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H^p SPACES ON BOUNDED SYMMETRIC DOMAINS[1]

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1. **Introduction.** Let D be a bounded symmetric domain in the complex vector space C^N and $0 \in D$. Any bounded symmetric domain D, furnished with the Bergman metric M, is a hermitian symmetric space (D, M) of noncompact type and is necessarily simply connected [3, p. 311]. Let Γ be the group of holomorphic automorphisms of D; Γ is transitive on D and extends continuously to the topological boundary ∂D of D [7, p. 269]. The isotropy group $\Gamma_0 = \{\gamma \in \Gamma : \gamma(0) = 0\}$ of Γ is a compact subgroup of Γ and contains no normal subgroup of Γ. Thus D can be identified with the coset space Γ/Γ_0. This realization of bounded symmetric domains enables us to study the structure of D, using the algebraic machinery of Lie groups. Any bounded symmetric domain may be represented as the topological product of irreducible bounded symmetric domains; the class of irreducible bounded symmetric domains consists of four types of classical Cartan domains and two exceptional ones.

A bounded symmetric domain D is circular and star-shaped with respect to the origin, that is, $tz \in D$ when $z \in D$ and $t \in C$ with $|t| \leqq 1$ [7]. It has Bergman-Šilov boundary b which is circular and invariant under Γ [7]. The group Γ_0 is transitive on b [13, p. 922] and b has a unique normalized Γ_0-invariant measure μ, which is given by $d\mu_t = V^{-1}\, ds_t$, V the euclidean volume of b and ds_t the euclidean volume element at $t \in b$.

A complex-valued function $h: D \to C$ is harmonic on D if $\Delta h = 0$ for each Γ-invariant differential operator Δ of the hermitian space (D, M) [6, p. 340].

For $p > 0$ the Hardy space H^p is defined on D by

$$H^p \equiv H^p(D) = \left\{ f : f \text{ holomorphic on } D \text{ and } \sup_{0 \leqq r < 1} \left(\frac{1}{V} \int_b |f(rt)|^p\, ds_t \right)^{1/p} = M < \infty \right\}$$

and the space $\tilde{H}^p$ by

$$\tilde{H}^p \equiv \tilde{H}^p(D) = \{ f : f \text{ holomorphic on } D \text{ and } |f(z)|^p \leqq h(z) \text{ on } D,\ h \in \mathfrak{H}(D)\},$$

where

$$\mathfrak{H} \equiv \mathfrak{H}(D) = \left\{ h : h(z) = \int_b P(z, t)\phi(t)\, ds_t \equiv (P_z, \phi)_b,\ z \in D,\ \phi \in L(b) \right\},$$

$P(z, t)$ the Poisson kernel of the domain D.

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In §2 we give several properties of the spaces H^p and $\tilde{H}^p$ and prove that these spaces are equivalent for bounded symmetric domains. Rudin pointed out this equivalence for C^1 in [8]. In §3 we show that H^p is a Banach space for $p \geqq 1$ and a complete linear Hausdorff space for $0<p<1$, thus generalizing the results for the unit disc [11]. §§4 and 5 consider properties of linear functionals on H^p. For other treatments of H^p spaces ($p \geqq 1$) on bounded symmetric domains see [6].

2. Properties of the spaces H^p and $\tilde{H}^p$.

1. Let

$$D_r = \{rz : z \in D\}, \qquad b_r = \{rz : z \in b\}.$$

Since D is star-shaped with respect to 0, $\bar{D}_r \subset D$ if $0<r<1$ and $\lim_{r\to 1} D_r = D$. Also any compact subset K of $D \subset D_r$ for some $r<1$.

Theorem 1. *Let $u(z)$ be defined on D_r and $v(\zeta)=u(r\zeta R^{-1})$. Then $u \in \mathfrak{H}(D_r)$ if and only if $v \in \mathfrak{H}(D_R)$.*

Proof. This follows since under the transformation $z=r^{-1}\zeta R$, $u(z)=(P_{r,z}, \phi)_{b_r}$, $\phi \in L(b_r)$, goes into $v(\zeta)=(P_{R,z}, \psi)_{b_R}$, where $\psi(v)=\phi(rvR^{-1}) \in L(b_R)$ and conversely ($P_{R,z}$ and $P_{r,z}$ the respective Poisson kernels of D_R and D_r).

