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ON THE EXISTENCE OF NORMAL SUBGROUPS CONTAINING THEIR CENTRALIZER

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Every finite p-group G has a characteristic subgroup N such that $G \circ N \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ with an elementary abelian factorgroup $N/\mathfrak{z}N$ (see W. Feit and J. G. Thompson [3, Lemma 8.2]). Every hyper-abelian group G has a normal subgroup N such that $N' \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ (see R. Baer [1, Lemma 4.1]). The proof of Baer's lemma is applicable in much more general situations which can best be described by what we call subgroup theoretical properties. From a general theorem we shall not only derive both of the above mentioned lemmas but also a variety of similar results. Here we mention only the following generalization of the Feit-Thompson lemma and of the case $\pi = \{p\}$ of a result of P. Hall and G. Higman ([4, Lemma 1.2.3]):

For every abelian, characteristic subgroup M of a p-solvable, finite group G whose only normal p'-subgroup is 1, *there exists a characteristic p-subgroup N of G with $M \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ and $N/\mathfrak{z}N \subseteq \Omega_1 \mathfrak{z}(\mathfrak{O}_p G/\mathfrak{z}N)$* (see §6.VII).

I am greatly indebted to Professor R. Baer for many remarks including an extension of his lemma to certain almost hyper-abelian groups which suggested the introduction of the property β below.

Notation will be standard with $\langle X, Y\rangle$ denoting the subgroup generated by X and Y.

1. **Statement of main theorem.** Throughout the paper a group G will be fixed and ν will be a set of subgroups of the group G, called ν-subgroups, such that:

(1) 1 is a ν-subgroup.

(2) $X, Y \in \nu$ implies $\langle X, Y\rangle \in \nu$.

A *subgroup theoretical property on* ν is a set of triples (X, Y, G) of ν-subgroups $X \subseteq Y$ of G. We say that the subgroup theoretical property α *is derived from the group theoretical property e* if ν is the set of all normal subgroups of G and α as a subgroup theoretical property on ν is characterized by

$$(X, Y, G) \in \alpha \quad \text{if and only if } Y/X \text{ is an } e\text{-group.}$$

The following conditions will be placed on ν, subgroup theoretical properties α and β on ν, and a ν-subgroup R of G in the main theorem:

(3) $X, Y \in \nu$ implies $X \cap Y \in \nu$.

(4) The union of a tower of ν-subgroups is a ν-subgroup.

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(5) $(Z, X, G) \in \alpha$, $(Z, Y, G) \in \alpha$, $Z = X \cap Y$ implies $(Z, \langle X, Y\rangle, G) \in \alpha$.

(6) For ν-subgroups J and Y the following statements are equivalent:

(i) $(J \cap Y, Y, G) \in \alpha$,

(ii) $(J, \langle J, Y\rangle, G) \in \alpha$,

(iii) there exists a ν-subgroup $L \supseteq \langle J, Y\rangle$ with $(J, L, G) \in \alpha$.

(7) $A, C \in \nu$, $R \cap C \subseteq A \subseteq C$, $(R, G, G) \in \beta$ implies $(A, C, G) \in \beta$.

For a discussion of these conditions see §§3 and 4.

DEFINITION. Let R be a ν-subgroup of G and α a subgroup theoretical property on ν. We say R is $[\nu, \alpha]$-*immersed in* G, in symbols $R[\nu, \alpha]G$, if

(8) $R \nsubseteq K \in \nu$ implies that there exists a ν-subgroup D with $K \subset D \subseteq \langle K, R\rangle$ and $(K, D, G) \in \alpha$.

Therefore, if α is derived from the group theoretical property e, $G[\nu, \alpha]G$ if and only if G is a hyper-e group.

DEFINITION. A *centralizer function* c on ν is a single valued mapping from ν to ν such that after introducing the *c-center* $c_0X = X \cap cX$ for $X \in \nu$ the following condition is satisfied:

(9) $N, B \in \nu$, $N \cap B = c_0N \subseteq B \subseteq cN$ implies $c_0N \subseteq c\langle N, B\rangle$.

Note that by definition cX is always a ν-subgroup for $X \in \nu$.

We are now in a position to state the main theorem.

THEOREM. *Let α and β be subgroup theoretical properties on ν and R a ν-subgroup of G such that* (1)–(7) *are satisfied. Suppose $R[\nu, \alpha]G$ and $(R, G, G) \in \beta$. Let c be a centralizer function on ν and assume that there exist ν-subgroups $A \subseteq N$ such that A is maximal among the $c_0\nu$-subgroups and N is maximal among the ν-subgroups T with $A = c_0T$ and $(A, T, G) \in \alpha$. Then $(c_0N, cN, G) \in \beta$.*

The proof appears in the next section.

2. The abstract situation.

We break up the proof of the theorem into several steps.

PROPOSITION 1. *Let α and β be subgroup theoretical properties on ν and c a centralizer function on ν. Assume that there exist ν-subgroups $A \subseteq N$ of G such that A is maximal among the $c_0\nu$-subgroups and N is maximal among the ν-subgroups T with $A = c_0T$ and $(A, T, G) \in \alpha$. Let* (5) *be satisfied by $Z = A$, $X = N$ and suppose*

(10) *if $A \subseteq C \in \nu$ and $(A, C, G) \notin \beta$, then there exists a ν-subgroup B with $A \subset B \subseteq C$ and $(A, B, G) \in \alpha$.*

Then $(c_0N, cN, G) \in \beta$.

