A GENERALIZATION OF THE CLASSICAL PARTITION THEOREMS

BY

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1. Introduction. Let $P_d(n)$ denote the number of partitions of n into positive integers of the form $n = b_1 + \cdots + b_s$ with $b_i - b_{i+1} \geq d$ where strict inequality holds if $d | b_i$. $P_1(n)$ appears in the first Rogers-Ramanujan identity [5, p. 291]. $P_2(n)$ appears in the first of the Göllnitz-Gordon partition theorems [2, p. 945], and $P_3(n)$ appears in a partition theorem of I. J. Schur [6, p. 489].

For $d > 3$, we have the following result due to H. L. Alder [1, p. 713].

(1.1) Let S be any fixed set of positive integers. Then $P_d(n)$ is not identically equal to the number of partitions of n into parts taken from S if $d > 3$.

In a different approach, B. Gordon [4, p. 394] has made the following generalization of the Rogers-Ramanujan identities.

(1.2) Let a and k be integers with $0 < a \leq k$. Let $A_{k,a}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm a \pmod{2k+1}$. Let $B_{k,a}(n)$ denote the number of partitions of n of the form $n = b_1 + \cdots + b_s$ with $b_i \geq b_{i+1}$, $b_i - b_{i+k-1} \geq 2$, and with 1 appearing as a summand at most $a-1$ times. Then $A_{k,a}(n) = B_{k,a}(n)$.

The Göllnitz-Gordon identities have also been generalized in this manner [2, p. 946].

(1.3) Let a and k be integers with $0 < a \leq k$. Let $A_{k,a}^\#(n)$ denote the number of partitions of n into parts which are neither $\equiv 2 \pmod{4}$ nor $\equiv 0, \pm(2a-1) \pmod{4k}$. Let $B_{k,a}^\#(n)$ denote the number of partitions of n of the form $n = b_1 + \cdots + b_s$, with $b_i \geq b_{i+1}$, no odd parts are repeated, $b_i - b_{i+k-1} \geq 2$ with strict inequality if $2 | b_i$ and at most $a-1$ parts not exceeding 2 appear. Then $A_{k,a}^\#(n) = B_{k,a}^\#(n)$.

This leads us to define the following partition function. Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $n = b_1 + \cdots + b_s$, with $b_i \geq b_{i+1}$, no parts $\not\equiv 0 \pmod{\lambda+1}$ are repeated, $b_i - b_{i+k-1} \geq \lambda+1$ with strict inequality if $\lambda+1 | b_i$ and at most $a-1$ parts not exceeding $\lambda+1$ appear. We note that $B_{0,k,a}(n)$ appears in (1.2); $B_{1,k,a}(n)$ appears in (1.3), and $B_{a-1,2,2}(n) = P_a(n)$.

QUESTION. If k is sufficiently large with respect to λ , does $B_{\lambda,k,a}(n)$ appear in a partition theorem of the Rogers-Ramanujan type?

We answer this question affirmatively with the following result.

THEOREM 1. Let $0 \leq \lambda \leq a \leq k$ be integers, and $2\lambda-1 \leq k$. If λ is even, define

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$A_{\lambda,k,a}(n)$ to be the number of partitions of n into parts such that no part $\not\equiv 0 \pmod{\lambda+1}$ may be repeated and no part is $\equiv 0, \pm(a - \frac{1}{2}\lambda)(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}$. If λ is odd, define $A_{\lambda,k,a}(n)$ to be the number of partitions of n into parts such that no part $\not\equiv 0 \pmod{\frac{1}{2}(\lambda+1)}$ may be repeated, no part is $\equiv \lambda+1 \pmod{2\lambda+2}$, and no part is $\equiv 0, \pm(2a-\lambda)\frac{1}{2}(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}$. Then $A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n)$.

There is numerical evidence to suggest that the condition " $k \geq 2\lambda - 1$ " is unnecessary; but, the proof given here strongly relies on this condition. Actually the case $k = \lambda = a = 2$ violates this condition; however, Theorem 1 becomes Schur's theorem in this case.

In §2, we shall develop results concerning auxiliary partition functions. In §3, we shall prove Theorem 1; our work will be based on the generalized well-poised basic hypergeometric functions studied in [3]. In §4, we prove a more general theorem than Theorem 1 in which the parameter a is allowed to take values in the interval $(\frac{1}{2}\lambda, k]$. In §5, we discuss possibilities for further work.

2. Auxiliary partition functions. Let $\pi(k, \lambda, t, j, \mu; n)$ denote the number of partitions of n of the form $n = \sum_{i=1}^{2\lambda+1} f_i \cdot i$ (here f_i is the number of times the summand i appears) where

$$(2.1) \quad f_1 + \cdots + f_{\lambda+1} = k - \lambda + j;$$

$$(2.2) \quad f_{\lambda+1} + \cdots + f_{2\lambda+1} = k - t;$$

$$(2.3) \quad f_m + \cdots + f_{m+\lambda} \geq k \text{ for some } m;$$

$$(2.4) \quad f_a > 1 \text{ implies } a = \lambda + 1;$$

$$(2.5) \quad f_1 + \cdots + f_{2\lambda+1} = k + \mu.$$

Throughout our discussion we shall assume k, λ, t, j , and μ are all nonnegative integers with $j \leq \lambda$, $k \geq \max(1, 2\lambda - 1)$, and $\pi(k, \lambda, t, j, \mu; 0) = 0$.

Let $g(k, \lambda, t, j, \mu; q) = \sum_{n=0}^{\infty} \pi(k, \lambda, t, j, \mu; n) q^n$.

LEMMA 2.1.

$$(2.6) \quad g(k, \lambda, t, j, \mu; q) = q^{(k-t)(\lambda+1)} \sigma_{\mu}(\lambda) \sigma_{\lambda+\mu+t-j}(\lambda),$$

where

$$\begin{aligned} \sigma_i(\lambda) &= q^{i(i+1)/2} \begin{bmatrix} \lambda \\ i \end{bmatrix}, \\ &= q^{i(i+1)/2} \frac{(q^{\lambda}-1) \cdots (q^{\lambda-i+1}-1)}{(q^i-1) \cdots (q-1)}, & 0 \leq i \leq \lambda, \\ &= 0, & \text{otherwise;} \end{aligned}$$

thus $\sigma_i(\lambda)$ is just the generating function for partitions into exactly i distinct parts with all parts $\leq \lambda$.

Proof. We proceed by induction on the second argument of g , namely λ . If $\lambda = 0$, then since $j \leq \lambda$, $j = 0$. The conditions (2.1)–(2.5) imply $f_1 = k + j = k - t$

$=k+\mu \geq k$; thus $j=t=\mu=0$. Thus the only allowable partition is of k into k "ones". Therefore

$$g(k, 0, t, j, \mu; q) = q^k \quad \text{if } \mu = j = t = 0, \\ = 0 \quad \text{otherwise.}$$

On the other hand, consider the right-hand side of (2.6); call it $G(k, 0, t, j, \mu; q)$. The only way $G(k, 0, t, j, \mu; q)$ will be nonzero is if $\mu=0$, $t=j$, and since $j \leq \lambda$ we have $t=j=0$. Hence

$$G(k, 0, t, j, \mu; q) = q^k \quad \text{if } \mu = j = t = 0, \\ = 0 \quad \text{otherwise.}$$

Thus the lemma holds for $\lambda=0$.