Theorem 2. *A function $u \in \mathfrak{H}(D)$ is harmonic on D.*

Proof. By definition the Poisson kernel is

$$P(z, t) = |S(z, \bar{t})|^2/S(z, \bar{z}), \tag{1}$$

where S is the Szegö (or Cauchy) kernel of D. By [5, p. 88]

$$S(z, \bar{t}) = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k} \phi_v^{(k)}(z)\overline{\phi_v^{(k)}(t)}, \qquad ((z, t) \in D \times b), \tag{2}$$

where $\{\phi_v^{(k)}\}$ is a complete orthonormal system of homogeneous polynomials on D, orthonormalized with respect to b, and $\phi_0=\phi_0^{(0)}=V^{-1}$. Since the convergence of series (2) is uniform on compact subsets of $D \times \bar{D}$ [5, p. 89], $S(z, \bar{t})$ is holomorphic in $(z, \bar{t})$ on $D \times D$ and continuous on $D \times \bar{D}$. In particular, $S(z, \bar{z})$ is holomorphic in $(z, \bar{z})$ for $z \in D$ and from (2)

$$S(z, \bar{z}) \geqq |\phi_0(z)|^2 = V^{-1}. \tag{3}$$

By the Weierstrass theorem [4, p. 6] $D_j S(z, \bar{t})$ is holomorphic on $D \times D$ and continuous on $D \times \bar{D}$ and $D_{jk}^2 S(z, \bar{z})$ is holomorphic in $(z, \bar{z})$ for $z \in D$. Thus the derivatives $D_{jk}^2 P(z, t)$ are bounded on $K \times b$, K a compact subset of D, and by the Lebesgue dominated convergence theorem $\Delta_z u=(\Delta_z P_z, \phi)=0$ ($\Delta=\Delta_z$ at z), since $P(z, t)$ is harmonic in z for $z \in D$, $t \in b$ [6, Theorem 3.5].

2. *Equivalence of H^p and $\tilde{H}^p$.*

Lemma 1. *Let u be a plurisubharmonic (psh) function on D and set $u_r(z)=u(rz)$ for $0<r<1$ and $z \in D$. Then*

$$u_r(z) \leqq (P_z, u_r). \tag{4}$$

Proof. Since u is psh on D, u_r is psh on $\bar{D}$ for $r<1$. Thus

$$u_r(0) \leqq \frac{1}{2\pi}\int_0^{2\pi} u_r(te^{i\theta})\,d\theta \tag{5}$$

for $t\in b$ [10, p. 73]. Let $\gamma\in\Gamma$ and $\gamma(z)=0$. Set $U_r(w)=u_r(\gamma^{-1}(w))$. Since psh functions are invariant under biholomorphic mappings [10, p. 81], U_r is psh on $\bar{D}$ so that $U_r(0)$ satisfies (5). Integrate over b and use Fubini's theorem

$$U_r(0)V \leqq \frac{1}{2\pi}\int_b ds_t \int_0^{2\pi} U_r(te^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_b U_r(te^{i\theta})\,ds_t = \int_b U_r(t')\,ds_{t'}, \tag{6}$$

since b is circular and $ds_t=ds_{t'}$ under $t'=te^{i\theta}$. Thus

$$u_r(z) \leqq \frac{1}{V}\int_b u_r(\gamma^{-1}(t'))\,ds_{t'}.$$

Set $t=\gamma^{-1}(t')$. Then $b=\gamma^{-1}(b)$ and by the invariance of the measure $P(z,t)\,ds_t$ under γ

$$(1/V)\,ds_{t'} = P(0,t')\,ds_{t'} = P(z,t)\,ds_t \tag{7}$$

[6, p. 339]. Thus (4) follows.

LEMMA 2. *The function*

$$M(r) = \frac{1}{V}\int_b |f(rt)|^p\,ds_t \tag{8}$$

is a monotone nondecreasing function of r on $[0, 1)$.

Proof. Since f is holomorphic on D, f_r is holomorphic on $\bar{D}$ and $|f_r|^p$ is psh on $\bar{D}$ for $p>0$ [10, p. 74]. As in (6)

$$I = \frac{1}{2\pi}\int_b ds_t \int_0^{2\pi} |f(rte^{i\theta})|^p\,d\theta = \int_b |f(rt)|^p\,ds_t = VM(r). \tag{9}$$

By the definition of psh, $|f(\lambda t)|^p$ is subharmonic with respect to λ in every component of the open set $O_t=\{\lambda : \lambda t\in D\}$. From [10, p. 62] the function

$$m(rt) = \frac{1}{2\pi}\int_0^{2\pi} |f(rte^{i\theta})|^p\,d\theta, \tag{10}$$

which is the mean value of a subharmonic function, is a nondecreasing function of r and convex with respect to $\log r$ for all t. By (9) and (10) for $r<r'$

$$M(r) = \frac{1}{V}\int_b m(rt)\,ds_t \leqq \frac{1}{V}\int_b m(r't)\,ds_t = M(r').$$

Also $M(r)$ is convex with respect to $\log r$ in (0, 1).

