

THE POTENTIAL KERNEL AND HITTING PROBABILITIES FOR THE GENERAL STABLE PROCESS IN R^N

BY

W. E. PRUITT⁽¹⁾ AND S. J. TAYLOR

1. **Introduction.** Let $X(t)$ be a stable process and $p(t, x)$ be the density of $X(t)$. For all the stable distributions of index α on the line, $|x|^{1+\alpha}p(1, x)$ is bounded [11]. This ensures that the hitting probabilities of spheres for these processes (as long as the processes are type A) are of the same order of magnitude as for the symmetric stable process of index α . In higher dimensions the situation is more complicated, and there has been little study of the asymptotic behaviour of the density function except in the symmetric case and in the case of Brownian motion. There is no problem for Brownian motion since the density function is known so we may as well assume that $\alpha < 2$. For the symmetric stable density of index α in R^N , $|x|^{N+\alpha}p(1, x)$ is bounded (and, in fact, tends to a limit as $|x| \rightarrow \infty$) [1]. To see that this is not the case in general consider the stable density of index α in R^2 given by

$$p_2(1, x) = p_1(1, x_1)p_1(1, x_2)$$

where $x = (x_1, x_2)$ and p_1 denotes the density of the symmetric stable process of index α in R^1 . If we let x tend to infinity along one of the axes, $p_2(1, x)$ decays like $|x|^{-1-\alpha}$, while if we go along any other ray from the origin it behaves like $|x|^{-2-2\alpha}$. This example extends to R^N and we shall prove that it illustrates the worst possible behaviour of the density $p(1, x)$ so that $|x|^{1+\alpha}p(1, x)$ is bounded for any stable process. (Our treatment of the stable processes of index one is restricted to those which satisfy the scaling property; see §2.)

One of our objectives is to obtain conditions on the form of the stable distribution which will ensure a decay rate for the density closer to that for the symmetric process. In terms of the measure μ in the usual representation of the characteristic function (see §2), we use a condition that μ is "not too concentrated" near any point to improve the estimate of the asymptotic behaviour. The form of this condition is fairly satisfactory for $N=2$, but it does not give enough information for $N \geq 3$. It seems plausible that, at least when the defining measure μ has a bounded density on the unit sphere, $|x|^{N+\alpha}p(1, x)$ should be bounded. Our method of proof does not get us anywhere near this result for general N . However, it does imply

Received by the editors March 6, 1969.

⁽¹⁾ Research was supported in part by NSF Grant GP-7490.

that for such a μ , $|x|^{2+\alpha-\varepsilon}p(1, x)$ is bounded for every fixed $\varepsilon > 0$, which is close to the conjecture when $N=2$. The proof of the main Theorem 1, as given in §3, is rather involved as it requires the use of N -dimensional Fourier transforms.

It was pointed out to us by S. Port that the alleged proof of the estimate (16) in [12] is invalid. The example we have already considered—the convolution of symmetric 1-dimensional distributions of index α on each axis—shows that the kernel $U(x)$ may take the value $+\infty$ for some x with $|x|=1$, if $\alpha \leq (N-1)/2$. It follows that Lemma 4 of [12] is false in general. Thus the hitting probabilities for spheres in dimension $N \geq 2$ are not of the same order of magnitude for all type A stable processes of index α . The main object of §5 of the present paper is to obtain valid estimates of these hitting probabilities. In order to do this, we first have to estimate the average value of $U(x)$ over a small neighbourhood; this is done in §4.

There are other consequences of Theorem 1. For example, it shows that if $\alpha > (N-1)/2$, then $U(x)$ is continuous on the unit sphere and it is then a consequence of the discussion in Port [8, p. 367] that his Theorem 2 is valid if the process is type A . This theorem concludes that the limit as the initial point tends to infinity of the conditional hitting distribution of a bounded Borel set exists and is the normalized cocapacitory measure of the set.

§6 contains a detailed discussion of the example where μ consists of point masses. This discussion shows that the results of §§3, 4, and 5 are best possible in general. An intuitive explanation of why the hitting probabilities are different for this example is also provided.

It is worth remarking that the inaccurate estimates of hitting probabilities in [12] do not invalidate the main results of that paper. The evaluation of the correct Hausdorff measure function for measuring the range of the process is accurate as one can use Lemma 6.1 of [10] to give an estimate of the number of cubes entered by the process. Furthermore, Theorems 2 and 3, stated without proof, are correct because it turns out that if we estimate the “delayed hitting probability” of a sphere, i.e. the probability of entering the sphere *after* unit time has elapsed, then this does have the same order of magnitude for all stable processes of type A and index α in R^N . This estimate of the “delayed hitting probability” is given in §7. Some remaining problems are discussed in §8.

2. Preliminaries. The N -dimensional characteristic function of a stable process $X(t)$ of index α has the form $\exp [t\psi(\xi)]$ where

$$\psi(\xi) = i(\xi, b) - c|\xi|^\alpha \int_{S_N} w_\alpha(\xi, \theta) \mu(d\theta),$$

with $b \in R^N$, $c > 0$,

$$w_\alpha(\xi, \theta) = [1 - i \operatorname{sgn}(\xi, \theta) \tan \pi\alpha/2] |(\xi/|\xi|, \theta)|^\alpha, \quad \alpha \neq 1,$$

$$w_1(\xi, \theta) = |(\xi/|\xi|, \theta)| + (2i/\pi)(\xi/|\xi|, \theta) \log |(\xi, \theta)|,$$

and μ is a probability measure on the surface of the unit sphere S_N in R^N [5]. We shall assume that μ is not supported by a proper subspace of R^N , and that $b=0$, $c=1$. The process is called the symmetric stable process of index α in R^N when μ is uniform.

The density function $p(t, x)$ of $X(t)$ is continuous and bounded in x for fixed t . It also satisfies the scaling property (except for some nonsymmetric processes of index 1)

$$(2.1) \quad p(t, x) = p(rt, r^{1/\alpha}x)r^{N/\alpha}$$

for all $r>0$, or in terms of the process itself, $X(rt)$ and $r^{1/\alpha}X(t)$ have the same distribution. The scaling property is satisfied when $\alpha=1$ if and only if the μ distribution satisfies

$$(2.2) \quad \int_{S_N} \theta \mu(d\theta) = 0.$$

We shall consider only those processes which satisfy (2.2) when $\alpha=1$. The stable processes have been classified in [12] as being of type *A* if $p(1, 0)>0$ and of type *B* otherwise. When $\alpha \geq 1$, only processes of type *A* can occur.

We assume that our process satisfies the conditions of Hunt's hypothesis (A) [3]. Thus the sample functions $X(t)$ will be right continuous and have left limits everywhere. The process will also have the strong Markov property. Since we only need to consider $N \geq 2$, and since there is no problem with planar Brownian motion we can and will assume $N>\alpha$. This allows us to use the ordinary potential kernel for the process. Since we have a continuous density $p(t, x)$, the kernel of the process has a density with respect to Lebesgue measure given by $u(x, y)=U(y-x)$ where

$$(2.3) \quad U(x) = \int_0^\infty p(t, x) dt,$$

and Hunt's hypothesis (F) is also satisfied. This means that, at least for compact sets E of positive natural capacity, there is a nonnull measure ν , concentrated on E , such that

$$(2.4) \quad \Phi(x, E) = P^x[X(t) \in E \text{ for some } t > 0] = \int_E u(x, y)\nu(dy).$$

The defining integral in (2.3) may diverge for some x , but it is well defined. We will see in §4 that (2.3) converges for a.e. x .

We shall use the notation

$$(2.5) \quad g(\xi) = \int_{S_N} w_\alpha(\xi, \theta)\mu(d\theta)$$

and note that for the processes being considered this is actually a function of $\xi/|\xi|$. Let the real valued function a_α be defined by

$$(2.6) \quad \begin{aligned} a_\alpha(y) &= y^\alpha, & \text{if } y \geq 0, \\ &= 0, & \text{if } y < 0, \end{aligned}$$

and let

$$(2.7) \quad \begin{aligned} g_1(\xi) &= \int_{S_N} a_\alpha((\xi/|\xi|, \theta)) \mu(d\theta), \\ g_2(\xi) &= \int_{S_N} a_\alpha(-(\xi/|\xi|, \theta)) \mu(d\theta). \end{aligned}$$

Then for $\alpha \neq 1$, we have

$$g(\xi) = \sec(\pi\alpha/2) \{g_1(\xi)e^{-i\pi\alpha/2} + g_2(\xi)e^{i\pi\alpha/2}\}.$$

Furthermore

$$\operatorname{Re} g(\xi) = g_1(\xi) + g_2(\xi) = \int_{S_N} |(\xi/|\xi|, \theta)|^\alpha \mu(d\theta).$$