If $\mu=0$, there are just k parts appearing in the partition by (2.5), and since (2.3) must hold we see that there exists a particular m such that

$$f_m + \cdots + f_{m+\lambda} = k, \quad f_i = 0 \quad \text{for } i < m \text{ or } i > m + \lambda.$$

If we set $A=f_1 + \cdots + f_\lambda$, $B=f_{\lambda+2} + \cdots + f_{2\lambda+1}$, then (2.1), (2.2), and (2.5) imply $A=t+\mu$, $f_{\lambda+1}=k+j-\lambda-t-\mu$, $B=\lambda+\mu-j$; thus in case $\mu=0$, $A=t$, $f_{\lambda+1}=k+j-\lambda-t$, $B=\lambda-j$. Let us now consider any partition of the type described in this paragraph. If we subtract $\lambda+1$ from each of the $\lambda-j$ parts greater than $\lambda+1$, we get a partition with $\lambda+1$ appearing $k+j-\lambda-t$ times with $\lambda+t-j$ remaining parts which are distinct and are all $\leq \lambda$. Conversely if we have a given partition in which $\lambda+1$ appears $k+j-\lambda-t$ times and there are $\lambda+t-j$ remaining parts which are distinct and are $\leq \lambda$, we define m to be the integer such that $f_1 + \cdots + f_{m-1} = \lambda-j$, and we add $\lambda+1$ to each part $< m$; this gives us precisely the type of partition described at the beginning of this paragraph. Hence by the definition of $\sigma_j(\lambda)$,

$$g(k, \lambda, t, j, 0; q) = q^{(k+j-\lambda-t)(\lambda+1) + (\lambda-j)(\lambda+1)} \sigma_{\lambda+t-j}(\lambda) \\ = q^{(k-t)(\lambda+1)} \sigma_{\lambda+t-j}(\lambda) \\ = G(k, \lambda, t, j, 0; q),$$

and (2.6) is verified in this case.

We now proceed by induction on λ (>0), and we may also assume $\mu>0$. Our first task is to verify

$$(2.7) \quad \begin{aligned} \pi(k, \lambda, t, j, \mu; n) = & \pi(k, \lambda-1, t, j-1, \mu; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j, \mu; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j-2, \mu-1; n+j-k-2\mu-\lambda) \\ & + \pi(k, \lambda-1, t, j-1, \mu-1; n+j-k-2\mu-\lambda). \end{aligned}$$

To verify (2.7) we proceed as follows. Let us consider a partition of n of the type enumerated by $\pi(k, \lambda, t, j, \mu; n)$. Let us subtract 1 from every summand $\leq \lambda+1$, and let us subtract 2 from each summand $> \lambda+1$. The number being partitioned now is $n-(k+j-\lambda)-2(\lambda+\mu-j)=n+j-k-2\mu-\lambda$. Four cases are to be

considered depending on: (1) $f_1 = f_{\lambda+2} = 0$; (2) $f_1 = 0, f_{\lambda+2} = 1$; (3) $f_1 = 1, f_{\lambda+2} = 0$; (4) $f_1 = f_{\lambda+2} = 1$.

We let f'_i denote the number of appearances of i in the transformed partition.

Thus for example in case (1), conditions (2.1)–(2.5) now read

$$(2.1)_1 \quad f'_1 + \cdots + f'_\lambda = k - (\lambda - 1) + j - 1;$$

$$(2.2)_1 \quad f'_\lambda + \cdots + f'_{2\lambda-1} = k - t;$$

$$(2.3)_1 \quad f'_m + \cdots + f'_{m+\lambda-1} \geq k \text{ for some } m;$$

$$(2.4)_1 \quad f'_a > 1 \text{ implies } a = \lambda;$$

$$(2.5)_1 \quad f'_1 + \cdots + f'_{2\lambda-1} = k + \mu.$$

Thus we have a partition of the type enumerated by

$$\pi(k, \lambda - 1, t, j - 1, \mu; n + j - k - 2\mu - \lambda).$$

Indeed the above procedure establishes a one-to-one correspondence between those partitions enumerated by $\pi(k, \lambda, t, j, \mu; n)$ which fall in case (1) and the partitions enumerated by $\pi(k, \lambda - 1, t, j - 1, \mu; n + j - k - 2\mu - \lambda)$. Similarly in the other three cases we get the other three partition functions appearing on the right-hand side of equation (2.7). Thus (2.7) is established.

(2.7) implies

$$(2.8) \quad \begin{aligned} g(k, \lambda, t, j, \mu; q) = & q^{k+\lambda+2\mu-j} (g(k, \lambda - 1, t, j - 1, \mu; q) + g(k, \lambda - 1, t, j, \mu; q) \\ & + g(k, \lambda - 1, t, j - 2, \mu - 1; q) \\ & + g(k, \lambda - 1, t, j - 1, \mu - 1; q)). \end{aligned}$$

Let us now examine the right-hand side of (2.6) which we have called $G(k, \lambda, t, j, \mu; q)$. Now

$$(2.9) \quad \sigma_j(\lambda) = q^j \sigma_j(\lambda - 1) + q^j \sigma_{j-1}(\lambda - 1);$$

this is merely a restatement of the well-known formula for Gaussian polynomials

$$(2.10) \quad \begin{bmatrix} \lambda \\ j \end{bmatrix} = \begin{bmatrix} \lambda - 1 \\ j - 1 \end{bmatrix} + q^j \begin{bmatrix} \lambda - 1 \\ j \end{bmatrix}.$$

Consequently

$$(2.11) \quad \begin{aligned} G(k, \lambda, t, j, \mu; q) &= q^{(k-t)(\lambda+1)} \sigma_\mu(\lambda) \sigma_{\lambda+\mu+t-j}(\lambda) \\ &= q^{k-t+\mu+\lambda+\mu+t-j} q^{(k-t)\lambda} (\sigma_\mu(\lambda - 1) + \sigma_{\mu-1}(\lambda - 1)) \\ &\quad \times (\sigma_{\lambda+\mu+t-j}(\lambda - 1) + \sigma_{\lambda+\mu+t-j-1}(\lambda - 1)) \\ &= q^{k+\lambda+2\mu-j} (G(k, \lambda - 1, t, j - 1, \mu; q) + G(k, \lambda - 1, t, j, \mu; q) \\ &\quad + G(k, \lambda - 1, t, j - 2, \mu - 1; q) \\ &\quad + G(k, \lambda - 1, t, j - 1, \mu - 1; q)). \end{aligned}$$

Now we know that $g(k, \lambda, t, j, \mu; q)$ and $G(k, \lambda, t, j, \mu; q)$ are identical if either λ or μ is zero. Thus since they both satisfy the recurrence (2.8), we have by induction on λ that they are equal in general. Thus Lemma 2.1 is established.