THEOREM 3. *For $p>0$, $f\in H^p$ if and only if $f\in \tilde{H}^p$.*

Proof. Since $|f|^p$ is psh on D for $p>0$, (4) holds for $|f_r|^p$, $0<r<1$.

Let $f \in H^p$. Since ds_t is a finite Borel measure and b is circularly invariant, by a result of Bochner [1, Theorems 2, 3] there exists a function ψ, measurable on b, such that

$$\lim_{r\to 1} \int_b |f(rt)-\psi(t)|^p \, ds_t = 0.$$

Since $P_z(t)$ is uniformly continuous and nonnegative on the compact set b,

$$(11) \qquad \lim_{r\to 1} \int_b |f(rt)-\psi(t)|^p P(z, t) \, ds_t = 0.$$

From (11) follows by Minkowski's inequality for $p \geqq 1$ and the inequality

$$(12) \qquad (a+b)^p \leqq a^p + b^p,$$

$a, b \geqq 0$ and $0<p<1$ [11] that

$$(13) \qquad \lim_{r\to 1} (P_z, |f_r|^p) = (P_z, |\psi|^p).$$

In particular since $P(0, t) = V^{-1}$ for $t \in b$, $\psi \in L^p(b)$. Let $r \to 1$. From the continuity of f on D and (4) and (13) follows

$$(14) \qquad 0 \leqq |f(z)|^p = \lim_{r\to 1} |f_r(z)|^p \leqq \lim_{r\to 1} (P_z, |f_r|^p) = (P_z, |\psi|^p) = u^*(z).$$

Since $u^* \in \mathfrak{H}(D)$, $f \in \tilde{H}^p$.

Remark. (14) is proved in [12] with reference to [1] for a method of proof of inequality (4) for $|f_r|^p$. However no details are given in [1].

Conversely let $f \in \tilde{H}^p$. Then $|f(z)|^p \leqq h(z) = (P_z, \phi)$, $\phi \in L(b)$ and hence $|f_r(t)|^p \leqq h_r(t)$ on $\bar{D}$ for $r<1$. Integrate over b and use Fubini's theorem. Then

$$\frac{1}{V}\int_b |f_r(t)|^p \, ds_t \leqq \frac{1}{V}\int_b h_r(t)\, ds_t = \frac{1}{V}\int_b \int_b P(rt, v)\phi(v)\, ds_v \, ds_t$$
$$= \frac{1}{V}\int_b \phi(v)\, ds_v = h(0) < \infty,$$

since $P(rt, v) = P(rv, t)$ for $v, t \in b$ [5, Theorem 4.5.2] and $\int_b P(rv, t)\, ds_t = 1$. Thus $f \in H^p$.

Another necessary and sufficient condition that $f \in H^p$ is given by

Theorem 4. *Let $z_0 \in D$ and f be holomorphic on D. Let r, $0<r<1$, be such that $z_0 \in D_r$. Then $f \in H^p(D)$ if and only if there exists a constant $B(z_0)$, independent of r, such that*

$$(15) \qquad (P_{z_0}, |f_r|^p) \leqq B(z_0).$$

Proof. The necessity of (15) follows from the uniform continuity of $P_{z_0}(t)$ on b and Lemma 2, namely

$$(16) \qquad (P_{z_0}, |f_r|^p) \leqq \max_{t\in b} P_{z_0}(t) VM(r) \equiv b_1(z_0) VM(r) \leqq b_1(z_0) MV \equiv B(z_0).$$

Conversely, let $b_2(z_0)=\min_{t\in b} P_{z_0}(t)$; $b_2(z_0)>0$ since $P_{z_0}(t)>0$ on b. Proof. For any $t\in b$ there exists a holomorphic automorphism γ_t of D which takes $z_0\to 0$ and $b\to b$. Let $t\to t'$. By (7) $P_{z_0}(t)>0$. Since $t\in b$ was arbitrary$P_{z_0}(t)$ >0 on b. Then by (14)

$$\int_b |f_r|^p\, ds_t \leqq b_2^{-1}(z_0)(P_{z_0}, |f_r|^p) \leqq b_2^{-1}(z_0)B(z_0)$$

so that $f\in H^p$.