Since we assumed that μ does not concentrate on a proper subspace of S_N , we have $\operatorname{Re} g(\xi) > 0$, and since it is continuous $\operatorname{Re} g(\xi)$ is bounded below by a positive constant γ . (Recall that g is actually a function on the compact S_N .) We also have $0 \leq g_1(\xi) \leq 1$, $0 \leq g_2(\xi) \leq 1$, so that all values of $g(\xi)$ are taken in the rectangular domain

$$D = \{\sigma = u + iv : \gamma \leq u \leq 1, |v| \leq \tan \pi\alpha/2\}.$$

It is important to note that while the rectangle D depends on μ through γ , γ (and hence D) is invariant under orthogonal transformations on μ . For $\alpha = 1$, a similar analysis with (assuming (2.2)) $g(\xi) = g_R(\xi) + ig_I(\xi)$ where

$$\begin{aligned} g_R(\xi) &= \int_{S_N} |(\xi/|\xi|, \theta)| \mu(d\theta), \\ g_I(\xi) &= (2/\pi) \int_{S_N} (\xi/|\xi|, \theta) \log |(\xi/|\xi|, \theta)| \mu(d\theta), \end{aligned}$$

shows that $g(\xi)$ takes its values in the rectangle

$$D = \{\sigma = u + iv : \gamma \leq u \leq 1, |v| \leq 2/\pi e\}.$$

The inversion formula allows us to write

$$p(1, x) = (2\pi)^{-N} \int_{R^N} \exp(-i(x, \xi) - |\xi|^\alpha g(\xi)) d\xi.$$

In estimating the asymptotic behaviour of the density, we are only interested (for the application to hitting probabilities) in letting x tend to infinity along rays from the origin and we want bounds which are independent of the particular direction of approach. Therefore there will be no loss in assuming that the direction we take to infinity is along the positive first coordinate axis since this can be achieved in any case by an orthogonal transformation on x , which can be converted to one on μ , and all the bounds obtained will be invariant under orthogonal

transformations on μ . We shall use the shorthand notation ρ for $(\rho, 0, 0, \dots, 0)$; then

$$p(1, \rho) = (2\pi)^{-N} \int_{R^N} \exp(-i\rho\xi_1 - |\xi|^\alpha g(\xi)) d\xi.$$

Making the usual change to spherical coordinates,

$$\xi_j = r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{j-1} \cos \varphi_j, \quad 1 \leq j \leq N-1,$$

$$\xi_N = r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{N-2} \sin \varphi_{N-1},$$

where $0 \leq r < \infty$, $0 \leq \varphi_j \leq \pi$ for $1 \leq j \leq N-2$, $0 \leq \varphi_{N-1} \leq 2\pi$, the Jacobian is

$$J(r, \varphi_1, \dots, \varphi_{N-1}) = r^{N-1} (\sin \varphi_1)^{N-2}, \dots, (\sin \varphi_{N-3})^2 \sin \varphi_{N-2},$$

and

$$(2.8) \quad p(1, \rho) = (2\pi)^{-N} \int_0^{2\pi} \dots \int_0^\pi \int_0^\infty \exp(-ir\rho \cos \varphi_1 - r^\alpha h(\varphi_1, \dots, \varphi_{N-1})) \\ J(r, \varphi_1, \dots, \varphi_{N-1}) dr d\varphi_1 \dots d\varphi_{N-1},$$

where we have let

$$(2.9) \quad h(\varphi_1, \dots, \varphi_{N-1}) = g(r \cos \varphi_1, \dots, r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{N-1}).$$

This is the formula which will prove most useful in estimating the asymptotic behaviour of $p(1, x)$.

We will use M , c , K to denote finite positive constants whose precise values are unimportant; these may differ when the same symbol occurs in distinct contexts.

3. Asymptotic bounds for the stable density. Our main aim is to give a proof of

THEOREM 1. *Suppose that for some $\beta \geq 0$, there is an M such that*

$$\int_{S_N} |(\xi, \theta)|^{-\beta} \mu(d\theta) \leq M$$

for all $\xi \in S_N$. Then there is a constant K such that $p(1, x) \leq K|x|^{-\alpha-\beta-1}$.

COROLLARY. *For any stable distribution of index α (satisfying (2.2) if $\alpha=1$) in R^N , $|x|^{1+\alpha}p(1, x)$ is bounded.*

The corollary follows immediately from the theorem since the hypothesis is trivially satisfied for $\beta=0$. Unfortunately the improvement in the bound on the rate of decay given by the theorem is limited since the hypothesis can only be satisfied if $\beta < 1$. In the planar case, this is not too important as we believe the result is close to best possible. We will see later what would be necessary in order to sharpen the method of proof to yield better results for $N \geq 3$. The proof of the theorem requires heavy computation so we break it into a sequence of lemmas.

The first lemma establishes a result about the order of magnitude of an integral involving Bessel functions. We give the proof because we have been unable to find

the exact result in the literature. However, the method of proof is that used to obtain the asymptotic behaviour of the symmetric stable density by Polya [7] in R^1 and Blumenthal and Gettoor [1] in R^N , $N \geq 2$. The notation and integration formulas for Bessel functions can be found in [2] or [13].

LEMMA 1. Let $J_\nu(y)$ denote the Bessel function of the first kind and let $\sigma \in D$. Then if $\rho > 0$, $\nu \geq 0$, $\beta + \nu > -1$,

$$\begin{aligned} \int_0^\infty J_\nu(r\rho) \exp(-\sigma r^\alpha) r^\beta dr &= O(\rho^{-\alpha-\beta-1}), \quad \text{if } \beta = \nu + 1, \\ &= O(\rho^{-\beta-1}), \quad \text{if } \beta \neq \nu + 1, \end{aligned}$$

the constants being uniform for $\sigma \in D$.

Proof. An integration by parts and change of variable yield

$$\begin{aligned} \int_0^\infty J_\nu(r\rho) \exp(-\sigma r^\alpha) r^\beta dr &= \rho^{-\alpha-\beta-1} \int_0^\infty J_{\nu+1}(s) \exp(-\sigma(s/\rho)^\alpha) \sigma \alpha s^{\alpha+\beta-1} ds \\ &\quad - (\beta - \nu - 1) \rho^{-\beta-1} \int_0^\infty J_{\nu+1}(s) \exp(-\sigma(s/\rho)^\alpha) s^{\beta-1} ds. \end{aligned}$$

Thus when $\beta = \nu + 1$, it only remains to show that the first integral on the right is bounded as $\rho \rightarrow \infty$. In order to do this, we introduce the Bessel function of the third kind $H_{\nu+1}^{(1)}(s) = J_{\nu+1}(s) + iY_{\nu+1}(s)$ and work separately with the real and imaginary parts of the integral. Letting $\sigma = u + iv$,

$$\begin{aligned} \operatorname{Re} \int_0^\infty J_{\nu+1}(s) \exp(-\sigma(s/\rho)^\alpha) \sigma \alpha s^{\alpha+\beta-1} ds \\ &= \int_0^\infty J_{\nu+1}(s) \exp(-u(s/\rho)^\alpha) \alpha \{u \cos[v(s/\rho)^\alpha] + v \sin[v(s/\rho)^\alpha]\} s^{\alpha+\beta-1} ds \\ &= \operatorname{Re} \int_0^\infty H_{\nu+1}^{(1)}(s) \exp(-u(s/\rho)^\alpha) \alpha \{u \cos[v(s/\rho)^\alpha] + v \sin[v(s/\rho)^\alpha]\} s^{\alpha+\beta-1} ds. \end{aligned}$$

Now choose a small positive angle φ_0 so that

$$\gamma \cos \alpha\varphi - \tan(\pi\alpha/2) \sin \alpha\varphi > 0$$

for all $\varphi \in [0, \varphi_0]$. (If $\alpha = 1$, $\tan \pi\alpha/2$ should be replaced by $2/\pi e$.) This choice of φ_0 ensures that

$$u \cos \alpha\varphi - |v| \sin \alpha\varphi > 0$$

for all $\varphi \in [0, \varphi_0]$ and $\sigma \in D$. By considering the usual contour integral, we see that we can change the integral to one along the ray $s \exp\{i\varphi_0\}$, $s \geq 0$. This gives the Bessel function a negative exponential behaviour so that the passage to the limit may be carried out. The imaginary part of the integral can be handled in exactly the same way.

For the other estimate (when $\beta \neq \nu + 1$), we start with a change of variable:

$$\int_0^\infty J_\nu(r\rho) \exp(-\sigma r^\alpha) r^\beta dr = \rho^{-\beta-1} \int_0^\infty J_\nu(s) \exp(-\sigma(s/\rho)^\alpha) s^\beta ds.$$

Again we must show that the integral is bounded, but if we proceed as before there are integrability difficulties near zero when $\beta \leq \nu - 1$. So we first observe that the integral from zero to one is bounded, independent of ρ and σ , since $|J_\nu(s)| \leq cs^\nu$. To handle the integral from one to infinity, we introduce $H_\nu^{(1)}(s)$ and work separately with the real and imaginary parts of the integral as before. With the same φ_0 , we now use the contour made up of a segment of the real axis, an arc of the circle $|z| = R$, a segment of the ray from the origin through $\exp\{i\varphi_0\}$, and an arc of the unit circle. The integral along the large arc and along the ray in the φ_0 direction are bounded as before, while it is easy to see that the integral along the arc of the unit circle is also bounded, independent of ρ and σ .