LEMMA 2.2. Let $\pi_1(k, \lambda, t, j; m; n)$ denote the number of partitions of n into m parts of the form $n = \sum_{i=1}^{2\lambda+1} f_i \cdot i$, where

$$(2.12) \quad f_1 + \cdots + f_{\lambda+1} \leq k - \lambda + j;$$

$$(2.13) \quad f_{\lambda+1} + \cdots + f_{2\lambda+1} = k - t;$$

$$(2.14) \quad f_v + \cdots + f_{v+\lambda} \geq k \text{ for some } v;$$

$$(2.15) \quad f_a > 1 \text{ implies } a = \lambda + 1.$$

Let

$$g_1(k, \lambda, t, j; x; q) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^m q^n \pi_1(k, \lambda, t, j; m; n).$$

Then

$$g_1(k, \lambda, t, j; x; q) = x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^{\mu} \sigma_{\mu}(\lambda) \sum_{r=0}^{j-\mu-t} \sigma_{\lambda-r}(\lambda).$$

Proof. By the definitions of π and π_1 , it is clear that

$$\pi_1(k, \lambda, t, j; k + \mu; n) - \pi_1(k, \lambda, t, j-1; k + \mu; n) = \pi(k, \lambda, t, j, \mu; n).$$

Hence

$$\begin{aligned} g_1(k, \lambda, t, j; x; q) - g_1(k, \lambda, t, j-1; x; q) &= \sum_{\mu=0}^{\infty} x^{k+\mu} g(k, \lambda, t, j, \mu; q) \\ &= \sum_{\mu=0}^{\infty} x^{k+\mu} q^{(k-t)(\lambda+1)} \sigma_{\mu}(\lambda) \sigma_{\lambda+\mu+t-j}(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} g_1(k, \lambda, t, j; x; q) &= \sum_{r=\lambda-k}^j \sum_{\mu=0}^{\infty} x^{k+\mu} q^{(k-t)(\lambda+1)} \sigma_{\mu}(\lambda) \sigma_{\lambda+\mu+t-r}(\lambda) \\ &= x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^{\mu} \sigma_{\mu}(\lambda) \sum_{r=\mu+t}^j \sigma_{\lambda+\mu+t-r}(\lambda) \\ &= x^k q^{(k-t)(\lambda+1)} \sum_{\mu=0}^{\lambda} x^{\mu} \sigma_{\mu}(\lambda) \sum_{r=0}^{j-\mu-t} \sigma_{\lambda-r}(\lambda). \end{aligned}$$

To conclude this section we prove a result concerning the multiplication of generating functions.

LEMMA 2.3. Let $p_1(A; c; m, n)$ denote the number of partitions of n into m parts all $< c$ where the summands are subject to a set of conditions, A . Let $p_2(B; c; m, n)$ denote the number of partitions of n into m parts all $\geq c$ where the summands are subject to a set of conditions, B . Let

$$\begin{aligned} f_1(x, q) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_1(A; c; m, n) x^m q^n, \\ f_2(x, q) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_2(B; c; m, n) x^m q^n, \end{aligned}$$

where we assume that $f_i(x, q)$ is absolutely convergent for $|x| < b_i$, $|q| < 1$.

Then for $|x| < \min(b_1, b_2)$, $|q| < 1$, if we let

$$f_1(x, q)f_2(x, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_3(A, B; c; m, n)x^m q^n,$$

then $p_3(A, B; c; m, n)$ is the number of partitions of n into m parts such that the summands $< c$ are subject to condition A and the summands $\geq c$ are subject to condition B .

Proof The result follows directly from

$$p_3(A, B; c; m, n) = \sum_{m_1+m_2=m; n_1+n_2=n} p_1(A; c; m_1, n_1)p_2(B; c; m_2, n_2).$$

3. Proof of Theorem 1. We begin by studying certain q -series considered in [3]. As in §2, we require that $k \geq \max(1, 2\lambda - 1)$. We define

$$(3.1) \quad J_{\lambda, k, i}(x) = J_{\lambda, k, i}(x; q) = J_{k, i}(-q, -q^2, \dots, -q^\lambda; x; q^{\lambda+1});$$

$$(3.2) \quad H_{\lambda, k, i}(x) = H_{\lambda, k, i}(x; q) = H_{k, i}(-q, -q^2, \dots, -q^\lambda; xq^{\lambda+1}; q^{\lambda+1}),$$

where (as in [3])

$$(3.3) \quad \begin{aligned} & J_{k, i}(a_1, \dots, a_\lambda; x; q) \\ &= \sum_{n=0}^{\infty} (-1)^{n(\lambda+1)} x^{kn} (a_1 a_2 \dots a_\lambda)^{-n} q^{(2k-\lambda+1)n^2/2 + (\lambda+1)n/2 + (k-i)n} \\ & \cdot \prod_{r=1}^n (1-q^r)^{-1} \prod_{u=n+1}^{\infty} (1-xq^u)^{-1} \prod_{s=1}^{\lambda} \left(\prod_{v=0}^{n-1} (1-a_s q^v) \prod_{w=n+1}^{\infty} \left(1 - \frac{xq^w}{a_s} \right) \right) \\ & \cdot \left\{ 1 + \frac{(-1)^{\lambda+1} x^i q^{(2n+1)i - \lambda n} (1-a_1 q^n)(1-a_2 q^n) \dots (1-a_\lambda q^n)}{a_1 a_2 \dots a_\lambda (1-xq^{n+1} a_1^{-1})(1-xq^{n+1} a_2^{-1}) \dots (1-xq^{n+1} a_\lambda^{-1})} \right\}; \end{aligned}$$

$$(3.4) \quad \begin{aligned} & H_{k, i}(a_1, \dots, a_\lambda; x; q) \\ &= \sum_{n=0}^{\infty} (-1)^{n(\lambda+1)} x^{kn} (a_1 a_2 \dots a_\lambda)^{-n} q^{(2k-\lambda+1)n^2/2 + (\lambda+1)n/2 - in} (1-x^i q^{2ni}) \\ & \cdot \prod_{r=1}^n (1-q^r)^{-1} \prod_{u=n}^{\infty} (1-xq^u)^{-1} \prod_{s=1}^{\lambda} \left(\prod_{v=0}^{n-1} (1-a_s q^v) \prod_{w=n+1}^{\infty} \left(1 - \frac{xq^w}{a_s} \right) \right). \end{aligned}$$

Then from [3, equation (2.1)], we have

$$(3.5) \quad H_{\lambda, k, i}(x) - H_{\lambda, k, i-1}(x) = (xq^{\lambda+1})^i J_{\lambda, k, k-i+1}(xq^{\lambda+1});$$

from [3, equation (2.2)], we have

$$(3.6) \quad J_{\lambda, k, i}(x) = \sum_{f=0}^{\lambda} x^f \sigma_f(\lambda) H_{\lambda, k, i-f}(x);$$

from [3, equation (2.3)], we have

$$(3.7) \quad H_{\lambda, k, -i}(x) = -(xq^{\lambda+1})^{-i} H_{\lambda, k, i}(x),$$

and from [3, equation (2.8)], we have

$$(3.8) \quad H_{\lambda,k,1}(x) = J_{\lambda,k,k}(xq^{\lambda+1}) = J_{\lambda,k,k+1}(xq^{\lambda+1}).$$