3. **The topology of H^p spaces.** For $p>0$ set

$$\|f\|_p = \sup_{0\leqq r<1}\left(\frac{1}{V}\int_b |f(rt)|^p\, ds_t\right)^{1/p}.$$

REMARK From (2.14) with $z=0$ and Lemma 2 follows $\|f\|_p=u^*(0)^{1/p}$, where $u^*\in\mathfrak{H}(D)$.

For $p\geqq 1$ the triangle inequality follows for $f, g\in H^p$ by Minkowski's inequality and properties of sup. For $0<p<1$ Minkowski's inequality does not hold but (2.12) applied to f, g gives

(1) $$\|f+g\|_p^p \leqq \|f\|_p^p+\|g\|_p^p \qquad (0<p<1).$$

LEMMA 3. *Let $f\in H^p$. For any $z\in D$ there exists a constant $C(z)$, depending on z, p, and D but not on f, such that*

(2) $$|f(z)| \leqq C(z)\|f\|_p.$$

For any compact set K of D there exists a constant $C=C(D, K, p)$, depending on D, K and p but not on f, such that for $z\in K$

(3) $$|f(z)| \leqq C\|f\|_p.$$

Hence if $\|f-f_n\|_p\to 0$ as $n\to\infty$, then $f_n\to f$ uniformly on compact subsets of D.

Proof. From (2.8) and (2.14) follows $|f_r(z)|^p\leqq b_1(z)M(r)\leqq b_1(z)V\|f\|_p^p$ for $z\in D$. Letting $r\to 1$ gives (2) with $C(z)=V^{1/p}b_1(z)^{1/p}$. If K is a compact subset of D, then

$$|f_r(z)|^p \leqq \max_{z\in K} P(z,t)V\|f\|_p^p \equiv C^p(K, D, p)\|f\|_p^p$$

and (3) follows when $r\to 1$.

From (2), $\|f\|_p=0$, implies $f(z)=0$ on D and conversely. Thus H^p is a metric space for $p\geqq 1$ and satisfies all the axioms except the triangle inequality for $0<p<1$. As usual a subset O of H^p is said to be open if for every $f_0\in O$ there exists $\rho>0$ such that $\{f:\ \|f-f_0\|_p<\rho\}\subset O$; for $p\geqq 1$ this gives the usual topology induced by the metric. It is easy to prove that the Hausdorff separation axiom holds. Thus H^p is a linear Hausdorff space. From the last statement of Lemma 3 the completeness of H^p follows by well-known procedures from the completeness of C^1, the triangle inequality for $p\geqq 1$, and inequality (1) for $0<p<1$.

Hence:

THEOREM 5. *For $p \geqq 1$ H^p is a Banach space and for $0<p<1$ a complete linear Hausdorff space.*

THEOREM 6. *H^p is equivalent to a closed subspace of $L^p(b)$ (that is, there exists an algebraic isomorphism σ of H^p onto a closed subspace of $L^p(b)$, the isomorphism being norm-preserving).*

Proof. From the monotonicity of $M(r)$ and (2.14) with $z=0$ follows

$$\|f\|_p = \lim_{r\to 1} M(r)^{1/p} = \left(\frac{1}{V}\int_b |\psi(t)|^p\, ds_t\right)^{1/p} \equiv \|\psi\|_p.$$

Define a mapping σ by $\sigma(f)=\psi$. From $\|f\|_p=\|\psi\|_p$ follows σ is 1-1 from H^p onto a subspace of $L^p(b)$. Also $\sigma(H^p)$ is closed in $L^p(b)$ [11, p. 802].

A final property is that H^p spaces are perfectly separable as follows by the same proof as in [11].

4. **Linear functionals.** Let γ be a functional in H^p. Then $\gamma \in (H^p)^*$ if and only if γ is bounded on the unit sphere in H^p. Topologize $(H^p)^*$ by setting $\|\gamma\| =\sup_{\|f\|=1} |\gamma(f)|$. Then $(H^p)^*$ is a Banach space [11]. The class $[L^p(0, 2\pi)]^*$ of linear functionals on $L^p(0, 2\pi)$, $0<p<1$, contains only the zero functional but, as in the case of the unit disc, $(H^p)^*$ contains other elements [11]. Set

$$\gamma_{n,z}^{(\nu)}(f) = (1/n!)D_z^n f(z), \qquad \nu = 1,\dots, m_n, \quad D_z^n = \partial^n/\partial z_1^{n_1}\cdots\partial z_N^{n_N},$$

where m_n is the number of derivatives of order n. Similar to Theorem 6 in [11], we obtain precise bounds for the norms $\|\gamma_{n,z}^{(\nu)}\|$. Such bounds will be used in the proof of Theorem 9 and later papers.