We now want to establish estimates (contained in Lemmas 2, 3, and 4) for integrals which will occur when we invert the characteristic function to obtain the stable density. The proof of Lemma 2 is very similar to that of Lemma 1, while the other two are easy consequences of Lemma 1.

LEMMA 2. Let $\sigma \in D$ and $\beta \geq 0$. Then

$$\int_0^\infty \exp(-iry) \exp(-\sigma r^\alpha) r^\beta dr = O(|y|^{-\beta-1}),$$

the constant being uniform for $\sigma \in D$.

Proof. If $y < 0$, make the change of variable $s = -ry$ and then use the same contour used in the proof of the first part of Lemma 1 to change the integral to the ray in the φ_0 direction. If $y > 0$, the change of variable is $s = ry$, and then the angle φ_0 should be the negative of the previous one.

LEMMA 3. Let $\sigma \in D$ and $\rho > 0$. Then

$$\int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi - \sigma r^\alpha) r^{N-1} (\sin \varphi)^{N-2} d\varphi dr = O(\rho^{-\alpha-N}),$$

the constant being independent of σ .

Proof. Since

$$\int_0^\pi \exp(-ir\rho \cos \varphi) (\sin \varphi)^{N-2} d\varphi = c J_{(N-2)/2}(r\rho) (r\rho)^{-(N-2)/2},$$

this is an immediate consequence of Lemma 1.

LEMMA 4. Let m, n be nonnegative integers and $\sigma \in D$. Then if $\beta \geq 0$, $\rho > 0$,

$$\int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi - \sigma r^\alpha) \sin^m \varphi \cos^n \varphi r^\beta d\varphi dr = O(\rho^{-\beta-1}),$$

the constant being independent of σ .

Proof. By use of the identity $\cos^2 \varphi = 1 - \sin^2 \varphi$, it suffices to consider $n=0, 1$. Then the formulas

$$\int_0^\pi \exp(-ir\rho \cos \varphi) \sin^m \varphi d\varphi = cJ_{m/2}(r\rho)(r\rho)^{-m/2},$$

$$\int_0^\pi \exp(-ir\rho \cos \varphi) \sin^m \varphi \cos \varphi d\varphi = cJ_{(m+2)/2}(r\rho)(r\rho)^{-m/2},$$

together with Lemma 1 complete the proof. \bullet

To estimate the integral involved in inverting the characteristic function, we shall use a finite number of terms from a power series expansion in the integrand. The next three lemmas establish the estimates we need for this.

LEMMA 5. Let $\sigma_1, \sigma_2 \in D$, $\rho > 0$. Then

$$\int_0^\infty \exp(-ir\rho \cos \varphi) \left(\exp(-\sigma_2 r^\alpha) - \exp(-\sigma_1 r^\alpha) \sum_{j=0}^{m-1} \frac{(\sigma_1 - \sigma_2)^j}{j!} r^{j\alpha} \right) r^{N-1} dr$$

$$= \begin{cases} O(|\sigma_2 - \sigma_1|^m) \\ O(\{\rho|\cos \varphi|\}^{-N-m\alpha} |\sigma_2 - \sigma_1|^m). \end{cases}$$

Proof. The first bound follows easily from the observation that

$$\exp(-\sigma_2 r^\alpha) - \exp(-\sigma_1 r^\alpha) \sum_{j=0}^{m-1} \frac{(\sigma_1 - \sigma_2)^j}{j!} r^{j\alpha}$$

$$(3.1) \quad = (-1)^m \int_{\sigma_1}^{\sigma_2} \int_{\sigma_1}^{z_m} \cdots \int_{\sigma_1}^{z_2} \exp(-z_1 r^\alpha) r^{m\alpha} dz_1 \cdots dz_m,$$

and then using integration contours on the straight line from σ_1 to σ_2 which has the effect of keeping the contours in D and also of making intermediate distances smaller. Thus

$$\left| \exp(-\sigma_2 r^\alpha) - \exp(-\sigma_1 r^\alpha) \sum_{j=0}^{m-1} \frac{(\sigma_1 - \sigma_2)^j}{j!} r^{j\alpha} \right| \leq \exp(-\gamma r^\alpha) r^{m\alpha} |\sigma_2 - \sigma_1|^m$$

and this suffices for the first bound. Now, if we substitute in the integral we are estimating the right-hand side of (3.1) and do the r -integration first, we have the integral equal to

$$(-1)^m \int_{\sigma_1}^{\sigma_2} \int_{\sigma_1}^{z_m} \cdots \int_{\sigma_1}^{z_2} \int_0^\infty \exp(-ir\rho \cos \varphi - z_1 r^\alpha) r^{m\alpha + N-1} dr dz_1 \cdots dz_m.$$

Applying Lemma 2 to this expression yields the second bound.

LEMMA 6. Suppose $\alpha > 0$ and α is not an integer, $\beta \geq 0$, and k is chosen so that

$$(3.2) \quad k-1 < \alpha + \beta \leq k.$$

Then

$$\left| a_\alpha(v) - \sum_{j=0}^{k-1} a_{\alpha-j}(u) \frac{(\alpha)_j}{j!} (v-u)^j \right| (\min\{|u|, |v|\})^\beta = O(|v-u|^{\alpha+\beta}).$$

(Here $(\alpha)_j = \alpha(\alpha-1)\dots(\alpha-j+1)$, and a_α is defined in (2.6).)

Proof. We give the proof only for $|v| \leq |u|$, the two cases being essentially the same. First suppose $0 \leq v \leq u$; then

$$\begin{aligned} \left| v^\alpha - \sum_{j=0}^{k-1} u^{\alpha-j} \frac{(\alpha)_j}{j!} (v-u)^j \right| v^\beta &= v^\beta \int_v^u \int_{y_k}^u \dots \int_{y_2}^u y_1^{\alpha-k} |(\alpha)_k| dy_1 \dots dy_k \\ &\leq \int_v^u \int_{y_k}^u \dots \int_{y_2}^u y_1^{\alpha+\beta-k} |(\alpha)_k| dy_1 \dots dy_k \\ &\leq \int_v^u \int_{y_k}^u \dots \int_0^{u-y_2} y_1^{\alpha+\beta-k} |(\alpha)_k| dy_1 \dots dy_k \\ &= \frac{|(\alpha)_k|}{(\alpha+\beta)_k} (u-v)^{\alpha+\beta}. \end{aligned}$$

Next, consider the case $0 \leq -v \leq u$. Then we must show that

$$\left\{ \sum_{j=0}^{k-1} u^{\alpha-j} \frac{(\alpha)_j}{j!} (u-v)^j \right\} |v|^\beta = O((u-v)^{\alpha+\beta}).$$

In this case, the bound is valid for each individual term, i.e.

$$u^{\alpha-j} (u-v)^j |v|^\beta \leq u^{\alpha+\beta-j} (u-v)^j \leq (u-v)^{\alpha+\beta}$$

since $u \leq u-v$ and $\alpha+\beta-j > 0$. Finally, if $u < 0$, the bound is trivial.

Next we give the lemma which will take the place of Lemma 6 when $\alpha=1$. In this case, $g_R(\xi)$ is easy to deal with, but $g_I(\xi)$ is complicated because of the log term. In order to get rid of the log term, we use symmetric second differences. To make the later work easier we will also use second differences for the real part. The first part of the lemma is valid for all β , and the second part could be made to work for $\beta > 1$ by subtracting more terms in the series expansion. We restrict our attention to $\beta \leq 1$ since that is the only case that can be used.