Proof. $A = c_0N$ by choice of N. c being a centralizer function on ν, $C = cN$ is a ν-subgroup. Assume by way of contradiction that $(A, C, G) \notin \beta$. Then by (10), there exists a ν-subgroup B with $A \subset B \subseteq C$ and $(A, B, G) \in \alpha$. We have $A \subseteq N \cap B \subseteq N \cap cN = A$, hence $A = N \cap B$, and (5) implies $(A, \langle N, B\rangle, G) \in \alpha$. Since c is a centralizer function we get $A = c_0N \subseteq c\langle N, B\rangle$. So $A \subseteq \langle N, B\rangle \cap c\langle N, B\rangle = c_0\langle N, B\rangle$

and $A = c_0\langle N, B\rangle$ by the maximality of A. But this implies $N = \langle N, B\rangle$ by the maximality of N. Hence $A \subset B \subseteq N \cap cN = A$, q.e.a.

LEMMA. (See R. Baer [2, Proposition 1.5].) *Let α be a subgroup theoretical property on ν such that* (1)–(4) *and* (6) *are satisfied. Let $S \subseteq R$ be ν-subgroups of G. Then $R[\nu, \alpha]G$ implies $S[\nu, \alpha]G$.*

Proof. Let K be a ν-subgroup that does not contain S. Then $S \cap K \subset S$. Since ν satisfies the tower condition (4) there exists, by Zorn's lemma, a subgroup J of G maximal among the subgroups X of G with

$$K \subseteq X \in \nu, \qquad S \cap K = S \cap X.$$

If $R \subseteq J$, then $S \subseteq J$ and $S = S \cap J = S \cap K \subset S$, q.e.a. So $R \nsubseteq J$. By hypothesis, R is $[\nu, \alpha]$-immersed in G and therefore there exists a ν-subgroup L with

$$J \subset L \subseteq \langle J, R\rangle \quad \text{and} \quad (J, L, G) \in \alpha,$$

and by the maximality of J we also have $S \cap K \subset S \cap L$. Put $Y = S \cap L$ and $D = \langle K, Y\rangle$ so that $K \subset D \subseteq \langle K, S\rangle$. D is a ν-subgroup. Moreover, $(J, L, G) \in \alpha$ implies $(J \cap Y, Y, G) \in \alpha$ by the implication (iii) $\to$ (i) in (6). But $J \cap Y = J \cap S = K \cap S = K \cap Y$ so that $(K \cap Y, Y, G) \in \alpha$ and $(K, D, G) = (K, \langle K, Y\rangle, G) \in \alpha$ by (6). This proves that S is $[\nu, \alpha]$-immersed in G.

PROPOSITION 2. *Let α and β be subgroup theoretical properties on ν such that* (1)–(4) *and* (6) *are satisfied. Suppose that R is a $[\nu, \alpha]$-immersed ν-subgroup of G with $(R, G, G) \in \beta$. Then* (7) *implies* (10) *for all ν-subgroups A.*

Proof. Let $A \subseteq C$ be ν-subgroups with $(A, C, G) \notin \beta$. Then (7) implies $R \cap C \nsubseteq A$. By the lemma $R \cap C$ is $[\nu, \alpha]$-immersed and therefore there exists a ν-subgroup B with $A \subset B \subseteq \langle A, R \cap C\rangle \subseteq C$ and $(A, B, G) \in \alpha$.

Proof of Theorem. By Proposition 2, Proposition 1 is applicable.

3. **Examples for properties α satisfying (5) and (6).** If ν consists of normal subgroups of G only then subgroup theoretical properties α on ν with (5) and (6) may be defined as follows:

I. For a group theoretical property e inherited by normal subgroups and direct products and ν-subgroups $A \subseteq B$ let $(A, B, G) \in \alpha$ if and only if B/A is an e-group.

Proof. A group isomorphic to an e-group is an e-group.

II. For a group theoretical property h that is inherited by subgroups, epimorphic images and direct products, and ν-subgroups $A \subseteq B$ let $(A, B, G) \in \alpha$ if and only if $G/\mathfrak{C}_G(B \bmod A)$ is an h-group.

Proof. (5) follows from the fact that $G/\mathfrak{C}_G(XY \bmod Z)$ is isomorphic to a subgroup of the direct product of $G/\mathfrak{C}_G(X \bmod Z)$ with $G/\mathfrak{C}_G(Y \bmod Z)$. In (6), (i) and (ii) are equivalent since $\mathfrak{C}_G(Y \bmod J \cap Y) = \mathfrak{C}_G(JY \bmod J)$, and (iii) implies (ii) because of $\mathfrak{C}_G(L \bmod J) \subseteq \mathfrak{C}_G(JY \bmod J)$.

III. Let $\alpha=\alpha_1 \cap \alpha_2$ for subgroup theoretical properties α_1 and α_2 that satisfy (5) and (6) for the same set ν.