THEOREM 7. *$\gamma_{n,z}^{(\nu)} \in (H^p)^*$ $(n=0, 1, 2, \dots; \nu=1,\dots, m_n)$ for $z\in D$ and*

$$\|\gamma_{0,z}\| \leqq 1/(1-r_z)^{2N/p}, \tag{1}$$

$$\|\gamma_{n,z}^{(\nu)}\| \leqq \frac{r_{n,z}^{N/2} V^{1/2}}{n!(r_{n,z}-r_z)^{N/2}(1-r_{n,z})^{2N/p}}[D_{z,\bar z}^{2n} S(Z_{n,r}, \bar Z_{n,r})]^{1/2} \qquad (n>0), \tag{2}$$

$(Z_{n,r}=(r_{n,z}r_z)^{-1/2}z)$, where r_z, $0<r_z<1$, depends on z only and $r_{n,z}$ is the value of r on $(r_z, 1)$, which minimizes the right side of (8). (*If $z=0$, replace r_z by $\frac{1}{2}r_{n,z}$.*)

Proof. (1) and (2) imply that $\gamma_{n,z}^{(\nu)} \in (H^p)^*$.

Proof of (1). Fix $r_1\in(0, 1)$. By (2.4) applied to $|f_{r_1}|^p$ and (2.16)

$$|f_{r_1}(z)|^p \leqq \max_{t\in b} P(z, t)V\|f\|_p^p. \tag{3}$$

From (2.1) and (2.2) and the fact that $\phi_\nu^{(k)}$ is homogeneous of degree k

$$P(z, t) = S(z, \bar z)^{-1} \lim_n \left|\sum_{k=0,\nu}^{n} \phi_\nu^{(k)}(zr^{-1})\overline{\phi_\nu^{(k)}(t)}r^k\right|^2,$$

where r is chosen so that $zr^{-1} \in \bar{D}$. By the maximum principle and Schwarz inequality the right side is

$$\leqq S(z, \bar{z})^{-1} \lim_n \max_{\zeta \in b} \sum_{k=0,\nu}^{n} |\phi_\nu^{(k)}(\zeta)|^2 r^k \sum_{k=0,\nu}^{n} |\phi_\nu^{(k)}(t)|^2 r^k$$
$$\leqq 1/V(1-r)^{2N}$$

by (2.3) and [5, Theorem 4.5.1]. Since D is circular and star-shaped, it is clear that there is a unique $r_z \in [0, 1)$ such that $z \in \bar{D}_{r_z}$ and $z \notin D_r$ for $r < r_z$.

Thus

$$P(z, t) \leqq 1/V(1-r_z)^{2N}; \tag{4}$$

and from (3) and (4)

$$|f_{r_1}(z)| \leqq (1-r_z)^{-2N/p} \|f\|_p.$$

(1) follows by letting $r_1 \to 1$.

Proof of (2). Since f is holomorphic on $\bar{D}_r$ for $r<1$, the Cauchy integral formula gives $f(z) = (S_{r,z}, f)_{b_r}$, $(z \in D_r)$, $S_{r,z}$ the Szegö kernel of D_r. By [4, p. 7, Corollary 2] D_z^n and $\int_{b_r}$ can be interchanged, giving

$$|D_z^n f(z)| \leqq \int_{b_r} |D_z^n S_r(z, \bar{t})|\, |f(t)|\, ds_t \leqq \max_{t \in b_r} |D_z^n S_r(z, \bar{t})| \int_{b_r} |f(t)|\, ds_t. \tag{5}$$