LEMMA 7. Suppose $0 \leq \beta \leq 1$. Then

$$\begin{aligned} &|u+v| + |u-v| - 2|u| \left| (\min\{|u|, |u+v|, |u-v|\})^\beta \right| = O(|v|^{1+\beta}) \\ &|(u+v) \log |u+v| + (u-v) \log |u-v| - 2u \log |u| \left| (\min\{|u|, |u+v|, |u-v|\})^\beta \right| \\ &= O(|v|^{1+\beta}). \end{aligned}$$

Proof. It clearly suffices to consider $v > 0$. The first statement is trivial. To prove the second, note that

$$(u+v) \log |u+v| + (u-v) \log |u-v| - 2u \log |u| = \int_0^v \log \left| \frac{u+z}{u-z} \right| dz.$$

The integrand is nonnegative for $u \geq 0$ and is an odd function of u . We will therefore consider only $u \geq 0$ and consider only upper bounds for the integral. If $u \geq 2v$, then

$$\int_0^v \log \left| \frac{u+z}{u-z} \right| dz \leq \int_0^v \left(\frac{u+z}{u-z} - 1 \right) dz \leq \frac{v^2}{u-v},$$

and since

$$|(\min \{|u|, |u+v|, |u-v|\})|^\beta \leq (u-v)^\beta,$$

combining these estimates gives the desired bound in this case. If $v \leq u \leq 2v$,

$$\int_0^v \log \left| \frac{u+z}{u-z} \right| dz \leq \int_0^u \log \left| \frac{u+z}{u-z} \right| dz = \int_1^\infty \log y \frac{2u}{(y+1)^2} dy \leq cu \leq 2cv,$$

and in this case we use the u term from the minimum function to get the final bound. Finally, if $0 \leq u \leq v$,

$$\begin{aligned} \int_0^v \log \left| \frac{u+z}{u-z} \right| dz &= \int_0^u \log \left| \frac{u+z}{u-z} \right| dz + \int_u^v \log \left| \frac{u+z}{u-z} \right| dz \\ &\leq cu + \int_{(u+v)/(v-u)}^\infty \log y \frac{2u}{(y-1)^2} dy \\ &\leq cu + (v-u) + \int_2^\infty \log y \frac{2u}{(y-1)^2} dy \leq c_1 v \end{aligned}$$

and the u term can again be used from the minimum function.

Now we are ready to apply Lemmas 6 and 7 to obtain estimates on the function g defined by (2.5). For $\alpha \neq 1$, we need to introduce some new functions. Let

$$\begin{aligned} g_{1j}(\xi, \eta) &= \frac{(\alpha)_j}{j!} \int_{S_N} a_{\alpha-j}((\xi, \theta))(\eta - \xi, \theta)^j \mu(d\theta), \\ g_{2j}(\xi, \eta) &= \frac{(\alpha)_j}{j!} \int_{S_N} a_{\alpha-j}(-(\xi, \theta))(\xi - \eta, \theta)^j \mu(d\theta), \end{aligned}$$

and

$$g_j(\xi, \eta) = \sec(\pi\alpha/2) \{g_{1j}(\xi, \eta)e^{-i\pi\alpha/2} + g_{2j}(\xi, \eta)e^{i\pi\alpha/2}\},$$

where we are assuming for the moment that the defining integrals converge. Then we have

LEMMA 8. Suppose $\alpha \neq 1$, and that for some $\beta \geq 0$ there is an M such that

$$(3.3) \quad \int_{S_N} |(\xi, \theta)|^{-\beta} \mu(d\theta) \leq M$$

for all $\xi \in S_N$. Then, choosing k as in (3.2), we have for all $\xi, \eta \in S_N$

$$g(\eta) - g(\xi) - \sum_{j=1}^{k-1} g_j(\xi, \eta) = O(|\xi - \eta|^{\alpha+\beta}).$$

Proof. First note that the defining integrals for $g_j(\xi, \eta)$ converge due to the assumption (3.3). Now apply Lemma 6 with $v=(\eta, \theta)$, $u=(\xi, \theta)$:

$$\left| a_\alpha((\eta, \theta)) - \sum_{j=0}^{k-1} a_{\alpha-j}((\xi, \theta)) \frac{(\alpha)_j}{j!} (\eta - \xi, \theta)^j \right| \leq K |(\eta - \xi, \theta)|^{\alpha+\beta} \{ |(\eta, \theta)|^{-\beta} + |(\xi, \theta)|^{-\beta} \}.$$

Integrating with respect to μ over S_N and using $|(\eta - \xi, \theta)| \leq |\eta - \xi|$, we obtain

$$\left| g_1(\eta) - g_1(\xi) - \sum_{j=1}^{k-1} g_{1,j}(\xi, \eta) \right| \leq 2MK |\eta - \xi|^{\alpha+\beta}.$$

The same technique works for g_2 and the estimates for g follow.

COROLLARY. For $\xi, \eta \in S_N$,

$$\begin{aligned} g(\eta) - g(\xi) &= O(|\xi - \eta|^\alpha), \quad \text{if } \alpha < 1, \\ &= O(|\xi - \eta|), \quad \text{if } \alpha > 1. \end{aligned}$$

Proof. Since (3.3) is always valid with $\beta=0$, the bound for $\alpha < 1$ is an immediate consequence of the lemma. For $\alpha > 1$, we have

$$g(\eta) - g(\xi) - g_1(\xi, \eta) = O(|\xi - \eta|^\alpha),$$

and it is clear that

$$|g_1(\xi, \eta)| \leq \alpha |\sec \pi\alpha/2| |\xi - \eta|$$

for $\alpha > 1$.

Before proving the sharp estimate that we need for $\alpha=1$, we shall consider what we could get by using the same technique used for $\alpha \neq 1$. The result corresponding to the corollary is not valid, but since $|\log |x||$ is dominated by $|x|^{-\epsilon}$ for small $|x|$ for any $\epsilon > 0$, one can prove very much as above that for $\alpha=1$ and any fixed $\epsilon > 0$,

$$(3.4) \quad g(\eta) - g(\xi) = O(|\xi - \eta|^{1-\epsilon}).$$

In fact, the estimate can even be improved to $O(|(\xi - \eta) \log |\xi - \eta||)$, but (3.4) is good enough for our purposes. However, we do need an improved estimate involving the symmetric second difference of g . This is the subject of

LEMMA 9. Suppose $\alpha=1$, and that for some $0 \leq \beta \leq 1$ there is an M such that (3.3) holds for all $\xi \in S_N$. Then

$$g(\xi + \zeta) + g(\xi - \zeta) - 2g(\xi) = O(|\zeta|^{1+\beta})$$

whenever $\xi \in S_N$, $|\xi + \zeta| \geq 1$, $|\xi - \zeta| \geq 1$.

Proof. We apply Lemma 7 with $u=(\xi, \theta)$, $v=(\zeta, \theta)$ to obtain

$$\begin{aligned} & | |(\xi + \zeta, \theta)| + |(\xi - \zeta, \theta)| - 2|(\xi, \theta)| | \\ & \leq K |(\zeta, \theta)|^{1+\beta} \{ |(\xi, \theta)|^{-\beta} + |(\xi + \zeta, \theta)|^{-\beta} + |(\xi - \zeta, \theta)|^{-\beta} \} \\ & \leq K |\zeta|^{1+\beta} \{ |(\xi, \theta)|^{-\beta} + |(\xi + \zeta/|\xi + \zeta|, \theta)|^{-\beta} + |(\xi - \zeta/|\xi - \zeta|, \theta)|^{-\beta} \}, \end{aligned}$$

the last inequality being a consequence of the fact that $|\xi + \zeta| \geq 1$, $|\xi - \zeta| \geq 1$. Integrating with respect to μ and using (3.3), we get

$$|\xi + \zeta|g_R(\xi + \zeta) + |\xi - \zeta|g_R(\xi - \zeta) - 2g_R(\xi) = O(|\zeta|^{1+\beta}).$$

Now by the parallelogram law,

$$2 + 2|\zeta|^2 = |\xi + \zeta|^2 + |\xi - \zeta|^2 \geq 1 + |\xi - \zeta|^2,$$

and thus

$$\begin{aligned} |(|\xi - \zeta| - 1)g_R(\xi - \zeta)| &\leq \frac{|\xi - \zeta|^2 - 1}{|\xi - \zeta| + 1} \leq \frac{2|\zeta|^2}{|\xi - \zeta| + 1} \leq \min\{2|\zeta|, |\zeta|^2\} \\ &\leq 2|\zeta|^{1+\beta}. \end{aligned}$$

In the same way we see that $|(|\xi + \zeta| - 1)g_R(\xi + \zeta)| \leq 2|\zeta|^{1+\beta}$. The imaginary part of g can be estimated in exactly the same way by using the other half of Lemma 7 and the fact that

$$(\xi + \zeta, \theta) \log |(\xi + \zeta)/|\xi + \zeta||, \theta) = (\xi + \zeta, \theta) \log |(\xi + \zeta, \theta)| - (\xi + \zeta, \theta) \log |\xi + \zeta|$$

so that when the θ integration is performed the last term is zero because of the assumption (2.2). Thus we have that both the real and imaginary parts of g satisfy the conclusion of the lemma.