4. **Examples for properties β satisfying (7).** If ν consists of normal subgroups of G only and if R is a ν-subgroup of G, then subgroup theoretical properties β which satisfy (7) may be defined as follows:

I. For a group theoretical property g that is inherited by normal subgroups and epimorphic images, and ν-subgroups $X \subseteq Y$ let $(X, Y, G) \in \beta$ if and only if Y/X is a g-group.

Proof. Under the hypothesis of (7) RY/R as a normal subgroup of G/R is a g-group. So C/A is a g-group as an epimorphic image of $C/R \cap C \cong RC/R$.

In many an application g will be the group theoretical property 1 (and $R=G$).

II. For ν-subgroups $X \subseteq Y$ let $(X, Y, G) \in \beta$ if and only if

$$\mathfrak{z}(G \bmod R) \subseteq \mathfrak{C}_G(Y \bmod X).$$

Proof. Under the hypothesis of (7) with $Z=\mathfrak{z}(G \bmod R)$ we have

$$Z \circ C \subseteq (Z \circ G) \cap C \subseteq R \cap C \subseteq A,$$

hence $Z \subseteq \mathfrak{C}_G(C \bmod A)$ and $(A, C, G) \in \beta$.

Note that with this β always $(R, G, G) \in \beta$ so that in the Theorem we always get $Z \subseteq \mathfrak{C}_G(cN \bmod c_0N)$. In this view the following example becomes useful.

III. Let $\beta=\beta_1 \cap \beta_2$ for subgroup theoretical properties β_1 and β_2 that satisfy (7) for the same set ν.

5. **Examples for centralizer functions.** All examples below give centralizer functions of a special type arising as follows:

5.1. Let σ be a relation on ν such that:

(a) the intersection of two ν-subgroups is a ν-subgroup (3),

(b) every set of ν-subgroups generates a ν-subgroup,

(c) if $N \in \nu$ and if $\mathfrak{X}$ is a set of ν-subgroups X with $X \sigma N$, then $\langle \mathfrak{X} \rangle \sigma N$,

(d) $U, X, N \in \nu$, $U \subseteq X \sigma N$ implies $U \sigma N$,

(e) $U, V \in \nu$, $U \sigma V$ implies $V \sigma U$.

For $N \in \nu$ define $cN=\langle X \mid X \in \nu, X \sigma N \rangle$. Then c is a centralizer function on ν.

Proof. By (b), c is a single valued mapping from ν to ν. To check (9), let $N, B \in \nu$ with $B \subseteq cN$. Then $cN \sigma N$ by (c), $N \cap cN \in \nu$ by (a), and $N \cap cN \sigma N$ by (d). Also $B \sigma N$ by (d), $N \sigma B$ by (e), and $N \cap cN \sigma B$ by (d). Hence $N \cap cN \sigma \langle N, B \rangle$ by (e) and (c). Therefore $c_0N \subseteq c\langle N, B \rangle$.

5.2. The theorem requires the existence of maximal $c_0\nu$-subgroups. Since $1 \in \nu$ by (1), there always exist $c_0\nu$-subgroups, e.g. $c_0 1=1$. So if

(11) the union of a tower of $c_0\nu$-subgroups is a $c_0\nu$-subgroup, then by the Maximum Principle of Set Theory, maximal $c_0\nu$-subgroups do exist. We shall check the validity of (11) in the following examples for 5.1.

Suppose that ν satisfies (1), 5.1(a), and 5.1(b).

I. For $U, V \in \nu$ let $U \sigma V$ if and only if U and V centralize each other. Then (a)–(e) in 5.1 are satisfied. If $\mathfrak{C}_G\nu \subseteq \nu$, then cN becomes the centralizer of N in G. In general however, cN may be smaller than $\mathfrak{C}_G N$. By 5.1(a), $c_0\nu \subseteq \nu$. A ν-subgroup is a $c_0\nu$-subgroup if and only if it is abelian. So by 5.1(b), $c_0\nu$ satisfies the tower condition (11).

II. For $U, V \in \nu$ let $U \sigma V$ if and only if every subgroup of U permutes with every subgroup of V. To prove (c) in 5.1 let $t \in \langle \mathfrak{X} \rangle$, $n \in N$. Then t is a product of elements x in various $X \in \mathfrak{X}$. But $xn = \bar{n}\bar{x}$ for some $\bar{n} \in \langle n \rangle \subseteq N$, $\bar{x} \in \langle x \rangle \subseteq X$, since $X \sigma N$. So $tn = \tilde{n}\tilde{t}$ with $\tilde{n} \in \langle n \rangle \subseteq N$, $\tilde{t} \in \langle \mathfrak{X} \rangle$ proving (c). (d) and (e) are obvious. By 5.1, $\mathfrak{P}N = cN = \langle X \in \nu \mid X \sigma N \rangle$ defines a centralizer function $c = \mathfrak{P}$ on ν. A ν-subgroup Z of G is a $c_0\nu$-subgroup if and only if $Z \sigma Z$. It is now easy to see that 5.1(b) implies the tower condition (11). Finite $c_0\nu$-subgroups are nilpotent (M. Suzuki [6, p. 7]).