But

$$S_r(z, \bar{t}) = \sum_{k,\nu} \phi_\nu^{(k)}(r^{-1}z)\overline{\phi_\nu^{(k)}(r^{-1}t)} = \sum_{k,\nu} \phi_\nu^{(k)}(z)\overline{\phi_\nu^{(k)}(r^{-1}t)}r^{-k},$$

where the convergence is uniform for $z \in$ compact subsets of D_r and $t \in b_r$. By Weierstrass's theorem [4, p. 6]

$$D_z^n S_r(z, \bar{t}) = \sum_{k \geqq n,\nu} D_z^n \phi_\nu^{(k)}(z)\overline{\phi_\nu^{(k)}(r^{-1}t)}r^{-k} = \sum_{k \geqq n,\nu} D_z^n \phi_\nu^{(k)}(r_z^{-1}z)\overline{\phi_\nu^{(k)}(r^{-1}t)}\left(\frac{r_z}{r}\right)^k,$$

if $z, \neq 0, \in \bar{D}_{r_z} \subset D_r$. By the Schwarz inequality

$$\begin{aligned} |D_z^n S_r(z, \bar{t})|^2 &\leqq \sum_{k \geqq n,\nu} |D_z^n \phi_\nu^{(k)}(r_z^{-1}z)|^2 \left(\frac{r_z}{r}\right)^k \sum_{k,\nu} |\phi_\nu^{(k)}(r^{-1}t)|^2 \left(\frac{r_z}{r}\right)^k \\ &\leqq \frac{r^N \mathscr{S}_n((rr_z)^{-1/2}z)}{V(r-r_z)^N}, \end{aligned} \tag{6}$$

where $\mathscr{S}_n(Z) = \sum_{k \geqq n,\nu} |D_z^n \phi_\nu^{(k)}(Z)|^2$. If $z=0$, r_z can be replaced by $\frac{1}{2}r$, say. Now if $t \in b_r$, then $t \in \bar{D}_r$ and $\notin D_\rho$ for $\rho < r$. Thus in (4) we can take $r_t = r$ for all t. Hence by (3) with $r_1 = 1$ and (4)

$$|f(t)| \leqq \|f\|_p/(1-r)^{2N/p}. \tag{7}$$

Using (6) and (7) in (5) gives

$$|D_z^n f(z)| \leqq \frac{V^{1/2} r^{N/2} \mathscr{S}_n^{1/2}(Z)}{(r-r_z)^{N/2}(1-r)^{2N/p}} \|f\|_p \qquad (Z = (rr_z)^{-1/2}z). \tag{8}$$

On $(r_z, 1)$ the function

$$Y(r) = r^{N/2}(r-r_z)^{-N/2}(1-r)^{-2N/p}\mathscr{S}_n^{1/2}(Z)$$

is positive and continuous and $\to \infty$ as $r \to 1^-$ and $\to r_z^+$, where $0 < \mathscr{S}_n^{1/2}(zr_z^{-1}) \leqq \infty$. Hence $Y(r)$ assumes its minimum value on $(r_z, 1)$ on the compact set $I_\sigma = \{r_z+\sigma, 1-\sigma\}$ if $\sigma > 0$ is sufficiently small, and this minimum value is positive. Suppose Y assumes its minimum value at $r = r_{n,z}$. Since $Z_{n,r} = (r_{n,z}r_z)^{-1/2}z \in D$, $\mathscr{S}_n(Z_{n,r}) = D_{z,\bar{z}}^{2n}S(Z_{n,r}, \bar{Z}_{n,r})$ and (2) follows.

Corollary 1 [11]. *There exists a countable collection of linear functionals $\{\eta_n\}$ on H^p such that if $f \in H^p$ and $f \neq 0$, then there is an n with $\eta_n(f) \neq 0$.*

Proof. Let $z_0 \in D$ and set $\eta_n = \gamma_{n,z_0}$. Since $f \neq 0$ on D, by the identity theorem $f(z) \not\equiv 0$ in any polydisc neighborhood N of $z_0 \subset D$. Hence by the power series expansion of f in N some derivative $D_{z_0}^n f(z_0) \neq 0$. Thus $\gamma_{n,z_0}(f) \neq 0$.

Theorem 8. *Let $F = \{f \in H^p : \gamma(f)$ is bounded on F for fixed $\gamma \in (H^p)^*\}$. Then there exists $B > 0$, independent of f, such that*

$$\text{(9)}\quad \begin{aligned} &\text{(a)}\quad |\gamma(f)| \leqq B\|\gamma\|, \\ &\text{(b)}\quad |f(z)| \leqq B/(1-r_z)^{2N/p}, \\ &\text{(c)}\quad |\gamma_{n,z}^{(\nu)}(f)| \leqq \frac{BV^{1/2}r_{n,z}^{N/2}}{n!(r_{n,z}-r_z)^{N/2}(1-r_{n,z})^{2N/p}}\{D_{z,\bar{z}}^{2n}S(Z_{n,r}, \bar{Z}_{n,r})\}^{1/2} \qquad (n > 0), \end{aligned}$$

$(Z_{n,r} = (r_{n,z}r_z)^{-1/2}z)$ for all $f \in F$. (If $z = 0$, set $r_0 = \frac{1}{2}r_{n,0}$.)