LEMMA 3.1. Let $|q| < 1$, and let $H_{\lambda,k,i}^*(x)$ ($0 \leq i \leq k$) and $J_{\lambda,k,i}^*(x)$ ($1 \leq i \leq k$) be any collection of functions analytic in x in the neighborhood of zero which satisfy

$$(3.9) \quad H_{\lambda,k,i}^*(x) - H_{\lambda,k,i-1}^*(x) = (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}^*(xq^{\lambda+1}), \quad 1 \leq i \leq k;$$

$$(3.10) \quad H_{\lambda,k,0}^*(x) = 0;$$

$$(3.11) \quad J_{\lambda,k,i}^*(x) = \sum_{j=0}^{\min(i,\lambda)} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}^*(x) - x^i \sum_{j=\min(i,\lambda)+1}^{\lambda} q^{(\lambda+1)(i-j)} \sigma_j(\lambda) H_{\lambda,k,j-i}^*(x);$$

$$(3.12) \quad H_{\lambda,k,i}^*(0) = J_{\lambda,k,i}^*(0) = 1, \quad 1 \leq i \leq k.$$

Then $H_{\lambda,k,1}^*(x) = H_{\lambda,k,1}(x)$, and $J_{\lambda,k,i}^*(x) = J_{\lambda,k,i}(x)$.

Proof. We need only show that solutions of the above equations are unique since (3.3)–(3.8) imply that the $H_{\lambda,k,i}(x)$ and $J_{\lambda,k,i}(x)$ fulfill the conditions of the lemma. If we can show that the $H_{\lambda,k,i}^*(x)$ are uniquely determined, then since the $J_{\lambda,k,i}^*(x)$ are defined in terms of the $H_{\lambda,k,i}^*(x)$ by (3.11), the $J_{\lambda,k,i}^*(x)$ will be uniquely determined also.

From (3.9) and (3.11) we have

$$(3.13) \quad \begin{aligned} & H_{\lambda,k,i}^*(x) - H_{\lambda,k,i-1}^*(x) \\ &= (xq^{\lambda+1})^{i-1} \sum_{j=0}^{\min(k-i+1,\lambda)} x^j q^{j(\lambda+1)} \sigma_j(\lambda) H_{\lambda,k,k-i+1-j}^*(xq^{\lambda+1}) \\ &\quad - (xq^{\lambda+1})^{k-i+1} \sum_{j=\min(k-i+1,\lambda)+1}^{\lambda} q^{(\lambda+1)(k-i+1-j)} \sigma_j(\lambda) H_{\lambda,k,j-k+i-1}^*(xq^{\lambda+1}). \end{aligned}$$

Hence if $H_{\lambda,k,i}^*(x) = \sum_{n=0}^{\infty} \eta_n(i) x^n$, then by (3.10) and (3.12)

$$\begin{aligned} \eta_0(i) &= 1, \quad 1 \leq i \leq k, \\ &= 0, \quad i = 0. \end{aligned}$$

From (3.13),

$$\begin{aligned} \eta_n(1) - \eta_n(k) q^{n(\lambda+1)} &= \text{polynomial in } \eta_j \text{'s with } j < n, \\ \eta_n(2) - \eta_n(1) &= \text{polynomial in } \eta_j \text{'s with } j < n, \\ &\vdots \\ \eta_n(k) - \eta_n(k-1) &= \text{polynomial in } \eta_j \text{'s with } j < n. \end{aligned}$$

We consider the above equations as a system of k equations in the k unknowns $\eta_n(1), \eta_n(2), \dots, \eta_n(k)$. The determinant of the system is $1 - q^{n(\lambda+1)} \neq 0$ since $|q| < 1$. Hence the $\eta_n(i)$ are uniquely determined by the η_j 's with $j < n$. Hence by mathematical induction the $\eta_n(i)$ are unique, and therefore the $H_{\lambda,k,i}^*(x)$ are unique. Thus Lemma 3.1 is proved.

LEMMA 3.2. Suppose $|q| < 1$, $k \geq \max(1, 2\lambda - 1)$. Let $H_{\lambda,k,i}^\#(x)$, $0 \leq i \leq k$ and $J_{\lambda,k,i}^\#(x)$, $1 \leq i \leq k$ be any collection of functions analytic in x in the neighborhood of zero which satisfy

$$(3.14) \quad H_{\lambda,k,i}^\#(x) - H_{\lambda,k,i-1}^\#(x) = (xq^{\lambda+1})^{i-1} J_{\lambda,k,k-i+1}^\#(xq^{\lambda+1}), \quad 1 \leq i \leq k;$$

$$(3.15) \quad H_{\lambda,k,0}^\#(x) = 0;$$

$$(3.16) \quad J_{\lambda,k,i}^\#(x) = \sum_{j=0}^{\min(i,\lambda)} x^j \sigma_j(\lambda) H_{\lambda,k,i-j}^\#(x), \quad 1 \leq i \leq k - \lambda + 1;$$

$$(3.17) \quad \begin{aligned} J_{\lambda,k,k-\lambda+1+j}^\#(x) &= \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^\#(x) \\ &\quad - \sum_{r=0}^j g_1(k, \lambda, r, j; x; q) H_{\lambda,k,r}^\#(xq^{\lambda+1}), \quad 1 \leq j < \lambda, \end{aligned}$$

where $g_1(k, \lambda, r, j; x; q)$ is defined in Lemma 2.2.

$$(3.18) \quad H_{\lambda,k,i}^\#(0) = 1, \quad 1 \leq i \leq k; \quad J_{\lambda,k,i}^\#(0) = 1, \quad 1 \leq i \leq k.$$

Then if we define

$$(3.19) \quad \begin{aligned} H_{\lambda,k,i}^*(x) &= H_{\lambda,k,i}^\#(x), \quad 0 \leq i \leq k - \lambda + 1, \\ &= H_{\lambda,k,i}^\#(x) - \sum_{r=1}^j H_{\lambda,k,r}^\#(xq^{\lambda+1}) x^k q^{(\lambda+1)(k-r)} \sum_{m=0}^{j-r} \sigma_{\lambda-m}(\lambda), \\ &\quad i = k - \lambda + 1 + j, \quad 1 \leq j < \lambda; \end{aligned}$$

$$(3.20) \quad J_{\lambda,k,i}^*(x) = x^{-k+i} (H_{\lambda,k,k-i+1}^*(xq^{-\lambda-1}) - H_{\lambda,k,k-i}^*(xq^{-\lambda-1})),$$

then $H_{\lambda,k,i}^*(x)$ and $J_{\lambda,k,i}^*(x)$ fulfill the equations of Lemma 3.1 and for $\lambda \leq i \leq k$, $J_{\lambda,k,i}^*(x) = J_{\lambda,k,i}^\#(x)$.