III. Let $\mathfrak{A}$ be a group of automorphisms of G that contains all inner automorphisms and suppose that ν consists of $\mathfrak{A}$-invariant subgroups only and satisfies (3) (e.g. the set of *all* $\mathfrak{A}$-invariant subgroups meets requirement (3)). An *$\mathfrak{A}$-composition factor $U_1\mathfrak{A}U_2$ of G contained in $U \subseteq G$* will be a pair (U_1, U_2) of $\mathfrak{A}$-invariant subgroups $U_2 \subseteq U_1 \subseteq U$ with no other $\mathfrak{A}$-invariant subgroup between them. $U_1\mathfrak{A}U_2$ *centralizes* $V_1\mathfrak{A}V_2$ if $U_1 \circ V_1 \subseteq U_2 \cap V_2$.

For $U, V \in \nu$ let $U \sigma V$ if and only if the $\mathfrak{A}$-composition factors of G contained in U centralize the $\mathfrak{A}$-composition factors of G contained in V. To prove 5.1(c) let $Z_1\mathfrak{A}Z_2$ be an $\mathfrak{A}$-composition factor in $Z = \langle \mathfrak{X} \rangle$ and $N_1\mathfrak{A}N_2$ an $\mathfrak{A}$-composition factor in N. Consider first the case where $\mathfrak{X}$ has just two members X and Y.

Clearly

$$Z_1 = Z_2(Z_1 \cap X) \quad \text{or} \quad Z_1 \cap X \subseteq Z_2,$$

since $Z_1\mathfrak{A}Z_2$ is an $\mathfrak{A}$-composition factor and $Z_2(Z_1 \cap X)$ is $\mathfrak{A}$-invariant. In the first case

$$Z_1 \circ N_1 \subseteq [Z_2 \circ N_1][(Z_1 \cap X) \circ N_1] \subseteq Z_2(Z_2 \cap X) \subseteq Z_2,$$

since $(Z_1 \cap X)\mathfrak{A}(Z_2 \cap X)$ is an $\mathfrak{A}$-composition factor in $X \sigma N$. Also $XZ_1\mathfrak{A}XZ_2$ and $(XZ_1 \cap Y)\mathfrak{A}(XZ_2 \cap Y)$ are $\mathfrak{A}$-composition factors, and

$$XZ_1 = (XZ_2)Z_1 = XZ_2(XZ_1 \cap Y).$$

Hence

$$XZ_1 \circ N_1 \subseteq [XZ_2 \circ N_1][(XZ_1 \cap Y) \circ N_1] \subseteq XZ_2(XZ_2 \cap Y) \subseteq XZ_2$$

so that in the second case

$$Z_1 \circ N_1 \subseteq Z_1 \cap (XZ_1 \circ N_1) \subseteq Z_1 \cap XZ_2 = Z_2(Z_1 \cap X) \subseteq Z_2.$$

Clearly $Z_1 \circ N_1 \subseteq XY \circ N_1 \subseteq [X \circ N_1][Y \circ N_1] \subseteq N_2$ since $X\mathfrak{A}X$ and $Y\mathfrak{A}Y$ are $\mathfrak{A}$-composition factors. This proves $XY \sigma N$.

In the general case there exists a ν-subgroup W of $\langle \mathfrak{X} \rangle$ maximal with $W \sigma N$ by the Maximum Principle of Set Theory and 5.1(b); the argument is similar to the one given below for the existence of maximal $c_0\nu$-subgroups. Let $X \in \mathfrak{X}$. Then $WX \sigma N$ by the special case, hence $X \subseteq W$ by the maximality of W. Therefore $\langle \mathfrak{X} \rangle = W \sigma N$ proving (c). (d) and (e) follow immediately. By 5.1, $cN = \mathfrak{K}N = \langle X \mid X \in \nu, X \sigma N \rangle$ defines a centralizer function $c = \mathfrak{K}$ on ν.

Let $\mathfrak{T}$ be a tower of $c_0\nu$-subgroups Z. Every $Z \in \mathfrak{T}$ is a ν-subgroup with $Z \sigma Z$. By 5.1(b), $W = \bigcup \mathfrak{T} \in \nu$. For an $\mathfrak{A}$-composition factor $W_1 \mathfrak{A} W_2$ of G in W let $w_1 \in W_1$ and $w \in W$. Then there exists $Z \in \mathfrak{T}$ which contains both w_1 and w. Moreover $(W_1 \cap Z)\mathfrak{A}(W_2 \cap Z)$ is an $\mathfrak{A}$-composition factor in Z. Therefore $w_1 \circ w \in W_2 \cap Z$ showing that $W_1 \circ W \subseteq W_2$ and hence that $W \sigma W$. (11) holds.

5.3. Let ν satisfy (1) and 5.1(b) and let f be a function from ν into a set $\bar{\nu}$ of subgroups of G. Suppose that $\bar{\nu}$ and a relation $\bar{\sigma}$ on $\bar{\nu}$ satisfy (1) and 5.1(b)–(e). Then the relation σ on ν defined by

$$U \sigma V \quad \text{if and only if } fU \,\bar{\sigma}\, fV$$

satisfies 5.1(c)–(e) provided that

(i) If $\mathfrak{X}$ is a set of ν-subgroups, then $f\langle \mathfrak{X} \rangle = \langle f\mathfrak{X} \rangle$.

