THE COHOMOLOGY OF THE COMPLEX PROJECTIVE STIEFEL MANIFOLD

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0. Let U_n be the unitary group of order n . We have inclusions

$$\cdots \supset U_n \supset U_{n-1} \supset \cdots \supset U_1 \approx S^1.$$

We denote by $W_{n,n-m}$ the complex Stiefel manifold $W_{n,n-m} = U_n/U_m$ and by $Y'_{n,n-m}$ the complex projective Stiefel manifold which we defined as follows: S^1 , regarded as the set of complex numbers of module 1, acts by multiplication on U_n . This action being compatible with the above inclusions defines an action of S^1 on $W_{n,n-m}$ and we define $Y'_{n,n-m}$ as the set of orbits.

In particular we have

$$\begin{aligned} W_{n,1} &= U_n/U_{n-1} \approx S^{2n-1}, \\ W_{n,n} &= U_n, \\ Y'_{n,n} &= PU_n, \text{ the projective unitary group.} \end{aligned}$$

In this paper we compute $(H^* Y'_{n,k})$. Baum and Browder [1] have obtained our result in the special case $n=p^r$, $m=0$.

In order to state our main result we need some notation: Let ω be the generator of $H^*(CP^\infty) = \mathbb{Z}[\omega]$ and z_i the generators of $H^*(W_{n,k}) = \bigwedge (z_{m+1}, \dots, z_n)$. Let $b_i = \text{G.C.D.}((C_{n,m+1}, \dots, C_{n,i}))$. Finally in §1, we will show there is a fibration

$$(B) \quad W_{n,n-m} \xrightarrow[i]{} Y_{n,n-m} \xrightarrow{\pi} CP^\infty$$

with $Y_{n,n-m}$ of the same homotopy type as $Y'_{n,n-m}$. Then our main theorem is

THEOREM A.

$$H^*(Y_{n,n-m}) = \mathbb{Z}[y]/I \otimes \bigwedge (v_{m+2}, \dots, v_n), \text{ where } \pi^* \omega = y;$$

$i^* v_i = (b_{i-1}/b_i) z_i$ and I is the ideal generated by $b_i y^i$, $i = m+1, \dots, n$.

In §1 we compute $H^*(Y_{n,k}; \mathbb{Z}_p)$ and $H^*(Y_{n,k}; \mathbb{Q})$ following the Gitler and Handel proof for real case [3]. In §2 we determine $\text{Ker } \pi^*$ and $\text{Im } i^*$. In §3 we show that this information is enough to determine all relevant Bockstein homomorphisms

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and compute the Bockstein spectral sequence for every prime p . This determines completely $H^*(Y_{n,k})$.

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1. First we construct a space $Y_{n,k}$ of the same homotopy type as $Y'_{n,k}$.

Observe that we have a principal bundle $S^1 \rightarrow W_{n,n-m} \rightarrow Y'_{n,n-m}$.

Let ξ be the associate line bundle.

PROPOSITION 1. $n\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ has $n-m$ $\mathbf{C}$ -linearly independent sections and it is the universal bundle for bundles $n\zeta$ having $n-m$ $\mathbf{C}$ -linearly independent sections, where ζ is a line bundle.

Proof. The proof is similar to the real case given in [3].

The inclusion $U_m \subset U_n$ gives rise to a fibration of classifying spaces

$$(A) \quad W_{n,n-m} \xrightarrow{i_1} BU_m \xrightarrow{\pi_1} BU_n$$

and the transgression satisfies

$$(1) \quad \tau z_i = \sigma_i \quad \text{where } \sigma_i \text{ is the universal Chern class.}$$

Now let γ be the canonical line bundle over CP^∞ and let f_n be the classifying map of $n\gamma$ and let

$$(B) \quad W_{n,n-m} \xrightarrow{i} Y_{n,n-m} \xrightarrow{\pi} CP^\infty$$

be the fibration induced from (A) by f_n .

It is easy to see that the bundle induced by $f_n \pi$ is a universal bundle for the n -plane bundles satisfying the conditions of Proposition 1. Thus we have

PROPOSITION 2. $Y_{n,n-m}$ and $Y'_{n,n-m}$ have the same homotopy type.

From (1) and naturality of Chern classes and transgression

$$(2) \quad \tau z_i = C_{n,i} \omega^i.$$

If $x \in H^*(E)$ we denote by $\bar{x}$ (resp. $\tilde{x}$) its image in $H^*(E; \mathbf{Z}_p)$ (resp. $H^*(E; \mathbf{Q})$).

Let $N(p)$ be the smallest i such that $m+1 \leq i < n$ and $C_{n,i}$ is not zero, mod p . The following theorem is similar to [3, Theorem 1.6].