Proof. Obviously (3.9) and (3.10) are fulfilled from (3.20) and (3.15). Clearly for $\lambda \leq i \leq k$,

$$\begin{aligned} J_{\lambda,k,i}^*(x) &= x^{-k+i} (H_{\lambda,k,k-i+1}^*(xq^{-\lambda-1}) - H_{\lambda,k,k-i}^*(xq^{-\lambda-1})) \\ &= x^{-k+i} (H_{\lambda,k,k-i+1}^\#(xq^{-\lambda-1}) - H_{\lambda,k,k-i}^\#(xq^{-\lambda-1})) \\ &= J_{\lambda,k,i}^\#(x). \end{aligned}$$

Now if $\lambda \leq i \leq k - \lambda + 1$, (3.19) and (3.16) show that (3.11) is valid. If $i = k - \lambda + 1 + j$, $1 \leq j < \lambda$, then

$$\begin{aligned}
 J_{\lambda,k,k-\lambda+1+j}^*(x) &= J_{\lambda,k,k-\lambda+1+j}^\#(x) \\
 &= \sum_{r=0}^{j-1} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^\#(x) + \sum_{r=j}^{\lambda} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^\#(x) \\
 &\quad - \sum_{r=0}^j g_1(k, \lambda, r, j; x; q) H_{\lambda,k,r}^\#(xq^{\lambda+1}) \\
 &= \sum_{r=0}^{j-1} x^r \sigma_r(\lambda) \left(H_{\lambda,k,k-\lambda+1+j-r}^*(x) + \sum_{v=1}^{j-r} H_{\lambda,k,v}^*(xq^{\lambda+1}) (xq^{\lambda+1})^k q^{-v(\lambda+1)} \right. \\
 &\quad \left. \sum_{w=0}^{j-r-v} \sigma_{\lambda-w}(\lambda) \right) \\
 &\quad + \sum_{r=j}^{\lambda} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^*(x) \\
 &\quad - \sum_{r=0}^j g_1(k, \lambda, r, j; x; q) H_{\lambda,k,r}^*(xq^{\lambda+1}) \\
 &= \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^*(x) \\
 &\quad + \sum_{v=1}^j H_{\lambda,k,r}^*(xq^{\lambda+1}) x^k q^{(k-v)(\lambda+1)} \sum_{r=0}^{j-v} x^r \sigma_r(\lambda) \sum_{w=0}^{j-r-v} \sigma_{\lambda-w}(\lambda) \\
 &\quad - \sum_{r=0}^j g_1(k, \lambda, r, j; x; q) H_{\lambda,k,r}^*(xq^{\lambda+1}) \\
 &= \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H_{\lambda,k,k-\lambda+1+j-r}^*(x),
 \end{aligned}$$

where the last equation follows by Lemma 2.2; thus again we have (3.11).

Finally, if $1 \leq i < \lambda$, then

$$\begin{aligned}
 J_{\lambda,k,i}^*(x) &= x^{-k+i} \left(H_{\lambda,k,k-i+1}^\#(xq^{-\lambda-1}) - \sum_{v=1}^{\lambda-i} H_{\lambda,k,v}^\#(x) x^k q^{-v(\lambda+1)} \sum_{w=0}^{\lambda-i-v} \sigma_{\lambda-w}(\lambda) \right. \\
 &\quad \left. - H_{\lambda,k,k-i}^\#(xq^{-\lambda-1}) + \sum_{v=1}^{\lambda-i-1} H_{\lambda,k,v}^\#(x) x^k q^{-v(\lambda+1)} \sum_{w=0}^{\lambda-i-1-v} \sigma_{\lambda-w}(\lambda) \right) \\
 &= J_{\lambda,k,i}^\#(x) - x^i \sum_{v=1}^{\lambda-i} H_{\lambda,k,v}^\#(x) q^{-v(\lambda+1)} \sigma_{i+v}(\lambda) \\
 &= \sum_{j=0}^i x^j \sigma_j(\lambda) H_{\lambda,k,i-j}^\#(x) - x^i \sum_{v=i+1}^{\lambda} H_{\lambda,k,v-i}^\#(x) q^{(i-v)(\lambda+1)} \sigma_v(\lambda) \\
 &= \sum_{j=0}^i x^j \sigma_j(\lambda) H_{\lambda,k,i-j}^*(x) - x^i \sum_{v=i+1}^{\lambda} H_{\lambda,k,v-i}^*(x) q^{(i-v)(\lambda+1)} \sigma_v(\lambda),
 \end{aligned}$$

which again is (3.11).

Now from (3.19) it is clear that $H_{\lambda,k,i}^*(0) = 1$, $1 \leq i \leq k$, and since $J_{\lambda,k,i}^*(x)$ satisfies (3.11), $J_{\lambda,k,i}^*(0) = 1$, $1 \leq i \leq k$. The analytic nature of these functions is now obvious from (3.19) and (3.11). Thus we have Lemma 3.2.

COROLLARY 3.3. *The set of functions described in the hypothesis of Lemma 3.2 is unique and exists.*

Proof. By Lemma 3.1, the $H_{\lambda,k,i}^*(x)$ and $J_{\lambda,k,i}^*(x)$ defined in Lemma 3.2 are unique, and thus by (3.19), (3.14), and (3.16) so must $H_{\lambda,k,i}^\#(x)$ and $J_{\lambda,k,i}^\#(x)$ be. For existence take $H_{\lambda,k,i}^\#(x) = H_{\lambda,k,i}(x)$, $0 \leq i \leq k - \lambda + 1$,

$$H_{\lambda,k,i}^\#(x) = H_{\lambda,k,i}(x) + \sum_{r=1}^j H_{\lambda,k,r}(xq^{\lambda+1})x^kq^{(\lambda+1)(k-r)} \sum_{m=0}^{j-r} \sigma_{\lambda-m}(\lambda),$$

$$i = k - \lambda + 1 + j, \quad 1 \leq j < \lambda,$$

$$J_{\lambda,k,i}^\#(x) = J_{\lambda,k,i}(x), \quad \lambda \leq i \leq k,$$

$$J_{\lambda,k,i}^\#(x) = \sum_{j=0}^i x^j \sigma_j(\lambda) H_{\lambda,k,i-j}(x), \quad 1 \leq i < \lambda.$$

Thus we have the corollary.

We are now in a position to establish our main result.

Proof of Theorem 1. Let $P_{\lambda,k,a}(m, n)$, $1 \leq a \leq k$, denote the number of partitions of n into m parts of the form $n = \sum_{i=1}^{\infty} f_i \cdot i$, where $f_i + \dots + f_{i+\lambda} \leq k-1$, $f_{i(\lambda+1)} + \dots + f_{(i+1)(\lambda+1)} \leq k-1$, $f_1 + \dots + f_{\lambda+1} \leq a-1$, $f_b > 1$ then $\lambda+1|b$ (equivalently, $n = b_1 + \dots + b_m$, $b_i \geq b_{i+1}$, $b_i - b_{i+k-1} \geq \lambda+1$ with strict inequality if $\lambda+1|b_i$, the total number of parts $\leq \lambda+1$ does not exceed $a-1$, and only multiples of $\lambda+1$ may be repeated).

Let $Q_{\lambda,k,a}(m, n)$ denote the number of partitions described above in which $f_1 = f_2 = \dots = f_\lambda = 0$ (i.e., the smallest part is $\geq \lambda+1$).