So if 5.1(a) holds for ν and $\bar{\nu}$, we have a centralizer function c on $\bar{\nu}$ defined by $\bar{\sigma}$ and a centralizer function $c[f]$ on ν defined by σ.

Proof. For a set $\mathfrak{X}$ of ν-subgroups X with $X \sigma N$, i.e. $fX \,\bar{\sigma}\, fN$, we get $\langle f\mathfrak{X} \rangle \,\bar{\sigma}\, fN$ by (c) for $\bar{\nu}$ and $\bar{\sigma}$. Therefore $f\langle \mathfrak{X} \rangle \,\bar{\sigma}\, fN$ by (i), i.e. $\langle \mathfrak{X} \rangle \sigma N$, proving (c) for ν and σ. (d) follows from

(ii) $U, V \in \nu$, $U \subseteq V$ implies $fU \subseteq fV$,

which is a consequence of (i). (e) is obvious.

5.4. Let γ be a set of triples (X, Y, G) with $X \in \nu$, $Y \in \bar{\nu}$. Note, however, that we do *not* require $X \subseteq Y$. For $X \in \nu$ let

$$\gamma X = G \cap \bigcap \{T \mid (X, T, G) \in \gamma\}.$$

We make the following assumptions:

(γ.a) ν satisfies (1), 5.1(a), and 5.1(b).

(γ.b) If $\mathfrak{Y}$ is a set of $\bar{\nu}$-subgroups Y with $(X, Y, G) \in \gamma$, then $(X, G \cap \bigcap \mathfrak{Y}, G) \in \gamma$.

(γ.c) $X_1 \in \nu$, $X_1 \subseteq X_2$, $(X_2, Y, G) \in \gamma$ implies $(X_1, Y, G) \in \gamma$.

(γ.d) If $\mathfrak{X}$ is a set of ν-subgroups, then $(\langle \mathfrak{X} \rangle, \langle \gamma \mathfrak{X} \rangle, G) \in \gamma$.

Then $(X, \gamma X, G) \in \gamma$ for $X \in \nu$ by (γ.b), and $\gamma \langle \mathfrak{X} \rangle \subseteq \langle \gamma \mathfrak{X} \rangle$ by (γ.d). Let $(\langle \mathfrak{X} \rangle, T, G) \in \gamma$. Then $(X, T, G) \in \gamma$ for all $X \in \mathfrak{X}$ by (γ.c), hence $\gamma X \subseteq T$ and $\langle \gamma \mathfrak{X} \rangle \subseteq T$. Therefore $\langle \gamma \mathfrak{X} \rangle \subseteq \gamma \langle \mathfrak{X} \rangle$. This shows that

$$fX = \gamma X$$

satisfies (i) in 5.3. The following functions f arise in this manner:

I. If every ν-subgroup of G is a normal subgroup of G, and if ν satisfies (1), 5.1(a), and 5.1(b), then a function f with property (i) is defined by

$$fX = X \circ G \quad \text{for } X \in \nu, \qquad \bar{\nu} = \text{set of all normal subgroups of } G.$$

f may be obtained from the set γ defined by

$$(X, Y, G) \in \gamma \quad \text{if and only if } X \in \nu,\ Y \in \bar{\nu},\ X \circ G \subseteq Y.$$

If c is a centralizer function on $\bar{\nu}$ defined by a relation $\bar{\sigma}$ according to 5.1, then

$$c[f]N = \langle X \mid X \in \nu,\ X \circ G\ \bar{\sigma}\ N \circ G \rangle.$$

In particular,

$$\mathfrak{C}_G[f]N = \mathfrak{z}(G \operatorname{mod} \mathfrak{C}_G(N \circ G)) \quad \text{for } N \in \nu$$

provided that $\mathfrak{z}(G \operatorname{mod} \mathfrak{C}_G(N \circ G))$ is a ν-subgroup.

II. Let ν satisfy (1), 5.1(a), and 5.1(b) and let $\bar{\nu}$ be the set of all $\mathfrak{A}$-invariant subgroups of G for a fixed group $\mathfrak{A}$ of automorphisms of G. Then

$$(X, Y, G) \in \gamma \quad \text{if and only if } X \in \nu,\ Y \in \bar{\nu}, \text{ and } X \subseteq Y$$

defines a set of triples with properties (γ.a)–(γ.d) and consequently a function f with property (i); $fX = \gamma X$ is the $\mathfrak{A}$-invariant hull of the ν-subgroup X.

III. One may derive functions f from suitable group theoretical properties e via

$$(X, Y, G) \quad \text{if and only if } X,\ Y \trianglelefteq G,\ X \subseteq Y,\ G/Y \text{ is an } e\text{-group}.$$

However, the resulting centralizer functions $\mathfrak{C}_G[f]$ do not lead to interesting applications in connection with the theorem.

6. Some applications of the theorem.

I. (R. Baer [1].) *For every abelian, normal subgroup M of a hyperabelian group G there is a normal subgroup N of G with $M \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ and $N' \subseteq \mathfrak{z}N$.*

Proof. We let ν be the set of all normal subgroups of G. The group theoretical property e of being abelian is inherited by normal subgroups and direct products. Hence the subgroup theoretical property α derived from e satisfies (5) and (6) (see 3.I). $R = G$ satisfies (8) by definition of "hyperabelian". β may be defined as in 4.I with $g = 1$ so that (7) holds trivially. There exists a maximal $\mathfrak{z}\nu$-subgroup $A \supseteq M$ as indicated in 5.2.I and there exist subgroups N maximal among the normal subgroups of G with $N' \subseteq A = \mathfrak{z}N$. By the theorem $\mathfrak{z}N = \mathfrak{C}_G N$.

