. Now consider the exact homology sequence for the pair $(N, \mathrm{Bd}\, N)$,

$$H_k(N) \to H_k(N, \mathrm{Bd}\, N) \to H_{k-1}(\mathrm{Bd}\, N) \to H_{k-1}(N).$$

Since N is $(n-3)$-connected, $H_k(N, \mathrm{Bd}\, N) \approx H_{k-1}(\mathrm{Bd}\, N)$ for $2 \leqq k \leqq n-3$; and hence $H_r(\mathrm{Bd}\, N)=0$ for $2 \leqq r \leqq n-4$. Since each component of Bd N is 1-connected, it now follows that each component of Bd N is $(n-4)$-connected. But since $n \geqq 6$,

$n-4 \geqq [(n-1)/2]$ and $n-1 \geqq 5$. Hence, by a strong form of the Poincaré conjecture (essentially [17]), each component of Bd N is a PL $(n-1)$-sphere. Let $S_1, S_2, \ldots, S_m$ be the components of Bd N. Let N' be the closed PL n-manifold obtained from N by inductively coning over the components of Bd N, i.e.,

$$N' = N \cup C(S_1) \cup \cdots \cup C(S_m)$$

where $C(S_i)$ is the cone over S_i and $\operatorname{Cl}(N'-C(S_i)) \cap C(S_i) = S_i$. By inductively applying the van Kampen theorem it is easy to see that N' is 1-connected. Suppose we have shown that $H_r(N')=0$ where $1 \leqq r < [n/2]$. Consider the triad

$$(N'; N, \bigcup C(S_i))$$

and the following portion of the resulting exact Mayer-Vietoris sequence,

$$H_{r+1}(N) \oplus H_{r+1}(\bigcup C(S_i)) \to H_{r+1}(N') \to H_r(\bigcup S_i).$$

Since $n \geqq 6$, $n-3 \geqq [n/2]$, and since N is $(n-3)$-connected, $H_{r+1}(N)=0$. Since each $C(S_i)$ is a ball and $C(S_i) \cap C(S_j) = \varnothing$ if $i \neq j$, $H_{r+1}(\bigcup C(S_i))=0$. Also $1 \leqq r < [n/2] < n-1$, and hence $H_r(\bigcup S_i)=0$. It follows that $H_{r+1}(N')=0$, and N' is $[n/2]$-connected. Again by the Poincaré conjecture for spheres, N' is a PL n-sphere, and hence N is a cored n-ball.

THEOREM 4.3. *An open* PL *n-manifold U $(n>5)$ is* PL *homeomorphic to S^n minus a nonempty tame compact* 0*-dimensional subset K of cardinality α if and only if there are positive integers p and q such that $p \geqq [n/2]$, $p+q \geqq n-2$, U is (p, q)-connected, and $\mathscr{E}U$ has cardinality α.*

SUFFICIENCY. Since U is 1-connected at infinity, let $\{C_i\}$ be a sequence of compact subsets of U such that $U = \bigcup \operatorname{Int} C_i$, $C_i \subset C_{i+1}$, and each component of $U - C_i$ is 1-connected for each i. Now by Proposition 4.2, U is the monotone union of cored n-balls. By Theorem 2.2, U is PL homeomorphic to $S^n - K$ for some nonempty tame compact 0-dimensional set K. By Lemma 2.1 and the fact that homeomorphic manifolds have the same number of ends, $\sigma\mathscr{E}U = \sigma K$.

NECESSITY. Let $U \approx S^n - K$ where K is a nonempty tame compact subset of cardinality α. By Lemma 2.1, $\sigma K = \sigma\mathscr{E}U$. A theorem of McMillan [8], implies that U is the monotone union of cored n-balls, and hence it follows easily that U is $(n-2, n-2)$-connected.

REMARK. The conditions on the integers p and q in Theorem 4.3 cannot be made any weaker. Consider the following examples, all of which have exactly one end, but none of which are homeomorphic to E^n.