Let

$$J'_{\lambda,k,a}(x) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{\lambda,k,a}(m, n) x^m q^n, \quad 1 \leq a \leq k;$$

$$H'_{\lambda,k,a}(x) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{\lambda,k,a}(m, n) x^m q^n, \quad 1 \leq a \leq k;$$

$$H'_{\lambda,k,0}(x) = 0.$$

By comparison with the generating function for the number of ordinary partitions of n into m parts, we see that we have convergence for $|q| < 1$, $|x| < |q|^{-1}$; thus, for given $|q| < 1$, the above are analytic in a circle of radius > 1 around $x=0$.

Also $H'_{\lambda,k,a}(0) = J'_{\lambda,k,a}(0) = 1$, $1 \leq a \leq k$.

Our next object is to show that these functions satisfy (3.14)–(3.17).

Beginning with (3.14), we consider $Q_{\lambda,k,i}(m, n) - Q_{\lambda,k,i-1}(m, n)$; this expression enumerates those partitions considered by $Q_{\lambda,k,i}(m, n)$ in which $\lambda+1$ appears

exactly $i-1$ times. Let us now subtract $\lambda+1$ from each part in such a partition; the number being partitioned is reduced to $n-m(\lambda+1)$, and the number of parts appearing is now $m-i+1$. Since we had originally

$$f_{\lambda+2} + \cdots + f_{2\lambda+2} = k-1-f_{\lambda+1} = (k-i+1)-1,$$

we have after subtraction

$$f_1 + \cdots + f_{\lambda+1} = (k-i+1)-1.$$

Otherwise the conditions on the summands are just shifted but unaltered. Thus we are now considering a partition of the type enumerated by

$$P_{\lambda, k, k-i+1}(m-i+1, n-m(\lambda+1)).$$

The above procedure establishes a one-to-one correspondence between the partitions enumerated by $Q_{\lambda, k, i}(m, n) - Q_{\lambda, k, i-1}(m, n)$ and those enumerated by $P_{\lambda, k, k-i+1}(m-i+1, n-m(\lambda+1))$. Hence

$$Q_{\lambda, k, i}(m, n) - Q_{\lambda, k, i-1}(m, n) = P_{\lambda, k, k-i+1}(m-i+1, n-m(\lambda+1)).$$

In terms of generating functions, this implies

$$H'_{\lambda, k, i}(x) - H'_{\lambda, k, i-1}(x) = (xq^{\lambda+1})^{i-1} J'_{\lambda, k, k-i+1}(xq^{\lambda+1}),$$

which is (3.14).

(3.15) is by definition.

We now treat (3.16). I claim that for $1 \leq i \leq k-\lambda+1$

$$(3.21) \quad x^j \sigma_j(\lambda) H'_{\lambda, k, i-j}(x), \quad 0 \leq j \leq \min(i, \lambda),$$

is the generating function for the number of partitions of the type enumerated by $P_{\lambda, k, i}(m, n)$ in which exactly j parts $\leq \lambda$ appear.

By Lemma 2.3, (3.21) is the generating function for partitions in which there are j distinct parts $\leq \lambda$ and the parts $\geq \lambda+1$ satisfy the desired conditions. Is it possible then that any of the conditions on the partitions enumerated by $P_{\lambda, k, i}(m, n)$ could be violated? This is not possible because for $1 \leq v \leq \lambda$

$$f_v + \cdots + f_{\lambda} \leq j; \quad f_{\lambda+1} \leq i-j-1; \quad f_{\lambda+2} + \cdots + f_{\lambda+v} \leq v-1;$$

hence

$$f_v + \cdots + f_{v+\lambda} \leq j+i-j-1+v-1 = i+v-2.$$

If $v=1$, this is $f_1 + \cdots + f_{\lambda+1} \leq i-1$ as desired, and if $1 < v \leq \lambda$, this is $f_v + \cdots + f_{v+\lambda} \leq i+v-2 \leq k-\lambda+1+\lambda-2 = k-1$ as desired.

Conversely it is easily seen that any partition of the type enumerated by $P_{\lambda, k, i}(m, n)$ with exactly j parts $\leq \lambda$ is of the type considered by (3.21).

Thus clearly

$$\begin{aligned} J'_{\lambda,k,i}(x) &= \sum_{j=0}^i x^j \sigma_j(\lambda) H'_{\lambda,k,i-j}(x) \\ &= \sum_{j=0}^{\min(i,\lambda)} x^j \sigma_j(\lambda) H'_{\lambda,k,i-j}(x), \quad 1 \leq i \leq k-\lambda+1, \end{aligned}$$

which is (3.16).

For (3.17) we note that the above procedure must be modified since now in considering (3.21) with $i > k-\lambda+1$ it is indeed possible that there exists v with $1 < v \leq \lambda$, such that $f_v + \cdots + f_{v+\lambda} \geq k$.

Indeed by the preceding argument for $1 \leq j < \lambda$

$$(3.22) \quad \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H'_{\lambda,k,k-\lambda+1+j-r}(x) - J'_{\lambda,k,k-\lambda+1+j}(x)$$

is the generating function for partitions of the type enumerated by

$$P_{\lambda,k,k-\lambda+1+j}(m, n)$$

except for the fact that for some v , $1 < v \leq \lambda$, $f_v + \cdots + f_{v+\lambda} \geq k$.

On the other hand, let us consider

$$(3.23) \quad g_1(k, \lambda, r, j; x; q) H'_{\lambda,k,r}(xq^{\lambda+1}), \quad 0 \leq r < \lambda.$$

I claim that this is the generating function for partitions of the type described in the preceding paragraph in which $f_{\lambda+1} + \cdots + f_{2\lambda+1} = k-r$.

By Lemma 2.3, Lemma 2.2 (and the fact that $H'_{\lambda,k,r}(xq^{\lambda+1})$ is the generating function for partitions of the type enumerated by $P_{\lambda,k,r}(m, n)$ with the added restriction that $f_1 = f_2 = \cdots = f_{2\lambda+1} = 0$, $f_{2\lambda+2} \leq r-1$), we see that (3.23) is the generating function for partitions of n into m parts of the form $n = \sum_{i=1}^{\infty} f_i \cdot i$, where

$$(3.24) \quad f_1 + \cdots + f_{\lambda+1} \leq k - \lambda + j;$$

$$(3.25) \quad f_{\lambda+1} + \cdots + f_{2\lambda+1} = k - r;$$

$$(3.26) \quad f_v + \cdots + f_{v+\lambda} \geq k \quad \text{for some } v, \quad 1 < v \leq \lambda;$$

$$(3.27) \quad f_a > 1 \quad \text{implies } \lambda+1 \mid a;$$

$$(3.28) \quad f_m + \cdots + f_{m+\lambda} \leq k-1, \quad \text{for } m \geq 2\lambda+2;$$

$$(3.29) \quad f_{m(\lambda+1)} + \cdots + f_{(m+1)(\lambda+1)} \leq k-1, \quad m \geq 2;$$

$$(3.30) \quad f_{2\lambda+2} \leq r-1.$$

We shall establish the claim made for (3.23) if we can show that the following two conditions hold for the partitions considered.

$$(3.31) \quad f_m + \cdots + f_{m+\lambda} \leq k-1, \quad \text{for } \lambda+1 < m < 2\lambda+2;$$

$$(3.32) \quad f_{\lambda+1} + \cdots + f_{2\lambda+2} \leq k-1.$$

Now (3.32) is clear from (3.25) and (3.30). As for (3.31) we have

$$f_m + \cdots + f_{2\lambda+1} + f_{2\lambda+3} + \cdots + f_{m+\lambda} \leq \lambda; \quad f_{2\lambda+2} \leq r-1;$$

therefore

$$f_m + \cdots + f_{m+\lambda} \leq \lambda + r - 1 \leq \lambda + \lambda - 2 = 2\lambda - 2 \leq k - 1.$$

Thus (3.23) is the desired generating function.