These estimates on g convert directly to bounds on the function h defined in (2.9). The situation which will interest us is when

$$\begin{aligned} \xi_1 &= 0, \\ (3.5) \quad \xi_j &= \sin \varphi_2 \dots \sin \varphi_{j-1} \cos \varphi_j, \quad 2 \leq j \leq N-1, \\ \xi_N &= \sin \varphi_2 \dots \sin \varphi_{N-2} \sin \varphi_{N-1}, \end{aligned}$$

and

$$\begin{aligned} \eta_1 &= \cos \varphi_1, \\ (3.6) \quad \eta_j &= \sin \varphi_1 \dots \sin \varphi_{j-1} \cos \varphi_j, \quad 2 \leq j \leq N-1, \\ \eta_N &= \sin \varphi_1 \dots \sin \varphi_{N-2} \sin \varphi_{N-1}. \end{aligned}$$

For this particular case, it is useful to note that $|\xi - \eta| \leq |\varphi_1 - \pi/2|$. Then, for example, we have from the corollary to Lemma 8 that

$$\begin{aligned} h(\varphi_1, \varphi_2, \dots, \varphi_{N-1}) - h(\pi/2, \varphi_2, \dots, \varphi_{N-1}) &= O\left(\left|\varphi_1 - \frac{\pi}{2}\right|^\alpha\right), \quad \text{if } \alpha < 1, \\ (3.7) \quad &= O\left(\left|\varphi_1 - \frac{\pi}{2}\right|\right), \quad \text{if } \alpha > 1, \end{aligned}$$

the bounds being uniform in $\varphi_2, \dots, \varphi_{N-1}$. For the case of $\alpha = 1$, we have

COROLLARY. *If $\alpha = 1$, $0 \leq \beta \leq 1$, and there is an M such that (3.3) holds for all $\xi \in S_N$, then*

$$\begin{aligned} h(\varphi_1, \varphi_2, \dots, \varphi_{N-1}) + h(\pi - \varphi_1, \varphi_2, \dots, \varphi_{N-1}) - 2h(\pi/2, \varphi_2, \dots, \varphi_{N-1}) \\ = O(|\varphi_1 - \pi/2|^{1+\beta}). \end{aligned}$$

Proof. Let ξ be as in (3.5) and $\zeta = (\text{ctn } \varphi_1, 0, \dots, 0)$. Then $|\xi + \zeta| = (1 + \text{ctn}^2 \varphi_1)^{1/2} = (\sin \varphi_1)^{-1}$ and we have $(\xi + \zeta)|\xi + \zeta|^{-1} = \eta$ as given in (3.6). Thus

$$g(\xi + \zeta) = g((\xi + \zeta)|\xi + \zeta|^{-1}) = h(\varphi_1, \varphi_2, \dots, \varphi_{N-1}),$$

and in a similar way

$$g(\xi - \zeta) = h(\pi - \varphi_1, \varphi_2, \dots, \varphi_{N-1}).$$

Lemma 9 now yields the corollary for $\varphi_1 \in [\pi/4, 3\pi/4]$, say; there is no problem for φ_1 outside this interval since h is bounded.

Proof of Theorem 1. We consider first the situation when $\alpha \neq 1$. Recalling (2.8), let

$$I(\rho) = \int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi_1 - r^\alpha h(\varphi_1)) r^{N-1} (\sin \varphi_1)^{N-2} d\varphi_1 dr$$

where the arguments $\varphi_2, \dots, \varphi_{N-1}$ have been suppressed in the function h . In order to prove the theorem, it will suffice to show this integral is $O(\rho^{-\alpha-\beta-1})$ uniformly in $\varphi_2, \dots, \varphi_{N-1}$. First choose m so that

$$\begin{aligned} m &\geq (N-1+\alpha)/\alpha, & \text{if } \alpha < 1, \\ &\geq N-1+\alpha, & \text{if } \alpha > 1. \end{aligned}$$

Then let

$$A_1(r, \varphi_1) = \exp(-r^\alpha h(\pi/2)) \sum_{j=1}^{m-1} \frac{\{h(\pi/2) - h(\varphi_1)\}^j}{j!} r^{j\alpha},$$

$$A_2(r, \varphi_1) = \exp(-r^\alpha h(\pi/2)),$$

$$A_3(r, \varphi_1) = \exp(-r^\alpha h(\varphi_1)) - A_1(r, \varphi_1) - A_2(r, \varphi_1),$$

and

$$(3.8) \quad I_n(\rho) = \int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi_1) A_n(r, \varphi_1) r^{N-1} (\sin \varphi_1)^{N-2} d\varphi_1 dr,$$

for $n=1, 2, 3$, the remaining φ arguments still being suppressed. Now

$$(3.9) \quad I(\rho) = I_1(\rho) + I_2(\rho) + I_3(\rho),$$

and

$$(3.10) \quad I_2(\rho) = O(\rho^{-\alpha-N})$$

by Lemma 3. If $\alpha < 1$, by Lemma 5 and (3.7)

$$\begin{aligned} I_3(\rho) &\leq K \int_{|\varphi_1 - \pi/2| \leq \rho^{-1}} |h(\varphi_1) - h(\pi/2)|^m d\varphi_1 \\ &\quad + K \int_{|\varphi_1 - \pi/2| > \rho^{-1}} |\rho \cos \varphi_1|^{-N-m\alpha} |h(\varphi_1) - h(\pi/2)|^m d\varphi_1 \\ &\leq c\rho^{-m\alpha-1} + c'\rho^{-N-\alpha}. \end{aligned}$$

In the same way, one shows that if $\alpha > 1$,

$$(3.11) \quad I_3(\rho) = O(\rho^{-\alpha-N}).$$

To find suitable estimates for $I_1(\rho)$ we must expand $h(\varphi_1) - h(\pi/2)$ to $k-1$ terms where k is determined by (3.2). Define $h_n(\varphi_1) = g_n(\xi, \eta)$, where ξ and η are as in (3.5) and (3.6). Then by Lemma 8

$$h(\varphi_1) - h(\pi/2) - \sum_{n=1}^{k-1} h_n(\varphi_1) = O(|\varphi_1 - \pi/2|^{\alpha+\beta}).$$

Furthermore,

$$\begin{aligned} & \{h(\pi/2) - h(\varphi_1)\}^j - \left\{ - \sum_{n=1}^{k-1} h_n(\varphi_1) \right\}^j \\ &= \left\{ h(\pi/2) - h(\varphi_1) + \sum_{n=1}^{k-1} h_n(\varphi_1) \right\} \sum_{v=0}^{j-1} \{h(\pi/2) - h(\varphi_1)\}^v \left\{ - \sum_{n=1}^{k-1} h_n(\varphi_1) \right\}^{j-1-v} \\ &= O(|\varphi_1 - \pi/2|^{\alpha+\beta}) \end{aligned}$$

for $j \leq m-1$, since m and k depend only on α , β , and N , and h , h_n are bounded. Thus we define

$$A_4(r, \varphi_1) = \exp(-r^\alpha h(\pi/2)) \sum_{j=1}^{m-1} \left(\left\{ - \sum_{n=1}^{k-1} h_n(\varphi_1) \right\}^j / j! \right) r^{j\alpha}$$

and $I_4(\rho)$ by (3.8). We have shown that

$$A_1(r, \varphi_1) - A_4(r, \varphi_1) = O\left(|\varphi_1 - \pi/2|^{\alpha+\beta} \exp(-\gamma r^\alpha) \sum_{j=1}^{m-1} r^{j\alpha}\right),$$

and it follows that

$$(3.12) \quad \int_0^\infty \exp(-ir\rho \cos \varphi_1) \{A_1(r, \varphi_1) - A_4(r, \varphi_1)\} r^{N-1} dr = O(|\varphi_1 - \pi/2|^{\alpha+\beta}).$$

By applying Lemma 2 to the difference of the j th terms in the sum, we find that for large values of $\rho |\cos \varphi_1|$ (3.12) is also

$$O(|\rho \cos \varphi_1|^{-N-\alpha} |\varphi_1 - \pi/2|^{\alpha+\beta}).$$

Performing the φ_1 integration yields

$$\begin{aligned} |I_1(\rho) - I_4(\rho)| &\leq K \int_{|\varphi_1 - \pi/2| \leq \rho^{-1}} |\varphi_1 - \pi/2|^{\alpha+\beta} d\varphi_1 \\ &\quad + K \int_{|\varphi_1 - \pi/2| > \rho^{-1}} |\rho \cos \varphi_1|^{-N-\alpha} |\varphi_1 - \pi/2|^{\alpha+\beta} d\varphi_1 \\ (3.13) \quad &\leq c\rho^{-\alpha-\beta-1}, \end{aligned}$$

where at the last step we have used $\beta+1 < N$. (As pointed out in the introduction, the hypothesis of the theorem can only be satisfied if $\beta < 1$ and we have $N \geq 2$.)