II. *For every abelian, normal subgroup M of a finite, solvable group G there is a normal subgroup N of G with $M \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ and $N/\mathfrak{z}N$ abelian of squarefree exponent.*

Proof as in I where, however, e-groups are abelian of squarefree exponent.

Since N is nilpotent this result implies that the fitting subgroup of a solvable group contains its centralizer.

III. *Let ν be the set of all normal subgroups of the group G, g a group theoretical property that is inherited by normal subgroups and epimorphic images, and α the subgroup theoretical property derived from $e =$ abelian as in* I. *Suppose that G has a* [ν, α]*-immersed normal subgroup R whose factor group G/R is a g-group. Then for* *every abelian, normal subgroup M of G there exists a normal subgroup N of G with the following properties*:

$$M \subseteq \mathfrak{z}N,$$
$$N/\mathfrak{z}N \text{ is abelian},$$
$$\mathfrak{C}_G N/\mathfrak{z}N \text{ is a } g\text{-group},$$
$$\mathfrak{C}_G(\mathfrak{C}_G N \bmod \mathfrak{z}N) \supseteq \mathfrak{z}(G \bmod R).$$

Proof. Define β_1 as β in 4.I, β_2 as β in 4.II, and $\beta=\beta_1 \cap \beta_2$ as in 4.III so that $(R, G, G) \in \beta$ and (7). ν satisfies (1)–(4) and α meets requirements (5) and (6) as pointed out in 3.I. As in the proof of 6.I there are normal subgroups A and N with $M \subseteq A$ and the properties mentioned in the main theorem, namely

$$A = \mathfrak{z}N, \qquad (A, N, G) \in \alpha, \qquad (\mathfrak{z}N, \mathfrak{C}_G N, G) \in \beta.$$

This means that $M \subseteq \mathfrak{z}N$, $N/\mathfrak{z}N$ is abelian, $\mathfrak{C}_G N/\mathfrak{z}N$ is a g-group, and

$$\mathfrak{C}_G(\mathfrak{C}_G N \bmod \mathfrak{z}N) \supseteq \mathfrak{z}(G \bmod R).$$

In III, ν, α and $\delta=\beta_1$ satisfy the hypothesis of Proposition 4 in §7 below. Therefore, if every epimorphic image of G is a g-group or has an abelian, normal subgroup, not 1, then the $[\nu, \alpha]$-hypercenter $\mathfrak{h}_{[\nu,\alpha]}G$ as defined in §7 has the properties imposed on R above.

Of course, 6.I is a special case of III.

IV. *Let G have a $[\nu, \alpha]$-immersed, normal subgroup R as in* III *whose factor group G/R is abelian. Then there exists for every abelian, normal subgroup M of G a normal subgroup N of G with $M \cdot N' \cdot (G \circ \mathfrak{C}_G N) \subseteq \mathfrak{z}N$.*

Proof. This is a special case of III.

V. *For every characteristic, abelian subgroup M of a hypercentral group G there is a characteristic subgroup N with $M(N \circ G) \subseteq \mathfrak{z}N=\mathfrak{C}_G N$.*

Proof. Let ν be the set of all characteristic subgroups of G and let α be defined as in 3.II with $h=1$. Then $R=G$ is $[\nu, \alpha]$-immersed in G by the definition of "hypercentral", and the main theorem is applicable.

VI. (W. Feit and J. G. Thompson [3].) *For every abelian, characteristic subgroup M of a finite p-group G there is a characteristic subgroup N of G with $M(N \circ G) \subseteq \mathfrak{z}N=\mathfrak{C}_G N$ and elementary abelian factor group $N/\mathfrak{z}N$.*

Proof. As for V. Special case of VII below.

Remark. In the situation of V and VI, the (characteristic) subgroup A in the main theorem uniquely determines a subgroup N as described in the theorem. In the notation of [5], $N=\mathfrak{X}_G(A)$. However, the generalizations of V and VI contained in [5] though of a similar formal nature do not fit in our present context and certainly cannot be expected to follow from such a simple observation as Proposition 1 in §2.

VII. *Result on finite, p-solvable groups as stated in the introduction.*

Proof. Let ν be the set of all characteristic subgroups of G and define the subgroup theoretical property α on ν by

$$(X, Y, G) \in \alpha \quad \text{if and only if } Y/X = Y_1/X \otimes Y_2/X$$

with a p'-group Y_1/X and an elementary abelian p-group Y_2/X that centralizes $\mathfrak{O}_p G$.