Consequently for $1 \leq j < \lambda$,

$$\begin{aligned} \sum_{r=0}^{\lambda} x^r \sigma_r(\lambda) H'_{\lambda,k,k-\lambda+1+j-r}(x) - J'_{\lambda,k,k-\lambda+1+j}(x) \\ = \sum_{r=0}^{\lambda-1} g_1(k, \lambda, r, j; x; q) H'_{\lambda,k,r}(xq^{\lambda+1}) \\ = \sum_{r=0}^j g_1(k, \lambda, r, j; x; q) H'_{\lambda,k,r}(xq^{\lambda+1}), \end{aligned}$$

which is (3.17); the second to last equation is summed to $\lambda-1$ only because (3.25) and (3.24) contradict (3.26) if $r \geq \lambda$.

Thus by Corollary 3.3 and its proof,

$$(3.33) \quad J'_{\lambda,k,i}(x) = J_{\lambda,k,i}(x), \quad \lambda \leq i \leq k.$$

Finally

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} B_{\lambda,k,i}(n) q^n \\ = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P_{\lambda,k,i}(m, n) q^n \\ = J_{\lambda,k,i}(1) \\ = \prod_{j=1}^{\infty} (1+q^j)(1+q^{j(\lambda+1)})^{-1} (1-q^{j(\lambda+1)})^{-1} (1-q^{(2k-\lambda+1)(\lambda+1)j - (i-\frac{1}{2}\lambda)(\lambda+1)}) \\ \cdot (1-q^{(2k-\lambda+1)(\lambda+1)(j-1) + (i-\frac{1}{2}\lambda)(\lambda+1)}) (1-q^{(2k-\lambda+1)(\lambda+1)j}) \\ = \begin{cases} \prod_{\substack{j=1 \\ j \neq 0 \pmod{\lambda+1}}}^{\infty} (1+q^j) \prod_{\substack{s=0 \\ s \equiv 0 \pmod{\lambda+1} \\ s \neq 0, \pm(i-\frac{1}{2}\lambda)(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}}}^{\infty} (1-q^s)^{-1}, & \lambda \text{ even} \\ \prod_{\substack{j=1 \\ j \neq 0 \pmod{\frac{1}{2}(\lambda+1)}}}^{\infty} (1+q^j) \prod_{\substack{s=0 \\ s \equiv 0, \frac{1}{2}(\lambda+1), \frac{1}{2}(\lambda+1) \pmod{2\lambda+2} \\ s \neq 0, \pm(2i-\lambda)\frac{1}{2}(\lambda+1) \pmod{(2k-\lambda+1)(\lambda+1)}}}^{\infty} (1-q^s)^{-1}, & \lambda \text{ odd} \end{cases} \\ = 1 + \sum_{n=1}^{\infty} A_{\lambda,k,i}(n) q^n, \end{aligned}$$

where the third to last equation follows from [3, equation (2.9)]. Hence $A_{\lambda,k,i}(n) = B_{\lambda,k,i}(n)$ and Theorem 1 is established.

4. Extension of Theorem 1. We extend our definition of $B_{\lambda,k,a}(n)$ as follows. Let $B_{\lambda,k,a}^*(n)$ denote those partitions of n enumerated by $B_{\lambda,k,a}(n)$ such that for any integer j in the interval $[1, \frac{1}{2}(\lambda+1)]$, there must be at most $a-j$ parts of the partition in the interval $[j, \lambda-j+1]$ (in terms of frequency notation the new condition is $f_j + \cdots + f_{\lambda-j+1} \leq a-j$, $1 \leq j \leq \frac{1}{2}(\lambda+1)$). We define $P_{\lambda,k,a}^*(m, n)$ similarly.

We remark that for $a \geq \lambda$, $B_{\lambda,k,a}^*(n) = B_{\lambda,k,a}(n)$. This is because the new conditions are automatically fulfilled for $a \geq \lambda$, since the number of summands in the interval $[j, \lambda-j+1]$ is $\leq \min(a-1, \lambda-2j+2) \leq \min(a-1, a-2j+2) \leq a-j$.

In Theorem 1 we note that $A_{\lambda,k,a}(n)$ is well defined for $\frac{1}{2}\lambda < a \leq k$. We now have the following extension of Theorem 1.

THEOREM 2. $A_{\lambda,k,a}(n) = B_{\lambda,k,a}^*(n)$ for $\frac{1}{2}\lambda < a \leq k$, $k \geq 2\lambda - 1$.

This theorem will be easy to deduce from Theorem 1 after we digress briefly for a study of a new auxiliary partition function.

Let $\varphi(i, j, \lambda; n)$, $\frac{1}{2}\lambda < i \leq \lambda$, $i \leq j \leq \lambda$, be the number of partitions of n into $2i-j$ distinct parts of the form $n = \sum_{e=1}^{\lambda} f_e \cdot e$ where *at least one* of the following inequalities holds

$$(4.1)_b \quad f_b + \cdots + f_{\lambda-b+1} > i-b, \quad 1 \leq b \leq \frac{1}{2}(\lambda+1).$$

Let $\psi(i, j, \lambda; q) = \sum_{n=0}^{\infty} \varphi(i, j, \lambda; n) q^n$.

LEMMA 4.1. For $\frac{1}{2}\lambda < i \leq \lambda$, $i \leq j \leq \lambda$,

$$(4.2) \quad \psi(i, j, \lambda; q) = q^{(i-j)(\lambda+1)} \sigma_j(\lambda).$$

Proof. Now if $i=j$, then $2i-j=i$. Therefore $f_1 + \cdots + f_{\lambda} = i > i-1$. Hence $\psi(i, i, \lambda; q)$ is just the generating function for partitions into i distinct parts all $\leq \lambda$. Thus

$$(4.3) \quad \psi(i, i, \lambda; q) = \sigma_i(\lambda).$$

We now wish to show

$$(4.4) \quad \psi(i+1, j, \lambda; q) = q^{\lambda+1} \psi(i, j, \lambda; q);$$

(4.4) together with (4.3) imply (4.2) by mathematical induction.