EXAMPLE 1. In [5], for $n \geqq 5$, contractible open PL n-manifolds which are not 1-connected at infinity are shown to exist.

Here $q \geqq 1$ fails. (Note that $p = \infty$.)

EXAMPLE 2. $(S^{[n/2]} \times S^{[(n+1)/2]})$ minus a point is an $([n/2]-1, n-2)$-connected open PL n-manifold which is not homeomorphic to E^n. Here $p \geqq [n/2]$ just fails. (Note that for $n \geqq 6$, $([n/2]-1)+(n-2) \geqq n$.)

EXAMPLE 3. For $n\geqq 7$ and $[n/2]\leqq p\leqq n-4$, $S^{p+1}\times E^{n-p-1}$ is a $(p, n-p-3)$-connected open PL n-manifold which is not homeomorphic to E^n. Here $p+q \geqq n-2$ just fails.

(Note that we need $n\geqq 7$ here, because for $n=6$, $[n/2]+1=n-2$; and hence the theorem applies.)

5. On the Hauptvermutung.

THEOREM 5.1. *Let U and U' be homeomorphic open (p, q)-connected* PL *n-manifolds ($n>5$) where p and q are two positive integers such that $p\geqq[n/2]$, $q\geqq 1$ and $p+q\geqq n-2$. Then U and U' are* PL *homeomorphic.*

Proof. Let $h\colon U \twoheadrightarrow U'$ be a homeomorphism. Since U is 1-connected at infinity, let $\{C_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of U such that each component of $U-C_i$ is 1-connected for each i, $C_i\subset \operatorname{Int} C_{i+1}$, and $\bigcup \operatorname{Int} C_i=U$. For each i, let $C_i'=h(C_i)$. Let M_1 and M_1' be cored n-balls in U and U' respectively which satisfy the conclusion of Proposition 4.2 with respect to C_1 and C_1' respectively. Hence $C_1\subset\operatorname{Int} M_1$, $C_1'\subset\operatorname{Int} M_1'$, and (index of M_1)=(index of M_1'). Let $S_{11},\dots,S_{1k_1}$ be the boundary spheres of M_1. Now let $S_{11}',\dots,S_{1k_1}'$ be the boundary spheres of M_1' such that if Q_{1j} is the component of $U-C_1$ which contains S_{1j}, then $S_{1j}'\subset h(Q_{1j})$. Let $g_1\colon M_1\to S^n$ be a PL embedding. Using Lemma 2.5, it follows easily that there is a PL homeomorphism $g_1'\colon M_1'\twoheadrightarrow g_1(M_1)$ such that $g_1'(S_{1j}')=g_1(S_{1j})$ for $j=1,\dots,k_1$. Let $h_1=(g_1')^{-1}g_1$. Hence h_1 is a PL homeomorphism of M_1 onto M_1' such that $h_1(S_{1j})=S_{1j}'$ for $j=1,\dots,k_1$. Suppose now for $1\leqq i\leqq r-1$, we have found a positive integer $s(i)$, cored n-balls M_i and M_i' in U and U' respectively, and a PL homeomorphism $h_i\colon M_i\twoheadrightarrow M_i'$ such that

(1) M_i and M_i' satisfy the conclusion of Proposition 4.2 for $C_{s(i)}$ and $C_{s(i)}'$ respectively,

(2) $\operatorname{Int} M_i\supset M_{i-1}\cup C_i$,

(3) $\operatorname{Int} M_i'\supset M_{i-1}'\cup C_i'$,

(4) $h_i|M_{i-1}=h_{i-1}$, and

(5) $h_i(S_{ij})=S_{ij}'$, $1\leqq j\leqq k_i$ where the S_{ij}'s are the boundary spheres of M_i, and S_{ij}' is the boundary sphere of M_i' such that if Q_{ij} is the component of $U-C_{s(i)}$ which contains S_{ij}, then $S_{ij}'\subset h(Q_{ij})$.