THEOREM 3.

$$H^*(Y_{n,n-m}; \mathbf{Z}_p) = \mathbf{Z}_p[\bar{y}]/[y^{N(p)}] \otimes \bigwedge (\bar{x}_{m+1} \cdots \widehat{\bar{x}_{N(p)}} \cdots \bar{x}_n),$$

$$H^*(Y_{n,n-m}; \mathbf{Q}) = \mathbf{Q}[y]/[y^{m+1}] \otimes \bigwedge (\tilde{x}_{m+2} \cdots \tilde{x}_n),$$

where p is a prime and $\mathbf{Q}$ is the set of rational numbers; $i^*(\bar{x}_i) = \bar{z}_i$; $\pi^*(\bar{\omega}) = \bar{y}$; and $i^*(\tilde{x}_i) = \tilde{z}_i$, $\pi^*(\tilde{\omega}) = \tilde{y}$.

Proof. Let K be a field, $K = \mathbb{Z}_p$ or $K = \mathbb{Q}$ and let N be the smallest i such that $m+1 \leq i < n$ and $C_{n,i}$ is not zero in K .

In the spectral sequence of (B) with coefficients K we have

$$E_2 = K[\omega] \otimes \bigwedge (z_{m+1} \cdots z_n).$$

By (2) $d_r z_i = 0$, $r < 2i$, $d_{2i} z_i = \tau z_i = C_{n,i} \omega^i$, then $d_r = 0$, $r < 2N$ and then $E_2 = E_{2N}$; but

$$d_{2N} z_N = C_{n,N} \omega^N \neq 0,$$

thus, z_N does not survive in E_{2N+1} and the image of the ideal $[\omega^N]$ in E_{2N+1} is 0. Moreover, E_{2N+1} is still a tensor product:

$$E_{2N+1}^{*,0} \otimes E_{2N+1}^{0,*} = E_{2N+1}^{*,*}.$$

Now the following transgressions are zero (thus all the following differentials are zero), hence

$$E_\infty = E_{2N+1} = K[\omega]/[\omega^N] \otimes \bigwedge (z_{m+1} \cdots \hat{z}_N \cdots z_n).$$

The theorem now follows from Borel [2].

COROLLARY 4. $H^*(Y_{n,n-m})$ has p -torsion if and only if p divides $C_{n,m+1}$.

2. In this section we obtain the first results about $H^*(Y_{n,k})$. The key is Proposition 5 below on $\text{Ker } \pi^*$. In Corollary 7 we pick some elements in $H^*(Y_{n,k})$ and using them we choose new generators for $H^*(Y_{n,k}; \mathbb{Z}_p)$ and $H^*(Y_{n,k}; \mathbb{Q})$ ((7), (7')).

PROPOSITION 5.

$$\text{Ker } \pi^* = [b_{m+1} \omega^{m+1}; \dots, b_n \omega^n].$$

COROLLARY 6.

$$\text{Ker } \bar{\pi}^* = [b_{m+1} \omega^{m+2}; \dots, b_n \omega^{n+1}].$$

Let $c_i = b_{i-1}/b_i$, $i = m+2, \dots, n$.

COROLLARY 7.

$$\begin{aligned} T^{2m+1}/\text{Im } i^* &= \mathbb{Z}, & T^q/\text{Im } i^* &= 0, & q &\geq 2n, \\ T^{2i-1}/\text{Im } i^* &= \mathbb{Z}_{c_i}, & m+2 &\leq i \leq n. \end{aligned}$$

Thus, there exist elements v_i in $H^*(Y_{n,n-m})$ such that $i^* v_i = c_i z_i$, $m+2 \leq i \leq n$.

Before we give the proof we recall some facts about transgression in the spectral sequence of a fibration.

We use the transgression in (B) as defined in the following diagram, $\tau = j^* \circ (\bar{\pi}^*)^{-1} \circ \delta$

$$\begin{array}{ccccccc} H^*(Y_{n,n-m}) & \xrightarrow{i^*} & H^*(W_{n,n-m}) & \xrightarrow{\delta} & H^*(Y_{n,n-m}, W_{n,n-m}) & \longrightarrow & H^*(Y_{n,n-m}) \\ & & & & \uparrow \bar{\pi}^* & & \uparrow \pi^* \\ & & & & H^*(CP^\infty, *) & \xrightarrow{j^*} & H^*(CP^\infty). \end{array}$$