Even apart from this, the best one could hope for in (3.13) is to get $\beta = N - 1$.) The next step is to investigate the functions $h_n(\varphi_1)$. When ξ and η are given by (3.5) and (3.6), then

$$(\xi - \eta, \theta) = B_1(\theta) \cos \varphi_1 + B_2(\theta)(\sin \varphi_1 - 1),$$

where the dependence of B_2 on $\varphi_2, \dots, \varphi_{N-1}$ has been suppressed, but B_2 is bounded. Thus, for all n , $h_n(\varphi_1)$ is a polynomial in the two variables $\sin \varphi_1, \cos \varphi_1$. The same must then be true for $\{\sum_{n=1}^k h_n(\varphi_1)\}^j$. It follows that Lemma 4 may be applied to show that $I_4(\rho) = O(\rho^{-\alpha-N})$.

Combining this with (3.9)–(3.11) and (3.13) completes the proof except for a comment on the situation when $N=2$ and the case of $\alpha=1$. When $N=2$, the range of integration for φ_1 is $[0, 2\pi]$ and we have only discussed $[0, \pi]$. However, it is clear that the integral over $[\pi, 2\pi]$ can be split up in the same way with $h(3\pi/2)$ playing the role of $h(\pi/2)$.

We now sketch briefly the argument for $\alpha=1$. Consider

$$\begin{aligned} I(\rho) + I(-\rho) &= \int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi_1 - rh(\varphi_1)) r^{N-1} (\sin \varphi_1)^{N-2} d\varphi_1 dr \\ &\quad + \int_0^\infty \int_0^\pi \exp(ir\rho \cos \varphi_1 - rh(\varphi_1)) r^{N-1} (\sin \varphi_1)^{N-2} d\varphi_1 dr \\ &= \int_0^\infty \int_0^\pi \exp(-ir\rho \cos \varphi_1) \{ \exp(-rh(\varphi_1)) + \exp(-rh(\pi - \varphi_1)) \} \\ &\quad \times r^{N-1} (\sin \varphi_1)^{N-2} d\varphi_1 dr; \end{aligned}$$

the substitution $\varphi_1 \rightarrow \pi - \varphi_1$ having been made in the second integral. Since $I(\rho) \leq I(\rho) + I(-\rho)$, it will suffice to prove the last integral is $O(\rho^{-2-\beta})$. Let

$$A_5(r, \varphi_1) = \exp(-rh(\pi/2)) \{ 2h(\pi/2) - h(\varphi_1) - h(\pi - \varphi_1) \} r,$$

$$A_6(r, \varphi_1) = 2 \exp(-rh(\pi/2)),$$

$$A_7(r, \varphi_1) = \exp(-rh(\varphi_1)) + \exp(-rh(\pi - \varphi_1)) - A_5(r, \varphi_1) - A_6(r, \varphi_1),$$

and define $I_n(\rho)$ by (3.8). Then

$$I(\rho) + I(-\rho) = I_5(\rho) + I_6(\rho) + I_7(\rho),$$

and $I_6(\rho) = O(\rho^{-1-N})$ by Lemma 3 as before. Using $m=2$ in Lemma 5 and (3.4) with $\varepsilon < (1-\beta)/2$ and splitting the φ_1 -integration as we did with $I_3(\rho)$, we see that

$$I_7(\rho) = O(\rho^{-3+2\varepsilon}) = O(\rho^{-2-\beta}).$$

Finally, using the corollary which follows Lemma 9 together with Lemma 2, the estimate $I_5(\rho) = O(\rho^{-2-\beta})$ is obtained in the same way as the estimate for

$$I_1(\rho) - I_4(\rho).$$

REMARKS. In the proof of Theorem 1, note that good estimates were found for the integrals $I_2(\rho)$, $I_3(\rho)$, and $I_4(\rho)$. The difficulty in improving the theorem in

dimensions higher than two comes with the estimate for $I_1(\rho) - I_4(\rho)$. This required an expansion for g and more terms in the expansion would be required in order to obtain better results from the present method of proof.

4. The potential kernel of a stable process. If $\alpha > (N-1)/2$, the estimate of Theorem 1 shows that if t is small, $|x| = 1$, then

$$p(t, x) = p(1, t^{-1/\alpha}x)t^{-N/\alpha} \leq Kt^{1+(1-N)/\alpha} = Kt^c$$

with $c > -1$. This implies that $\int_0^1 p(t, x) dt$ converges uniformly for $|x| = 1$. Since the density is bounded, we also have $p(t, x) \leq Kt^{-N/\alpha}$ and since we are assuming $N > \alpha$ (see §2) $\int_1^\infty p(t, x) dt$ converges uniformly for $|x| = 1$. The argument given in [12] is therefore valid in this case so that

$$(4.1) \quad U(x) \leq K|x|^{\alpha-N}$$

for any stable process in R^N with index $\alpha > (N-1)/2$.

We now go on to consider the case $\alpha \leq (N-1)/2$. In order to show that the integral of the potential kernel is not too badly behaved (even though the kernel itself may be infinite on some rays) we will need to show that the average behaviour of the stable density is the same as in the symmetric case so that on average it decays like $|x|^{-N-\alpha}$ even though the decay may be slower in some directions. It is convenient to have this result in the following slightly different form.

THEOREM 2. *Let $p(1, x)$ be a stable density of index α in R^N and let $D(\rho, h)$ denote a spherical shell:*

$$D(\rho, h) = \{x \in R^N : \rho \leq |x| \leq \rho + h\}.$$

Then there is a constant K such that

$$\int_{D(\rho, h)} p(1, x) dx \leq K\rho^{-1-\alpha}h.$$

Proof. By the inversion formula

$$\int_{D(\rho, h)} p(1, x) dx = (2\pi)^{-N} \int_{R^N} \int_{D(\rho, h)} \exp(-i(x, \xi) - |\xi|^\alpha g(\xi)) d\xi dx.$$

Now for each fixed ξ we make an orthogonal transformation on x given by $\eta = Bx$ with $b_{1j} = \xi_j/|\xi|$, $j = 1, 2, \dots, N$. Then

$$(x, \xi) = \sum_{j=1}^N x_j \xi_j = |\xi| \sum_{j=1}^N b_{1j} x_j = |\xi| \eta_1$$

and since the orthogonal transformation has Jacobian ± 1 and preserves $D(\rho, h)$ we have

$$\int_{D(\rho, h)} p(1, x) dx = (2\pi)^{-N} \int_{R^N} \int_{D(\rho, h)} \exp(-i\eta_1 |\xi| - |\xi|^\alpha g(\xi)) d\eta d\xi.$$

Next change η to spherical coordinates:

$$\begin{aligned} \int_{D(\rho, h)} p(1, x) dx &= (2\pi)^{-N} \int_{R^N} \int_{\rho}^{\rho+h} \int_0^{2\pi} \cdots \int_0^{\pi} \exp(-ir \cos \varphi_1 |\xi| - |\xi|^{\alpha} g(\xi)) \\ &\quad \cdot J(r, \varphi_1, \dots, \varphi_{N-1}) d\varphi_1 \cdots d\varphi_{N-1} dr d\xi \\ &= c \int_{R^N} \int_{\rho}^{\rho+h} \int_0^{2\pi} \cdots \int_0^{\pi} J_{(N-2)/2}(r|\xi|)(r|\xi|)^{-(N-2)/2} \exp(-|\xi|^{\alpha} g(\xi)) \\ &\quad \cdot r^{N-1} (\sin \varphi_2)^{N-3} \cdots \sin \varphi_{N-2} d\varphi_2 \cdots d\varphi_{N-1} dr d\xi \\ &= c' \int_{R^N} \int_{\rho}^{\rho+h} J_{(N-2)/2}(r|\xi|) \exp(-|\xi|^{\alpha} g(\xi)) |\xi|^{-(N-2)/2} r^{N/2} dr d\xi. \end{aligned}$$

The next step is to change ξ to spherical coordinates and perform the integration with respect to $|\xi|$. The Jacobian contributes a factor of $|\xi|^{N-1}$. Recalling that $g(\xi)$ is independent of $|\xi|$ and applying Lemma 1, we see that this integration produces an integrand of order $r^{-\alpha-1-N/2} r^{N/2}$ times the angular components of the Jacobian from the change of ξ to spherical coordinates. After integrating the angular components of ξ , we have

$$\int_{D(\rho, h)} p(1, x) dx \leq K \int_{\rho}^{\rho+h} r^{-\alpha-1} dr \leq K \rho^{-\alpha-1} h.$$

If $(r, \varphi_1, \dots, \varphi_{N-1})$ denotes the representation of a point in R^N in spherical coordinates, then restricting the angles φ_i by

$$|\varphi_i - \psi_i| < \delta, \quad i = 1, 2, \dots, N-1,$$

where $\delta > 0$ and $\psi_1, \dots, \psi_{N-1}$ are fixed, gives a type of cone in R^N with vertex at the origin. Let

$$D(\rho, h, \delta) = \{(r, \varphi_1, \dots, \varphi_{N-1}) : \rho \leq r \leq \rho+h, |\varphi_i - \psi_i| < \delta, i = 1, \dots, N-1\},$$

the intersection of such a cone with the shell $D(\rho, h)$. The remainder of this section will be devoted to the application of Theorems 1 and 2 to the estimation of the integral of the potential kernel $U(x)$ (see (2.3)) over the set $D(\rho, h, \delta)$.