We wish to define $s(r)$, M_r, M_r', and h_r. Since $U=\bigcup \operatorname{Int} C_i$, $U'=\bigcup\operatorname{Int} C_i'$, and $M_{r-1}\cup C_r$, $M_{r-1}'\cup C_r'$ are compact; choose an integer $s(r)$ such that $\operatorname{Int} C_{s(r)}\supset M_{r-1}\cup C_r$ and $\operatorname{Int} C_{s(r)}'\supset M_{r-1}'\cup C_r'$. Now apply Proposition 4.2 to $C_{s(r)}$ and $C_{s(r)}'$, obtaining cored n-balls M_r and M_r' such that $C_{s(r)}\subset\operatorname{Int} M_r$ and $C_{s(r)}'\subset\operatorname{Int} M_r'$. It follows that M_r and M_r' have the same index. Let $S_{r1},\dots,S_{rk_r}$ be the boundary spheres of M_r, and let $S_{r1}',\dots,S_{rk_r}'$ be the boundary spheres of M_r' such that, if Q_{rj} is the component of $U-C_{s(r)}$ which contains S_{rj}, then $S_{rj}'\subset h(Q_{rj})$. Clearly M_r and M_r' satisfy (1), (2), and (3) for $i=r$. We must construct a PL homeomorphism

$h_r\colon M_r \twoheadrightarrow M'_r$ which satisfies (4) and (5). By Lemma 2.4, let $A_{r1}, \ldots, A_{rk_{r-1}}, A'_{r1}, \ldots, A'_{rk_{r-1}}$ be the pairwise disjoint cored n-balls such that

$$\bigcup_{j=1}^{k_{r-1}} A_{rj} = \mathrm{Cl}\,(M_r - M_{r-1}), \qquad \bigcup_{j=1}^{k_{r-1}} A'_{rj} = \mathrm{Cl}\,(M'_r - M'_{r-1}),$$

$A_{rj} \cap M_{r-1} = S_{(r-1)j}$, and $A'_{rj} \cap M'_{r-1} = S'_{(r-1)j}$. It now suffices, to complete the inductive step, to define for $1 \leqq j \leqq k_{r-1}$, a PL homeomorphism $h_{rj}\colon A_{rj} \twoheadrightarrow A'_{rj}$ such that

(a) $h_{rj}|S_{(r-1)j} = h_{r-1}|S_{(r-1)j}$ and

(b) $h_{rj}(S_{rm}) = S'_{rm}$ if $S_{rm} \subset A_{rj}$.

For if we can do this, then

$$\begin{aligned} h_r &= h_{r-1} \quad \text{on } M_{r-1}, \\ &= h_{rj} \quad \text{on } A_{rj} \end{aligned}$$

will clearly satisfy (4) and (5). We will need the following.

CLAIM. *$S_{rm} \subset A_{rj}$ if and only if $S'_{rm} \subset A'_{rj}$.*

Proof. $S_{rm} \subset A_{rj}$ implies that $S_{rm} \subset Q_{(r-1)j}$; since $A_{rj} \cap M_{r-1} = S_{(r-1)j}$, $Q_{(r-1)j}$ is a component of $U - C_{s(r-1)}$, and A_{rj} is connected and lies in $U - C_{s(r-1)}$. By our construction, $S_{rm} \subset Q_m$. Hence since $Q_{(r-1)j} \in \mathscr{C}(U - C_{s(r-1)})$, $Q_{rm} \in \mathscr{C}(U - C_{s(r)})$, and $C_{s(r-1)} \subset C_{s(r)}$; we have that $Q_{(r-1)j} \supset Q_{rm}$. Hence $h(Q_{(r-1)j}) \supset h(Q_{rm}) \supset S'_{rm}$. But $S'_{(r-1)j} \subset A'_{rj} \cap h(Q_{(r-1)j})$, and using the fact that $h(Q_{(r-1)j})$ is a component of $C'_{s(r-1)}$, we conclude that $A'_{rj} \subset h(Q_{(r-1)j})$. It follows that $A'_{rk} \cap h(Q_{(r-1)j}) = \varnothing$ if $k \neq j$ and hence it must be that $S'_{rm} \subset A'_{rj}$.