We recall that in the spectral sequence of (B) $E_2^{0,2i-1} \approx H^{2i-1}(W_{n,n-m})$. This isomorphism carries the subgroup $E_{2i}^{0,2i-1}$ onto T^{2i-1} , the subgroup of transgressive elements. Also $E_2^{2i,0} \approx H^{2i}(CP^\infty)$ and this isomorphism induces the isomorphism of quotient groups $E_{2i}^{2i,0} \approx H^{2i}(CP^\infty)/\text{Ker}^{2i} \bar{\pi}^*$. Moreover, via these isomorphisms, τ corresponds to $d_{2i}^{0,2i-1}$ and

$$(3) \quad \text{Im } d_{2i}^{0,2i-1} \approx \text{Ker}^{2i} \pi^* / \text{Ker}^{2i} \bar{\pi}^* \subset H^{2i}(CP^\infty) / \text{Ker}^{2i} \bar{\pi}^*.$$

Finally, τ induces an isomorphism

$$(4) \quad T^{q-1} / \text{Im } i^* \approx \text{Ker}^q \pi^* / \text{Ker}^q \bar{\pi}^*.$$

Consider the diagram:

$$(C) \quad \begin{array}{ccccc} H^*(W_{n,n-m}) & \xleftarrow{i^*} & H^*(Y_{n,n-m}) & \xleftarrow{\pi^*} & H^*(CP^\infty) \\ \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\ H^*(W_{n,n-m}; A) & \xleftarrow{i_A^*} & H^*(Y_{n,n-m}; A) & \xleftarrow{\pi_A^*} & H^*(CP^\infty; A). \end{array}$$

If $A = \mathbb{Z}_p$, then Theorem 2 and $i^* \circ \pi^* = 0$ yield

$$(5) \quad \text{Ker } i_{\mathbb{Z}_p}^* = [\bar{y}],$$

$$(6) \quad \text{Ker } \pi_{\mathbb{Z}_p}^* = [\omega^{N(p)}].$$

If $A = \mathbb{Q}$, we have

$$(5') \quad \text{Ker } i_{\mathbb{Q}}^* = [\tilde{y}],$$

$$(6') \quad \text{Ker } \pi_{\mathbb{Q}}^* = [\tilde{\omega}^{m+1}].$$

Proof of Proposition 5. The spectral sequence of (B) is trivial through E_{2m+2} . Thus $E_{2m+2}^{2m+2,0} \approx H^{2m+2}(CP^\infty)$ and $\text{Ker}^q \bar{\pi}^* = 0$, $q \leq 2m+2$.

From (2) and (3) we obtain

$$\text{Ker}^q \pi^* = 0, \quad q < 2m+2 \quad \text{and} \quad \text{Ker}^{2m+2} \pi^* = [C_{n,m+1} \omega^{m+1}]^{2m+2}.$$

Applying (2) repeatedly we have

$$[b_{m+1} \omega^{m+1}, \dots, b_n \omega^n] \subset \text{Ker } \pi^* \subset H^*(CP^\infty).$$

For the other inclusion, put $b_i = ap^r$, where p does not divide a . We use diagram (C) with $A = \mathbb{Z}_{p^r}$, if $p^s c \omega^i$ belongs to $\text{Ker } \pi^*$, $s < r$ and c divides a , then $p^s c \theta(\omega)^i$ belongs to $\text{Ker } \pi_{\mathbb{Z}_{p^r}}^*$ but it is not 0 because c is not a divisor of 0 in $\mathbb{Z}_{p^r}$. On the other hand in the spectral sequence of (B) with coefficients $\mathbb{Z}_{p^r}$

$$\tau z_k = C_{n,k} \omega^k = 0, \quad k < i,$$

because p^r divides $C_{n,k}$ for those k . Thus, the spectral sequence is trivial through $2i$ and then $\text{Ker}^{2i} \pi_{\mathbb{Z}_{p^r}}^* = 0$.

Proof of Corollary 6. Follows from (3) and Proposition 5.

Proof of Corollary 7. First part follows from (4), second part follows trivially from first part.

The elements v_i are not unique, we will choose a fixed set of such elements arbitrarily.

In diagram (C) with $Z_p = A$, we have:

(i) if p does not divide c_i , $\theta v_i = c_i \bar{x}_i + u_i$, $m+2 \leq i \leq n$ where $u_i \in \text{Ker } i_{Z_p}^*$;

(ii) if p does divide c_i , then $\theta v_i \in \text{Ker } i_{Z_p}^*$.

Let $I = \{i; p \text{ does not divide } c_i, m+2 \leq i \leq n\}$. Let $J = \{j; j \notin I, m+1 \leq j \leq n\}$. Then $I = \{i; b_i \text{ is divided by the same power of } p \text{ as } b_{i-1}\}$.