The first step is to use a change of scale to convert the estimate in Theorem 2 into one involving t :

$$\begin{aligned} (4.2) \quad \int_{D(\rho, h)} p(t, x) dx &= \int_{D(\rho, h)} p(1, t^{-1/\alpha} x) t^{-N/\alpha} dx = \int_{D(t^{-1/\alpha} \rho, t^{-1/\alpha} h)} p(1, \eta) d\eta \\ &\leq K \rho^{-1-\alpha} h t. \end{aligned}$$

Here we have used the change of variable $\eta = t^{-1/\alpha} x$. The volume of $D(\rho, h, \delta)$ is proportional to $\rho^{N-1} \delta^{N-1} h$ so the upper bound on the density given by Theorem 1 (with $\beta=0$) leads to

$$\int_{D(\rho, h, \delta)} p(1, x) dx \leq c \rho^{N-\alpha-2} \delta^{N-1} h.$$

Using the change of scale as before, this yields

$$(4.3) \quad \int_{D(\rho, h, \delta)} p(t, x) dx \leq c \rho^{N-\alpha-2} \delta^{N-1} h t^{1-(N-1)/\alpha}.$$

Now let us work on the kernel; under the assumption $\alpha < (N-1)/2$,

$$\begin{aligned} \int_{D(\rho, h, \delta)} U(x) dx &= \int_0^\infty dt \int_{D(\rho, h, \delta)} p(t, x) dx \\ &\leq \int_0^{(\rho\delta)^\alpha} K \rho^{-1-\alpha} h t dt + \int_{(\rho\delta)^\alpha}^\infty c \rho^{N-\alpha-2} \delta^{N-1} h t^{1-(N-1)/\alpha} dt \end{aligned}$$

using (4.2), (4.3), and $D(\rho, h, \delta) \subset D(\rho, h)$. Performing the integration,

$$(4.4) \quad \int_{D(\rho, h, \delta)} U(x) dx \leq c' \rho^{-1+\alpha} h \delta^{2\alpha}.$$

For the case $\alpha = (N-1)/2$, we need one more estimate which comes from the fact that the density is bounded

$$\begin{aligned} \int_{D(\rho, h, \delta)} p(t, x) dx &= \int_{D(\rho, h, \delta)} p(1, t^{-1/\alpha} x) t^{-N/\alpha} dx \\ &\leq c_1 t^{-N/\alpha} \rho^{N-1} \delta^{N-1} h. \end{aligned}$$

Now if $\delta \leq 1$,

$$\begin{aligned} \int_{D(\rho, h, \delta)} U(x) dx &\leq \int_0^{(\rho\delta)^\alpha} K \rho^{-1-\alpha} h t dt + \int_{(\rho\delta)^\alpha}^{\rho^\alpha} c \rho^{N-\alpha-2} \delta^{N-1} h t^{1-(N-1)/\alpha} dt \\ &\quad + \int_{\rho^\alpha}^\infty c_1 \rho^{N-1} \delta^{N-1} h t^{-N/\alpha} dt. \end{aligned}$$

Evaluating the integrals and remembering that $\alpha = (N-1)/2$ gives

$$(4.5) \quad \int_{D(\rho, h, \delta)} U(x) dx \leq c_2 \rho^{-1+\alpha} h \delta^{2\alpha} (1 + c_3 |\log \delta|).$$

If $\delta > 1$, the upper limit in the first integral can be changed to ρ^α and the middle integral ignored with the result that (4.5) is still valid.

5. Hitting probabilities. We start with an interpretation of the potential generated by Lebesgue measure on a set.

LEMMA 10. *If F is a set of positive Lebesgue measure, and T_F denotes the total time spent in F by the process, then*

$$E^x\{T_F\} = \int_F U(y-x) dy.$$

Proof.

$$\begin{aligned} E^x\{T_F\} &= \int_0^\infty P^x[X(t) \in F] dt \\ &= \int_0^\infty dt \int_F p(t, y-x) dy \\ &= \int_F U(y-x) dy. \end{aligned}$$

Now if τ is any stopping time, since shifting the path can only reduce T_F , the strong Markov property gives

$$E^x\{T_F\} \geq E^x E^{X(\tau)}\{T_F\}.$$

We will apply this when τ is the hitting time of a set E , i.e.

$$\tau_E = \inf \{t > 0 : X(t) \in E\}.$$

If E is closed, then $X(\tau_E) \in E$, so if we put $\lambda = \inf_{x \in E} E^x\{T_F\}$, then we will have

$$(5.1) \quad E^x\{T_F\} \geq \lambda \Phi(x, E)$$

where $\Phi(x, E)$ denotes the probability of hitting E , starting at x . This will now be used to obtain bounds for $\Phi(x, E)$ when E is a sphere.

THEOREM 3. *Suppose E is a sphere of radius r and x is at a distance d from E with $r \leq d$. For a fixed stable process of index $\alpha < N$ in R^N (restricted to satisfy (2.2) if $\alpha = 1$), there is a constant c such that*

- (i) if $\alpha > (N-1)/2$, $\Phi(x, E) \leq c(r/d)^{N-\alpha}$;
- (ii) if $\alpha = (N-1)/2$, $\Phi(x, E) \leq c(r/d)^{N-\alpha}(1 + \log d/r)$;
- (iii) if $\alpha < (N-1)/2$, $\Phi(x, E) \leq c(r/d)^{1+\alpha}$.

If the process is of type A, there is a constant K with $\Phi(x, E) \geq K(r/d)^{N-\alpha}$.

REMARK. The estimates in (ii) and (iii) can be improved if we know that the measure μ appearing in the characteristic function of the stable process satisfies the condition of Theorem 1 for some $\beta > 0$. We do not state the general result as it seems far from best possible; note, however, that if

$$\int_{S_N} |(\xi, \theta)|^{-\beta} \mu(d\theta) \leq M$$

for some $\beta > N-1-2\alpha$, then we have $\Phi(x, E) \leq c(r/d)^{N-\alpha}$, so that in this case the hitting probability of a sphere has the same order of magnitude as for the corresponding symmetric process. The example considered in the next section shows that the estimates (ii) and (iii) cannot be improved in general.

Proof. The lower bound for processes of type A was obtained in [12]. If $\alpha > (N-1)/2$, it is a consequence of Theorem 1, with $\beta = 0$, that the arguments in [12] are valid for the upper bound also as we have observed at the beginning of §4. We proceed to prove (ii) and (iii). First note that we can assume that $d \geq 2r$ since these estimates are trivially true otherwise. The total time spent in a sphere of radius r by a process starting at its centre has expectation Kr^α by the scaling property (2.1). If we take for F the sphere of radius $2r$, concentric with E , then the sphere with centre at x , radius r will be contained in F for every $x \in E$. Thus $\lambda = \inf_{x \in E} E^x\{T_F\} \geq Kr^\alpha$, so that by (5.1) and Lemma 10,

$$(5.2) \quad \Phi(x, E) \leq K^{-1}r^{-\alpha}E^x\{T_F\} = K^{-1}r^{-\alpha} \int_F U(y-x) dy.$$

To estimate the last integral, let $D(\rho, h, \delta)$ be a sector which contains $F-x$; it suffices to let $\rho=d-r$, $h=4r$, and $\delta=2r\rho^{-1}$. Applying (4.4) in case (iii) and (4.5) in case (ii) completes the proof since in (5.2) we have

$$\int_F U(y-x) dy \leq \int_{D(\rho, h, \delta)} U(z) dz.$$

The assumption $d \geq 2r$ lets us replace $d-r$ by $d/2$ at the last step.