Conversely, if $S_{rm} \not\subset A_{rj}$, then $S_{rm} \subset A_{rt}$ for $t \neq j$ since $S_{rm} \in \mathscr{C}$ Bd M_r and

$$\mathrm{Bd}\, M_r \subset \bigcup_{k=1}^{k_{r-1}} \mathrm{Bd}\, A_{rk}.$$

Hence by the first part $S'_{rm} \subset A'_{rt}$ and $A'_{rt} \cap A'_{rj} = \varnothing$. Hence $S'_{rm} \not\subset A'_{rt}$.

Now for $1 \leqq j \leqq k_{r-1}$, let $g_{rj}\colon A_{rj} \to S^n$ be a PL embedding. Hence

$$\bar{g}_{rj} = g_{rj}(h_{r-1}^{-1}|S'_{(r-1)j})$$

is a PL homeomorphism of $S'_{(r-1)j}$ onto $g_{rj}(S_{(r-1)j})$. By Lemma 2.5 and the claim, let g'_{rj} be an extension of $\bar{g}_{rj}$ to A'_{rj} such that $g'_{rj}(S'_{rm}) = g_{rj}(S_{rm})$ whenever $S'_{rm} \subset A'_{rj}$. Now let $h_{rj} = (g'_{rj})^{-1} g_{rj}$. One easily verifies (a) and (b).

Now define $H\colon U \to U'$ by $H(x) = h_i(x)$ whenever $x \in M_i$. It follows that H is a PL homeomorphism.

In view of Theorem 4.3, we obtain the following corollary.

COROLLARY. *The Hauptvermutung holds for* PL *manifolds whenever the manifolds are topologically S^n $(n > 5)$ minus a nonempty tame compact* 0*-dimensional subset.*

For example the manifolds may be topologically S^n minus a tame Cantor set, S^n minus any countable set, an open n-annulus, or E^n. (Stallings [19] has proven the corollary for E^n $(n \geqq 5)$.)

THEOREM 5.2. *For $n=6, 7$ homeomorphic differentiable manifolds which are topologically S^n minus a nonempty tame compact 0-dimensional subset are diffeomorphic.*

Proof. Let K be a nonempty tame compact 0-dimensional subset of S^n, and let U and V be differentiable manifolds which are homeomorphic to S^n-K. By [22], give U and V PL triangulations T_1 and T_2 respectively, which are compatible with their differentiable structures. By the preceding theorem, (U, T_1) and (V, T_2) are PL homeomorphic. By [8], U is the monotone union of cored n-balls; and hence it follows that $\tilde{H}_k(U)=0$ if $k\neq n-1$ and $H_{n-1}(U)$ is free. It follows from [12], [4], [17], [7], and [9] that the least positive integer a such that $\Gamma_a\neq 0$ is 7. For $n=6, 7$; $H_6(U)$ is free and $H_k(U)=0$ for $k\geqq 7$. Now applying Theorem 6.5 of [12], it follows that U and V are diffeomorphic.

REMARK. Theorem 5.2 does not hold for arbitrary $n>5$. For let Σ^n be an exotic n-sphere. Hence $\Sigma^n\times E^1$ is topologically an open $(n+1)$-annulus, but is not diffeomorphic to $S^n\times E^1$. For, if so, then Σ^n can be smoothly embedded in S^{n+1}. Now by 3.6 of [11], Σ^n is bicollared in S^{n+1} and by [3], the closure of a component of $S^{n+1}-\Sigma$ is a topological ball B. By the smooth Hauptvermutung for balls [18]; B, as a smooth manifold, is diffeomorphic to B^{n+1}. Hence Bd $B^{n+1}=\Sigma^n$ is diffeomorphic to S^n, contradicting the fact that Σ^n is exotic, i.e., the existence of an exotic n-sphere implies the existence of an exotic open $(n+1)$-annulus.

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