The important situation occurs when i belongs to J . For example, $m+1$ and $N(p)$ are the smallest and the greatest elements of J .

We will change the generators of $H^*(Y_{n,n-m}; Z_p)$ to the following

$$\begin{aligned} c_i x_i &= \theta v_i, & i \in I, \\ x_i &= \bar{x}_i, & i \in J, i \neq N(p); \end{aligned}$$

then we obtain:

$$H^*(Y_{n,n-m}; Z_p) = Z_p[\bar{y}]/[\bar{y}^{N(p)}] \otimes \bigwedge (x_{m+1}, \dots, \hat{x}_{N(p)}, \dots, x_n)$$

where

$$(7) \quad \theta v_i = c_i x_i, \quad i \in I; \quad i_{Z_p}^* x_i = \bar{z}_i, \quad m+1 \leq i \leq n, \quad i \neq N(p) \quad \text{and} \quad \pi_{Z_p}^*(\bar{\omega}) = \bar{y}.$$

Again in diagram (C), this time with $A = \mathbb{Q}$, we have

$$\theta v_i = c_i \tilde{x}_i + \tilde{u}_i, \quad \tilde{u}_i \text{ belongs to } \text{Ker } i_{\mathbb{Q}}^*, \quad m+2 \leq i \leq n.$$

We define $c_i w_i = \theta v_i$, $m+2 \leq i \leq n$ and we obtain

$$(7') \quad \begin{aligned} H^*(Y_{n,n-m}; \mathbb{Q}) &= \mathbb{Q}[\bar{y}]/[\bar{y}^{m+1}] \otimes \bigwedge (w_{m+2}, \dots, w_n) \\ \text{where } \theta v_i &= c_i w_i, \quad i_{\mathbb{Q}}^* w_i = \bar{z}_i, \quad \pi_{\mathbb{Q}}^*(\bar{\omega}) = \bar{y}. \end{aligned}$$

3. Next, we compute the Bockstein spectral sequence of the couple

$$\begin{array}{ccc} H^*(Y_{n,n-m}) & \xrightarrow{(\cdot p)^*} & H^*(Y_{n,n-m}) \\ \delta \swarrow & & \searrow \theta \\ & H^*(Y_{n,n-m}; Z_p) & \end{array}$$

It follows from (7) that

$$E_1 = H^*(Y_{n,n-m}; Z_p) = Z_p[y]/[y^N] \otimes \bigwedge (x_{m+1} \cdots \hat{x}_{N(p)} \cdots x_n)$$

and from (7') that

$$E_\infty = H^*(Y_{n,n-m})/\text{Torsion} \otimes Z_p = Z_p[y]/[y^{m+1}] \otimes \bigwedge (w_{m+2}, \dots, w_n).$$

Recall that the differentials are Bockstein homomorphisms β_r and an element $x \in E_1$, belongs to $\text{Im } \theta$ if and only if

$$(8) \quad \beta_r x = 0 \quad \text{for all } r.$$

An element $y \in H^*(Y_{n,n-m})$ has torsion p^r , that is $p^r a y = 0$ where p does not divide a , if and only if $\theta y \notin \text{Im } \beta_j$ for $j < r$, but

$$(9) \quad \theta y \in \text{Im } \beta_r.$$

First we will give some easy results:

If $x \in E_r$, call $\phi(x)$ its image in E_∞ , then

$$(10) \quad \phi(\bar{y}) = \bar{y}; \quad \phi(x_i) = w_i, \quad i \in I.$$

By (7) and (8)

$$(11) \quad \beta_r(\bar{y}) = 0, \quad \beta_r(x_i) = 0, \quad \text{all } r, i \in I.$$

By (10), since $w_i \neq 0$,

$$(12) \quad x_i \notin \text{Im } \beta_r, \quad \text{all } r, i \in I.$$

We arrange J so that $m+1 = i(0) < i(1) < \dots < i(j) < \dots < i(t) = N(p)$ and put $b_{i(j)} = p^{r(j)} a_j$, where p does not divide a_j ; then $r(j) > r(j+1)$ and $b_i = p^{r(j)} a_i$, $i(j) \leq i < i(j+1)$.