Proof. The proof of (a) uses only functional analysis and is the same as the proof of Theorem 7 in [11]. Inequalities (b) and (c) follow by setting $\gamma = \gamma_{n,z}^{(\nu)}$ and using (1) and (2).

5. **Weak convergence.** A sequence $\{f_n\} \subset H^p$ converges weakly to $f \in H^p$, $f_n \to^w f$, if $\lim_n \gamma(f_n) = \gamma(f)$ for every $\gamma \in (H^p)^*$. By Corollary 1 the limit is unique.

The following lemma is more general than necessary but has some independent interest. Let Δ_R be a polydisc of radius R and center 0.

Lemma 4 (Vitali's Convergence Theorem for C^N). *Let $\{f_n\}$ be a sequence of holomorphic functions on the closed polydisc $\bar{\Delta}_R$, which are bounded independently of z and n on $\bar{\Delta}_R$. Also $f_n \to$ a limit as $n \to \infty$ on a set $\{z'\}$ with limit point 0 and such that for each j, $1 \leqq j \leqq N$, $\{z'_j\}$ is an infinite set. Then $\{f_n\}$ tends uniformly to a limit on compact subsets of Δ_R.*

Proof. It is sufficient to consider the case $N = 2$. Then f_n, holomorphic on $\bar{\Delta}_R$, has a power series representation

$$f_n(z) = \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k \qquad (z \in \bar{\Delta}_R),$$

where the convergence is absolute and uniform on compact subsets of $\bar{\Delta}_R$. Following the proof in [9] for $N=1$, we show that $\lim_n a_{jk}^{(n)}$ exists for each j, k and equals a_{jk}, say. By Schwarz's Lemma [2] and the uniform boundedness it follows as in [9] that $\{a_{00}^{(n)}\}$ is a Cauchy sequence and hence convergent. The function

$$g_{10}^{(n)}(z_1) = \sum_{j=1}^{\infty} a_{j0}^{(n)} z_1^{j-1} = \frac{f_n(z_1, 0) - a_{00}^{(n)}}{z_1}$$

$(z_1 \neq 0)$ satisfies the same hypotheses as $f_n(z)$.

Proof. The functions $f_n(z_1, z_2')$ are holomorphic and uniformly bounded independently of n and z_1 on the closed disc $|z_1| \leqq R$. Also 0 is a limit point of the set (z_1') and $\lim_n f_n(z_1', z_2')$ exists. Thus by Vitali's convergence theorem for $N=1$, $\lim_n f_n(z_1, z_2')$ exists uniformly on compact subsets of $|z_1| < R$. Since also $\lim_{z_2' \to 0} f_n(z_1, z_2') = f_n(z_1, 0)$, the hypotheses of the Moore-Osgood theorem are satisfied so that $\lim_n \lim_{z_2' \to 0} f_n(z_1', z_2') = \lim_n f_n(z_1', 0)$ exists. Thus $\lim g_{10}^{(n)}(z')$ exists. Also $g_{10}^{(n)}$ is holomorphic on $\bar{\Delta}_R$ with a removable singularity at $z_1 = 0$. Now $|g_{10}^{(n)}(z_1)| \leqq 2MR^{-1}$ on $|z_1| = R$ so that by the maximum principle $|g_{10}^{(n)}(z_1)| \leqq 2MR^{-1}$ on $|z_1| \leqq R$. Hence similarly as for $\{a_{00}^{(n)}\}$, $\lim_n a_{10}^{(n)}$ exists. By an analogous argument $\lim_n a_{j0}^{(n)}$ and $\lim_n a_{0j}^{(n)}$ exist for all $j \geqq 1$. Next set

$$(1) \qquad g_{11}^{(n)}(z) = \sum_{j,k=1}^{\infty} a_{jk}^{(n)} z_1^{j-1} z_2^{k-1} = \frac{f_n(z_1, z_2) - f_n(z_1, 0) - f_n(0, z_2) + f_n(0, 0)}{z_1 z_2},$$