6. Example. Suppose $X^N(t) = (X_1^N(t), X_2^N(t), \dots, X_N^N(t))$ denotes the stable process in R^N obtained by assuming that the coordinate processes $X_i^N(t)$ are independent symmetric stable processes of index α in R^1 . This corresponds to a measure μ on S_N which consists of $2N$ equal point masses at the points where the coordinate axes intersect S_N . We could also think of $X^N(t)$ as the sum of N independent linear processes in different directions. It is clear that the density $p(t, x)$ for this process is of the form $p(t, x) = p_1(t, x_1)p_1(t, x_2) \dots p_1(t, x_N)$ where $x = (x_1, x_2, \dots, x_N)$ and $p_1(t, x_1)$ is the density for a linear symmetric stable process of index α . Since [11]

$$(6.1) \quad \begin{aligned} c_1 &\leq p_1(1, x) \leq c_2 && \text{for } |x| \leq 1, \\ c_3 &\leq |x|^{1+\alpha} p_1(1, x) \leq c_4 && \text{for } |x| \geq 1, \end{aligned}$$

it is easy to check that $U(x) = \int_0^\infty p(t, x) dt$ will be finite for $x \neq 0$ if $\alpha > (N-1)/2$, but will be infinite for x on a coordinate axis if $\alpha \leq (N-1)/2$. (The scaling property is used in this computation much as it was in (4.2), for example.) We now estimate $U(x)$ for points close to one of the axes. Consider the set

$$C = \{x : \rho \leq x_1 \leq \rho+h, |x_i| \leq a, i = 2, \dots, N\}$$

for a and h small compared to ρ , say $\max\{2a, h\} \leq \rho$. Then C will differ very little from $D(\rho, h, \delta)$ with $\delta = a\rho^{-1}$. For $x \in C$,

$$\begin{aligned} U(x) &= \int_0^\infty p(t, x) dt \geq \int_{a^\alpha}^{\rho^\alpha} \prod_{j=1}^N p_1(1, t^{-1/\alpha} x_j) t^{-N/\alpha} dt \\ &\geq c_3 c_1^{N-1} x_1^{-1-\alpha} \int_{a^\alpha}^{\rho^\alpha} (t^{-1/\alpha})^{-1-\alpha} t^{-N/\alpha} dt \end{aligned}$$

by (6.1), since $t^{-1/\alpha} x_1 \geq 1$ for $t \leq \rho^\alpha$ and $t^{-1/\alpha} |x_j| \leq 1$ for $t \geq a^\alpha$. Performing the integration, we obtain

$$(6.2) \quad U(x) \geq K_1 \rho^{-1-\alpha} a^{2\alpha-N+1}, \quad \text{if } \alpha < (N-1)/2,$$

$$(6.3) \quad U(x) \geq K_2 \rho^{-1-\alpha} \log \rho/a, \quad \text{if } \alpha = (N-1)/2.$$

These estimates could be used to show that (4.4) and (4.5) are best possible in general, but we will proceed directly to the estimation of the hitting probabilities. Recall that

$$\Phi(x, E) = \int_E U(y-x) \nu(dy)$$

where ν is the capacitory measure for the compact set E . If E is the sphere of radius r centred at the point $(d+r, 0, \dots, 0)$, then $E \subset C$ with $\rho=d$, $h=2r$, $a=r$. Thus, if $\alpha < (N-1)/2$ and $d \geq 2r$, it follows from (6.2) and the fact (Lemma 3 of [12]) that the capacity of a sphere of radius r is $K_3 r^{N-\alpha}$ that

$$\begin{aligned}\Phi(0, E) &\geq K_1 d^{-1-\alpha} r^{2\alpha-N+1} \nu(E) \\ &= K_1 K_3 d^{-1-\alpha} r^{\alpha+1}.\end{aligned}$$

Similarly, if $\alpha = (N-1)/2$ and $d \geq 2r$,

$$\Phi(0, E) \geq K_2 K_3 d^{-1-\alpha} r^{\alpha+1} \log(d/r).$$

These results show that the estimates in Theorem 3 (ii) and (iii) for $\Phi(x, E)$ are of the right order of magnitude.

Further examination of this example shows that while the hitting probabilities of spheres located in certain directions from the starting point can be larger than for the symmetric stable process of the same index as we have seen, in other directions these probabilities will be of the same order of magnitude as they are in the symmetric case. Thus, for the process we are considering, completely different orders of magnitude are possible for the hitting probabilities of spheres located in different directions. Intuitively this is explained by the possibility of a large jump in the preferred direction in a small time before the other components of the process have had a chance to grow. This intuitive idea becomes precise in the next section.

7. Delayed hitting probabilities. The purpose of this section is to state for purposes of comparison the result that if we consider the probability of hitting the sphere after a fixed positive time T (and if the starting point is not too far away) then the "delayed hitting probability" will be of the same order of magnitude for all stable processes of a given index. Let

$$Q(x, E, T) = P^x[X(t) \in E \text{ for some } t \geq T].$$

Then we have

THEOREM 4. *Suppose E is a sphere of radius r . For a fixed stable process of index $\alpha < N$ in R^N (restricted to satisfy (2.2) if $\alpha = 1$), there is a constant c such that*

$$Q(x, E, T) \leq c(r/T^{1/\alpha})^{N-\alpha}.$$

If the process is of type A, $x \in E$, and $r \leq T^{1/\alpha}$, then there is a constant K such that

$$Q(x, E, T) \geq K(r/T^{1/\alpha})^{N-\alpha}.$$

REMARK. The upper bound shows that for a sphere of radius r at a distance d , the probability of hitting this sphere after time d^α is no larger than $c(r/d)^{N-\alpha}$, as it is in the symmetric case. Thus, as we mentioned earlier, the large estimates in (ii) and (iii) of Theorem 3 must be due to large excursions in specific directions in

relatively small times. (Note that the expected length of time to first travel the distance d is of order d^α .)

Proof. The upper bound is Lemma 2.2 of [4]. From the proof of that lemma, we take the equality

$$(7.1) \quad Q(x, E, T) = \int_E \int_T^\infty p(1, s^{-1/\alpha}(z-x)) s^{-N/\alpha} ds \nu(dz)$$

where ν is the capacity measure for E . (There is a misprint in [4], the exponent for s there reading $N-\alpha$. However, (7.1) is the correct version.) Now $|s^{-1/\alpha}(z-x)| \leq T^{-1/\alpha} 2r \leq 2$ and since the process is type A , the density is bounded below on the sphere of radius 2, centre the origin. Thus

$$Q(x, E, T) \geq K_1 \nu(E) \int_T^\infty s^{-N/\alpha} ds.$$

Performing the integration and recalling that $\nu(E) = K_2 r^{N-\alpha}$ (Lemma 3 of [12]) completes the proof.

As we have pointed out in the introduction, Theorem 4 is sufficient to justify Theorems 2 and 3 of [12].

8. Further problems. The problem of improving the hypothesis of Theorem 1 (particularly for $N \geq 3$) has already been mentioned in the introduction. In addition, it would seem desirable to know whether it is possible to obtain the more complete information about the asymptotic behaviour of the stable density that is available in one dimension [11].

There are some interesting problems concerning polar sets. For $\alpha=1$, $N=1$, it is known that the nonsymmetric Cauchy processes have no nonempty polar sets [9], while the symmetric process has nonempty polar sets. If $N \geq 2$, does the class of polar sets for $\alpha=1$ vary with the process? It seems likely that (as in R^1) it may differ at least for processes not having the scaling property. A similar problem now exists for general α . Orey [6] showed that all stable processes of index α in R^N which satisfy

$$(8.1) \quad U(x) \leq c|x|^{\alpha-N}$$

have the same polar sets. However, we have seen that (8.1) is false in general. It remains an open question, therefore, whether the class of polar sets is the same for all stable processes with given $N \geq 2$, $\alpha \neq 1$. For $N=1$, there is no problem since (8.1) is then satisfied for $\alpha < 1$.

Another question we have ignored entirely is the behaviour of processes with index $\alpha=1$ not satisfying the scaling property. For example, in this case we have not even considered the asymptotic behaviour of the density. Of course, for $N=1$ this is given in [11]. Also the asymptotic behaviour of U in this case is obtained in [9].

REFERENCES

1. R. M. Blumenthal and R. K. Gettoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc. **95** (1960), 263–273.
2. A. Erdélyi et al., *Higher transcendental functions*, Vol. II, Bateman Manuscript Project, McGraw-Hill, New York, 1953.
3. G. A. Hunt, *Markov processes and potentials*. I, II, III, Illinois J. Math. **1** (1957), 44–93, 316–369; **2** (1958), 151–213.
4. N. Jain and W. E. Pruitt, *The correct measure function for the graph of a transient stable process*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 131–138.
5. P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris, 1937.
6. S. Orey, "Polar sets for processes with stationary independent increments" in *Markov processes and potential theory*, edited by J. Chover, Wiley, New York, 1967, pp. 117–126.
7. G. Polya, *On the zeroes of an integral function represented by Fourier's integral*, Messenger of Math. **52** (1923), 185–188.
8. S. C. Port, *A remark on hitting places for transient stable process*, Ann. Math. Statist. **39** (1968), 365–371.
9. S. C. Port and C. J. Stone, *The asymmetric Cauchy processes on the line*, Ann. Math. Statist. **40** (1969), 137–143.
10. W. E. Pruitt and S. J. Taylor, *Sample path properties of processes with stable components*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **12** (1969), 267–289.
11. A. V. Skorohod, *Asymptotic formulas for stable distribution laws*, Dokl. Akad. Nauk SSSR **98** (1954), 731–734; English transl., Selected Transl. in Math. Stat. and Prob. **1** (1961), 157–161.
12. S. J. Taylor, *Sample path properties of a transient stable process*, J. Math. Mech. **16** (1967), 1229–1246.
13. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1944.

UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINNESOTA
WESTFIELD COLLEGE,
LONDON, ENGLAND