By (9) and Proposition 5:

$$(13) \quad y^i \notin \text{Im } \beta_r, \quad r < r(j), \quad y^i \in \text{Im } \beta_{r(j)}, \quad i(j) \leq i < i(j+1).$$

Trivially

$$(14) \quad E_1^q = E_\infty^q, \quad q < 2i(0) - 1.$$

Now, we will compute β_r :

LEMMA 8. *The following formulae hold for every j*

$$(15) \quad \beta_r x_{i(j)} = 0, \quad r < r(j),$$

$$(16) \quad \beta_{r(j)} x_{i(j)} = k_j y^{i(j)}, \quad k_j \in \mathbb{Z}_p, k_j \neq 0,$$

$$(17) \quad E_1^q = E_{r(j)}^q, \quad q < 2i(j+1) - 1.$$

Proof. By (13) there is an element x such that $\beta_{r(0)} x = y^{i(0)}$ but x can only be a scalar multiple of $x_{i(0)}$, then (15) and (16) hold for $j=0$.

By the same argument (15) and (16) hold for $j=h$ provided that (17) holds for $j=h-1$.

In turn, (15) for every $j \leq h$ and (11) together imply (17) for $j=h$ because Bockstein homomorphisms are derivations.

COROLLARY 9. *For every j*

$$(18) \quad \beta_{r(j)}(x_{i(j)} y^s) = k_j y^{i(j)+s} \neq 0, \quad 0 \leq s < i(j+1) - i(j),$$

$$(19) \quad \beta_{r(j)}(x_{i(j)} y^{i(j+1)-i(j)}) = 0.$$

Proof. (18) follows from (16) and (17). (19) follows from (16).

We call $u_{i(j+1)}$ the image of $x_{i(j)}y^{i(j+1)-i(j)}$ in $E_{r(j)+1}$.

It remains to prove that $\beta_r=0$ unless $r=r(j)$ for some j . This is part of the following lemma.

LEMMA 10. $\beta_r=0$ unless $r=r(j)$ and $E_\infty=E_r(o)$.

We use induction. Assign $\tilde{y}$ to $\bar{y}$ and w_i to x_i for $i \in I, i < i(1)$.

By (15), ..., (19) and $\dim E_\infty \leq E_{r(0)}$, this correspondence determines an isomorphism from E_∞ onto $E_{r(0)}$, up to degree $2i(1)-2$.

Moreover, $\beta_r=0$ up to degree $2i(1)-2$ unless $r=r(0)$.

Suppose we have elements $\bar{u}_{i(j)}, j=1, \dots, h$, such that:

- (i) $\text{gr } \bar{u}_{i(j)} = 2i(j) - 1$.
- (ii) $\bar{u}_{i(j)}x_i = -x_i\bar{u}_{i(j)}$; $\bar{u}_{i(j)}u_{i(j')} = -\bar{u}_{i(j')} \bar{u}_{i(j)}, j' < j$; $(\bar{u}_{i(j)})^2 = 0$.
- (iii) If we assign $\tilde{y}$ to $\bar{y}$; w_i to x_i for $i \in I, i < i(j+1)$ and $w_{i(j)}$ to $\bar{u}_{i(j)}$ we determine an isomorphism from E_∞ onto $E_{r(0)}$, up to degree $2i(h+1)-2$.

Suppose besides that $\beta_r=0$ up to degree $2i(h+1)-2$ unless $r=r(j), j=0, \dots, t-1$.

From these assumptions and (15), ..., (19) we have $\dim E_{r(0)}^q = \dim E_\infty^q, q \leq 2i(h+1)-1$ and all differentials are determined on all elements of degree $\leq 2i(h+1)-1$ except $u_i(h+1)$ belonging to $E_{r(h)+1}$ and its images in $E_r, r > r(h)+1$.

Thus, for every $r, \beta_r u_{i(h+1)}$ must lie in the subspace of E_r spanned by $\{\beta_r a\}$, where a ranges over products. That means $\beta_r u_i(h+1) = 0$ for $r < r(h-1)$; and there is an element u' in $E_{r(h-1)}^{2i(h+1)-1}$, such that $\beta_{r(h-1)} u' = 0$ and u' does not belong to the subalgebra generated by elements with degree $< 2i(h+1)-1$. It is easy to see that u' satisfies (ii). Again, $\beta_r u' = 0$ for $r < r(h-2)$ and we repeat the argument until we reach $E_{r(0)}$, then we obtain an element $\bar{u}_{i(h+1)}$ in $E_{r(0)}$ to which we may assign $w_{i(h+1)}$. Now we assign w_i to x_i for $i \in I, i < i(h+2)$ and obtain an isomorphism up to degree $2i(h+2)-1$. Then we have finished the proof of (i), (ii), (iii) with $j=h+1$. From (11) we see $\beta_r=0$ up to degree $2i(h+2)-1$, unless $r=r(j)$ for $j=0, \dots, t-1$. This completes the proof.

We have identified $H^*(Y_{n,n-m}; \mathbb{Q})$ with $E_{r(0)}$ as algebras, for every prime p . Then we have completed the proof of Theorem A.

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