We have to show that G is $[\nu, \alpha]$-immersed in itself and that α satisfies (5) and (6). Let $K \neq G$ be a characteristic subgroup of G. Since G is p-solvable,

$$\mathfrak{O}_{p'}(G/K) \otimes \mathfrak{O}_p(G/K) \neq 1.$$

If $\mathfrak{O}_{p'}(G/K) \neq 1$, let $D/K = \mathfrak{O}_{p'}(G/K)$. $\mathfrak{O}_p G$ induces a p-group of automorphisms of $\mathfrak{O}_p(G/K)$. So if $\mathfrak{O}_{p'}(G/K) = 1$ then $\mathfrak{O}_p(G/K) \neq 1$ and the fixed elements in $\mathfrak{O}_p(G/K)$ under the action of $\mathfrak{O}_p G$ form a subgroup $F/K \neq 1$; it is invariant under all automorphisms of G and so is $D/K = \Omega_1 \mathfrak{z}(F/K) \neq 1$; D is a characteristic subgroup of G and D/K is elementary abelian and centralizes $\mathfrak{O}_p G$. Therefore D is in both cases a characteristic subgroup of G with $K \subset D$ and $(K, D, G) \in \alpha$. This proves $G[\nu, \alpha]G$.

One checks easily that α meets requirements (5) and (6).

Thus, G has a characteristic subgroup N with $M \subseteq \mathfrak{z}N = \mathfrak{C}_G N$ and $(\mathfrak{z}N, N, G) \in \alpha$. Since $\mathfrak{O}_{p'} G = 1$, $\mathfrak{z}N$ must be a p-group and $N/\mathfrak{z}N$ must be a p-group. Hence

$$N/\mathfrak{z}N \subseteq \Omega_1 \mathfrak{z}(\mathfrak{O}_p G/\mathfrak{z}N).$$

VIII. *Let e be a group theoretical property inherited by normal subgroups, direct products, and unions of towers of normal e-subgroups of a group. Then every hyper-e-group G has*

(1) *normal subgroups N_1 and Z_1 such that*

(a) *$Z_1 \circ G$ centralizes $N_1 \circ G$,*

(b) *if the normal subgroup X of G has the property that $X \circ G$ centralizes $N_1 \circ G$ then $X \subseteq Z_1$,*

(c) *$Z_1 \subseteq N_1$ and N_1/Z_1 is an e-group,*

(2) *normal subgroups N_2 and Z_2 such that*

(a) *every subgroup of Z_2 permutes with every subgroup of N_2,*

(b) *if the normal subgroup X of G has the property that every subgroup of X permutes with every subgroup of N_2, then $X \subseteq Z_2$,*

(c) *$Z_2 \subseteq N_2$ and N_2/Z_2 is an e-group,*

(3) *normal subgroups N_3 and Z_3 depending on the choice of a group $\mathfrak{A}$ of automorphisms of G containing all inner automorphisms such that*

(a) *every $\mathfrak{A}$-composition factor in Z_3 is centralized by N_3,*

(b) *if the normal subgroup X of G has the property that every $\mathfrak{A}$-composition factor of G in X centralizes every $\mathfrak{A}$-composition factor of G in N_3, then $X \subseteq Z_3$,*

(c) *$Z_3 \subseteq N_3$ and N_3/Z_3 is an e-group.*

Proof. Let ν be the set of all normal subgroups of G and define α as in 3.I so that (1)–(6) are satisfied. The centralizer functions $\mathfrak{C}_G[f]$, $\mathfrak{P}$, and $\mathfrak{K}$ on ν as described in 5.4.I, 5.2.II and III have the right properties to ensure the existence of a subgroup A as in the main theorem. Since the union of a tower of normal e-subgroups of G/A is an e-group, there will exist a normal subgroup N with the properties required in the main theorem. By the theorem and the definition of the centralizer function c to be used $Z_i = c_0 N$ and $N_i = N$ have the properties (a), (b), (c).

7. **Excursion on** $[\nu, \alpha]$-δ-**immersion.** Proposition 4 below will clarify the role of the hypothesis about R in the main theorem. Let ν always satisfy (1) and (2). For ν-subgroups $A \subseteq Q$ and subgroup theoretical properties α and δ on ν we define: Q is $[\nu, \alpha]$-δ-*immersed over* A *in* G, or $Q[\nu, \alpha]$-δG over A, if

(12) $A \subseteq K \in \nu$, $(K, \langle K, Q\rangle, G) \notin \delta$ implies the existence of a ν-subgroup L with $K \subset L \subseteq \langle K, Q\rangle$ and $(K, L, G) \in \alpha$.

We denote by 1 the subgroup theoretical property $\{(X, X, G) \mid X \in \nu\}$. Then $[\nu, \alpha]$-1-immersion over 1 amounts to $[\nu, \alpha]$-immersion as defined in §1. If $1 \subseteq \delta$ then every $[\nu, \alpha]$-immersed ν-subgroup is also $[\nu, \alpha]$-δ-immersed (over 1). We shall have to consider the following conditions on δ and α:

(13) $X, Y \in \nu$, $X \subseteq Y$, $(X, G, G) \in \delta$ implies $(Y, G, G) \in \delta$.

(14) If K is a ν-subgroup and if $\mathfrak{M}$ is a set of ν-subgroups X with $(K, \langle K, X\rangle, G) \in \delta$, then $(K, \langle K, \mathfrak{M}\rangle, G) \in \delta$.

(15) $X \subseteq K \in \nu$, $(X, Y, G) \in \alpha$ implies $(K, \langle K, Y\rangle, G) \in \alpha$.