$g_{11}^{(n)}(z)$ satisfies the same hypotheses as $f_n(z)$. $\lim_n g_{11}^{(n)}(z')$ exists since the four limits on the right side of (1) exist. Also $g_{11}^{(n)}$ is holomorphic on $\bar{\Delta}_R$ if $z_1 z_2 \neq 0$ and is locally bounded in the neighborhood of points $(z_1, 0)$, $(0, z_2)$, $(0, 0)$ since

$$\lim_{z_2 \to 0} g_{11}^{(n)}(z) = \sum_{j=1}^{\infty} a_{j1}^{(n)} z_1^{j-1} = z_1^{-1}[\partial f_n(z_1, 0)/\partial z_2 - \partial f_n(0, 0)/\partial z_2]$$

for $z_1 \neq 0$ and similarly for the other two limits. Thus by Riemann's theorem on removable singularities [2] $g_{11}^{(n)}$ is holomorphic on $\bar{\Delta}_R$. Also on $c = \{z : |z_j| = R, j = 1, 2\}$, $|g_{11}^{(n)}(z)| \leqq 4MR^{-2}$ and by the maximum principle for polydiscs $|g_{11}^{(n)}(z)| \leqq 4MR^{-2}$ on $\bar{\Delta}_R$. Thus as above $\lim_n a_{11}^{(n)}$ exists. Similarly $\lim_n a_{jk}^{(n)}$ exists.

Finally we show that

$$(2) \qquad \lim_n \sum_{j,k=0}^{\infty} a_{jk}^{(n)} z_1^j z_2^k = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k.$$

By the Cauchy inequality for derivatives [2] and the uniform boundedness follows $|a_{jk}^{(n)}| \leqq MR^{-j-k}$ and hence $|a_{jk}| \leqq MR^{-j-k}$. Thus the series on the right of (2) converges absolutely and uniformly on compact subsets of Δ_R. Now given $\varepsilon > 0$ there exists $K = K(\sigma, \varepsilon)$ such that

$$\sum_{j=K+1}^{\infty} \left(\frac{R-\sigma}{R}\right)^j < \tfrac{1}{2}\varepsilon A_\sigma, \qquad A_\sigma = \frac{\sigma}{4RM} \qquad (\sigma > 0),$$

and $N=N(K,\varepsilon)$ such that for $n>N$

$$|a_{jk}^{(n)}-a_{jk}| < \tfrac{1}{2}\varepsilon B_K, \qquad B_K = \left(\frac{R-1}{R^{K+1}-1}\right)^2 \qquad (j,k = 1,\ldots,K).$$

Then

$$\left|\sum_{j,k=0}^{\infty} a_{jk}^{(n)}z_1^j z_2^k - \sum_{j,k=0}^{\infty} a_{jk}z_1^j z_2^k\right|$$

$$\leqq \left|\sum_{j,k=0}^{K}(a_{jk}^{(n)}-a_{jk})z_1^j z_2^k\right| + \left|\left(\sum_{j=K+1}^{\infty}\sum_{k=0}^{\infty}+\sum_{j=0}^{K}\sum_{k=K+1}^{\infty}\right)(a_{jk}^{(n)}-a_{jk})z_1^j z_2^k\right|$$

$$\leqq \sum_{j,k=0}^{K}|a_{jk}^{(n)}-a_{jk}|R^{j+k}+2M\left(\sum_{j=K+1}^{\infty}\sum_{k=0}^{\infty}+\sum_{j=0}^{K}\sum_{k=K+1}^{\infty}\right)\left(\frac{R-\sigma}{R}\right)^j\left(\frac{R-\sigma}{R}\right)^k < \varepsilon$$

for $n>N$. Thus (2) holds and Lemma 4 is proved.

We have

THEOREM 9. *If $f_n \to {}^w f$ in H^p, then $\lim_n f_n(z)=f(z)$ uniformly on compact subsets of D.*

Proof. Since $\lim_n \gamma(f_n)=\gamma(f)$ for $\gamma \in (H^p)^*$, $\{\gamma(f_n)\}$ is bounded independently of n. From inequality (4.9b) follows $|f_n(z)| \leqq B(1-r)^{-2N/p}$ for $z \in \bar{D}_r$, which bound is independent of n and z. In particular $\gamma_{0,z}(f_n) \to \gamma_{0,z}(f)$, that is, $f_n(z) \to f(z)$ for $z \in \bar{D}_r$. Hence by Lemma 4, $f_n(z) \to f(z)$ uniformly on compact subsets of D_r $(r<1)$. Hence $f_n(z) \to f(z)$ uniformly on compact subsets of D. (Lemma 4 was proved for a polydisc but the compact set $\bar{D}_r$ can be covered by a finite number of closed polydiscs and the conclusion of Lemma 4 will hold for $\bar{D}_r$ also.)

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