We proceed as follows. For a given partition, π , of the type enumerated by $\varphi(i, j, \lambda; n)$, define

$$(4.5) \quad m(\pi) = \max_{1 \leq b \leq \frac{1}{2}(\lambda+1)} f_b + \cdots + f_{\lambda-b+1} - i + b;$$

$$(4.6) \quad U(\pi) = \min \{b \mid f_b + \cdots + f_{\lambda-b+1} - i + b = m(\pi)\};$$

$$(4.7) \quad L(\pi) = \max \{b \mid f_b + \cdots + f_{\lambda-b+1} - i + b = m(\pi)\}.$$

We now note that

$$(4.8) \quad f_{U(\pi)-1} = f_{\lambda-U(\pi)+2} = 0; \quad f_{L(\pi)} = f_{\lambda-L(\pi)+1} = 1;$$

(4.8) is correct because the falsity of the first assertion would contradict the minimality of $U(\pi)$, and the falsity of the second assertion would contradict the maximality of $L(\pi)$ (note that $U(\pi) > 1$ for $j > i$ because $i-1 \geq 2i-j = f_1 + \cdots + f_\lambda$ in contrast with the fact that (4.1)_b is valid for some b).

We now prove (4.4). If π is a partition enumerated by $\varphi(i, j, \lambda; n)$ delete from π the summands $L(\pi)$ and $\lambda - L(\pi) + 1$, and call the new partition π' . π' is a partition of $n - \lambda - 1$ with $2(i-1) - j$ parts.

$$(4.9) \quad U(\pi') = L(\pi) + 1;$$

this is because if $b < L(\pi) + 1$, then in π' we have

$$f'_b + \cdots + f'_{\lambda-b+1} - (i-1) + b = f_b + \cdots + f_{\lambda-b+1} - 2 - i + 1 + b \leq m(\pi) - 1,$$

while for $b \geq L(\pi) + 1$

$$f'_b + \cdots + f'_{\lambda-b+1} - (i-1) + b = f_b + \cdots + f_{\lambda-b+1} - i + 1 + b \leq m(\pi),$$

and

$$\begin{aligned} f'_{L(\pi)+1} + \cdots + f'_{\lambda-L(\pi)} - (i-1) + L(\pi) + 1 \\ = f_{L(\pi)} + \cdots + f_{\lambda-L(\pi)+1} - 2 - i + 1 + L(\pi) + 1 \\ = m(\pi). \end{aligned}$$

Now since originally $m(\pi) > 0$, we see that $m(\pi) = m(\pi') > 0$, and therefore π' is a partition of the type enumerated by $\varphi(i-1, j, \lambda; n - \lambda - 1)$.

Conversely if π' is a partition of the type enumerated by $\varphi(i-1, j, \lambda; n - \lambda - 1)$ then introduce the summands $U(\pi') - 1$, and $\lambda - U(\pi') + 2$ and call the new partition π'' . π'' is a partition of n into $2i - j$ parts.

$$(4.10) \quad L(\pi'') = U(\pi') - 1;$$

this is because if $b > U(\pi') - 1$

$$\begin{aligned} f''_b + \cdots + f''_{\lambda-b+1} - i + b &= f'_b + \cdots + f'_{\lambda-b+1} - (i-1) + b - 1 \\ &\leq m(\pi') - 1, \end{aligned}$$

while for $b \leq U(\pi') - 1$

$$\begin{aligned} f''_b + \cdots + f''_{\lambda-b+1} - i + b &= f'_b + \cdots + f'_{\lambda-b+1} - (i-1) + b + 2 - 1 \\ &\leq m(\pi') - 1 + 2 - 1 = m(\pi'), \end{aligned}$$

and

$$\begin{aligned} f''_{U(\pi')-1} + \cdots + f''_{\lambda-U(\pi')+2} - i + U(\pi') - 1 \\ = f'_{U(\pi')} + \cdots + f'_{\lambda-U(\pi')+1} + 2 - (i-1) + U(\pi') - 2 \\ = m(\pi'). \end{aligned}$$

Thus we see that the above procedure establishes a one-to-one correspondence between the partitions enumerated by $\varphi(i, j, \lambda; n)$ and those enumerated by $\varphi(i-1, j, \lambda; n - \lambda - 1)$. Therefore

$$(4.11) \quad \varphi(i+1, j, \lambda; n) = \varphi(i, j, \lambda; n - \lambda - 1).$$

We deduce (4.4) immediately from (4.11). Thus we have established Lemma 4.1.

Proof of Theorem 2. We have already noted that Theorem 2 is just Theorem 1 if $a \geq \lambda$. We thus need only treat $\frac{1}{2}\lambda < a < \lambda$. Under these conditions we let $J''_{\lambda,k,a}(x)$ be the generating function for $P^*_{\lambda,k,a}(m, n)$. Then as in the proof of Theorem 1,

$$\begin{aligned} J''_{\lambda,k,a}(x) &= J'_{\lambda,k,a}(x) - \sum_{j=a}^{\lambda} x^{2a-j} \psi(a, j, \lambda; q) H_{\lambda,k,j-a}(x) \\ (4.12) \quad &= \sum_{j=0}^a x^j \sigma_j(\lambda) H_{\lambda,k,a-j}(x) - \sum_{j=a+1}^{\lambda} x^{2a-j} \psi(a, j, \lambda; q) H_{\lambda,k,j-a}(x) \end{aligned}$$

for the first term on the right-hand side (from the proof of Theorem 1) generates the partitions enumerated by $P_{\lambda,k,a}(m, n)$ and the second term, by the definition of ψ and Lemma 2.3, just removes those partitions enumerated by $P_{\lambda,k,a}(m, n)$ which are not enumerated by $P^*_{\lambda,k,a}(m, n)$.

Now on the other hand from (3.6) and (3.7)

$$(4.13) \quad J_{\lambda,k,a}(x) = \sum_{j=0}^a x^j \sigma_j(\lambda) H_{\lambda,k,a-j}(x) - x^a \sum_{j=a+1}^{\lambda} q^{(a-j)(\lambda+1)} \sigma_j(\lambda) H_{\lambda,k,j-a}(x).$$

Consequently by Lemma 4.1

$$(4.14) \quad J''_{\lambda,k,a}(1) = J_{\lambda,k,a}(1).$$

The remainder of the proof is now identical with that of Theorem 1.

5. Conclusion. Let us define $K(\lambda)$ to be that integer such that for $k \geq K(\lambda)$ Theorem 2 (and consequently Theorem 1) holds.

CONJECTURE. $K(\lambda) = \lambda$.

It is fairly easy to show that $K(2) = 2$, and indeed numerical evidence indicates the truth of $K(\lambda) = \lambda$ for $\lambda = 3, 4, 5$. The main problem in proving the conjecture lies in the fact that the q -difference equations become much more involved for smaller values of k and the related partition identities become even more cumbersome.

A further interesting question seems to be whether or not even smaller values of k may be taken provided new restrictions are put on the partitions enumerated by $B_{\lambda,k,a}(n)$. As an example, Schur has proved [6, p. 495] that

$$A_{3,2,2}(n) = B_{3,2,2}^0(n)$$

where $B_{3,2,2}^0(n)$ denotes the number of partitions enumerated by $B_{3,2,2}(n)$ with the added restriction that no parts are $\equiv 2 \pmod{4}$.

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