Note that (15) is satisfied whenever α is derived from a group theoretical property that is inherited by epimorphic images.

PROPOSITION 3. (See R. Baer [2, PROPOSITION 1.1]). *Let* $A \subseteq Q$ *be* ν-*subgroups.*

(1) *If* Q *is* $[\nu, \alpha]$-δ-*immersed over* A *and if the* ν-*subgroup* B *contains* A, *then* $\langle Q, B\rangle$ *is* $[\nu, \alpha]$-δ-*immersed over* B.

(2) *If* P *is* $[\nu, \alpha]$-1-*immersed over* A *and* Q *is* $[\nu, \alpha]$-δ-*immersed over* $P \subseteq Q$, *then* Q *is* $[\nu, \alpha]$-δ-*immersed over* A.

(3) *Suppose* 5.1(b) *and* (14). *If* $\mathfrak{M}$ *is a set of* ν-*subgroups* X $[\nu, \alpha]$-δ-*immersed over* A, *then* $\langle\mathfrak{M}\rangle$ *is* $[\nu, \alpha]$-δ-*immersed over* A.

(4) *One may substitute* $[\nu, \alpha]$-δ *for* $[\nu, \alpha]$ *in the lemma of* §2 *if* δ *meets requirement* (6) (*with* α *replaced by* δ).

Proof. (1) and (3) are immediate applications of the definition of immersion.—With regard to (2), let K be a ν-subgroup with $A \subseteq K$ and $(K, \langle K, Q\rangle, G) \notin \delta$. Assume first that $P \subseteq K$ or equivalently that $(K, \langle K, P\rangle, G) \notin 1$. Since P is $[\nu, \alpha]$-1-immersed over A, there exists a ν-subgroup L with $K \subset L \subseteq \langle K, P\rangle \subseteq \langle K, Q\rangle$ and $(K, L, G) \in \alpha$ as desired. If however $P \subseteq K$ then we may use that Q is $[\nu, \alpha]$-δ-immersed over P.—To verify (4) define J as in the proof of the lemma. If $(J, \langle J, R\rangle, G) \in \delta$, then $(J \cap S, S, G) \in \delta$ by the implication (iii)$\rightarrow$(i) in (6). But $J \cap S = K \cap S$, hence $(K, \langle K, S\rangle, G) \in \delta$ by (6). Therefore we may assume $(J, \langle J, R\rangle, G) \in \delta$ and using $R[\nu, \alpha]$-δG proceed as in the lemma.

We define the $[\nu, \alpha]$-δ-*hypercenter* $\mathfrak{h}_{[\nu,\alpha]\text{-}\delta}G$ of G to be the subgroup generated by all $[\nu, \alpha]$-δ-immersed ν-subgroups (over 1) (see R. Baer [2]). Assuming 5.1(b) and (14) we conclude from Proposition 3(3) that $\mathfrak{h}_{[\nu,\alpha]\text{-}\delta}G$ is $[\nu, \alpha]$-δ-immersed. In particular, since $\delta = 1$ satisfies (14), $\mathfrak{h}_{[\nu,\alpha]}G$ is a $[\nu, \alpha]$-immersed ν-subgroup whenever 5.1(b) holds.

PROPOSITION 4. *Let ν satisfy* 5.1(b) *and suppose that the subgroup theoretical properties δ and α meet requirements* (13) *and* (15). *Then the following statements are equivalent*:

(a) *There exists a ν-subgroup R with $R[\nu, \alpha]G$ and $(R, G, G) \in \delta$.*

(b) $(\mathfrak{h}_{[\nu,\alpha]}G[\nu, \alpha]G$ *and*) $(\mathfrak{h}_{[\nu,\alpha]}G, G, G) \in \delta$.

(c) $G[\nu, \alpha]$-δG.

Proof. The equivalence of (a) and (b) follows from 5.1(b) and (13). Assume (b) and let $H = \mathfrak{h}_{[\nu,\alpha]}G$. Let K be a ν-subgroup with $(K, G, G) \notin \delta$. By (b) and (13), $H \nsubseteq K$ so that $K \subset \langle K, H\rangle$. But $H[\nu, \alpha]G$ guarantees the existence of a ν-subgroup L with $K \subset L \subseteq \langle K, H\rangle$ and $(K, L, G) \in \alpha$. We proved (c). Conversely, assume $G[\nu, \alpha]$-δG and by way of contradiction that $(H, G, G) \notin \delta$. Then there exists a ν-subgroup $L \subset H$ with $(H, G, G) \in \alpha$. This implies that L is $[\nu, \alpha]$-immersed over H, since for any ν-subgroup K with $H \subseteq K \subset \langle K, L\rangle$ we have $\langle K, L\rangle \in \nu$ with $K \subset \langle K, L\rangle \subseteq \langle K, L\rangle$ and $(K, \langle K, L\rangle, G) \in \alpha$ by (15). Now Proposition 3(2) yields that L is $[\nu, \alpha]$-immersed and is hence contained in $H = \mathfrak{h}_{[\nu,\alpha]}G$, a contradiction, since $H \subset L$. Therefore $(H, G, G) \in \delta$, and the proof is complete.

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