ON THE NUMBER OF MULTIPLICATIONS ON $SU(3)$ AND $Sp(2)$

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Dedicated to Professor A. Komatu for his sixtieth birthday

1. Introduction. A multiplication on a space X with a base point $*$ is a map $\mu: X \times X \rightarrow X$ such that $\mu(*, x) = \mu(x, *) = x$ for any x of X . Two multiplications μ_1 and μ_2 are called homotopic if μ_1 is homotopic to μ_2 relative to $X \vee X$.

The enumeration problem of the homotopy classes of multiplications on a given H -space was studied by James [5] for spheres and by Arkowitz and Curjel [1] for finite CW-complexes. In fact, they showed that there exists a one-to-one correspondence between the homotopy set $[X \wedge X, X]$ and the set of homotopy classes of multiplications of a finite CW-complex X with a homotopy associative multiplication.

Moreover they proved that the following Lie groups have a finite number of nonhomotopic multiplications:

$SO(n)$ for $n \leq 16$ and $n \neq 10, 14$; $SU(n)$ for $n \leq 5$; $Sp(n)$ for $n \leq 7$; the representative of G_2 , F_4 and E_7 ; and that all other classical groups and the representatives of the other exceptional structures have an infinite number of multiplications.

As is well known, $SU(2)$ and $Sp(1)$ have 12 distinct homotopy classes of multiplications, since these groups are homeomorphic to S^3 and since

$$[S^3 \wedge S^3, S^3] \cong \pi_6(S^3) \cong \mathbb{Z}_{12}.$$

Recently Naylor [11] calculated the order of $[SO(3) \wedge SO(3), SO(3)]$ and showed that $SO(3)$ has precisely 768 distinct homotopy classes of multiplications.

The purpose of this note is to compute the orders of $[SU(3) \wedge SU(3), SU(3)]$ and $[Sp(2) \wedge Sp(2), Sp(2)]$ and to prove

THEOREM 5.7. *There exist precisely $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ distinct homotopy classes of multiplications on $SU(3)$.*

THEOREM 6.6. *There exist precisely $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ distinct homotopy classes of multiplications on $Sp(2)$.*

Notation.

$[X, Y]$: the set of homotopy classes of continuous maps from X to Y .

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pG : the p -primary components of the abelian group G .

$\theta(G)$: the order of the group G .

$K^{(n)}$: the n -skeleton of a CW-complex K .

K/L : the quotient space of K by pinching L to a point for a CW-pair $K \supset L$.

$X \wedge Y$: the smash product $X \times Y / X \vee Y$.

$C_\alpha = Y \cup_\alpha CX$: the mapping cone of $\alpha \in [X, Y]$.

When we consider the mapping cone $Y \cup_f CX$ of a map $f: X \rightarrow Y$, we use often the homotopy class α of f in its place, since the homotopy type is determined by its homotopy class α .

The equality $X = Y$ for two spaces X and Y reads that X is homotopy equivalent to Y .

The paper is organized as follows. In §3 and §4, the cell structures of $SU(3) \wedge SU(3)$ and $Sp(2) \wedge Sp(2)$ are studied. Then the computations of the orders of $[SU(3) \wedge SU(3), SU(3)]$ and $[Sp(2) \wedge Sp(2), Sp(2)]$ are done in §5 and §6 respectively. The final section (§7) is devoted to the computations of ${}^2\pi_i(W_{5,2})$ for $i=15, 16$ and 17 which are needed in §5.

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2. Preliminaries. To begin with we recall some results on $\pi_{n+i}(S^n)$:

$\pi_n(S^n) \cong Z = \{\iota_n\}$ for $n \geq 1$. $\pi_{n+1}(S^n) \cong Z_2 = \{\eta_n\}$ for $n \geq 3$.

$\pi_{n+2}(S^n) \cong Z_2 = \{\eta_n^2\}$ for $n \geq 3$. $\pi_{n+3}(S^n) \cong Z_{24} = \{\omega_n\}$ for $n \geq 5$.

$\pi_{n+4}(S^n) = 0$ for $n \geq 6$. $\pi_{n+5}(S^n) = 0$ for $n \geq 7$.

$\pi_{n+6}(S^n) \cong Z_2 = \{\nu_n\}$ for $n \geq 5$, where $3\omega_n = \nu_n$.

$\pi_{13}(S^6) \cong Z_{60} = \{\Sigma^7\}$. $\pi_{14}(S^7) \cong Z_{120} = \{\Sigma^7\}$.

$\pi_{15}(S^8) \cong Z \oplus Z_{120} = \{\Sigma_8, S\Sigma^7\}$.

$\pi_{n+7}(S^n) \cong Z_{240} = \{\Sigma_n\}$ for $n \geq 9$,

where $S\Sigma^7 = 2\Sigma^7$, $S^2\Sigma^7 = 2\Sigma_9$. Furthermore, we use the following notations: $15\Sigma^7 = \sigma''$, $15\Sigma^7 = \sigma'$ and $15\Sigma_9 = \sigma_9$.

We also recall the well-known formulae

$$(2.1) \quad \pi_i(X \vee Y) \cong \pi_i(X) \oplus \pi_i(Y) \oplus \pi_{i+1}(X \times Y, X \vee Y).$$

$$(2.2) \quad [X \vee Y, Z] \cong [X, Z] \oplus [Y, Z] \quad \text{for a topological group } Z.$$

The groups $SU(3)$ and $Sp(2)$ are S^3 -bundles over S^n where $n=5$ and 7 respectively. So they have the cell structure as follows [7]:

$$G = S^3 \cup e^n \cup e^{n+3}.$$

It is also known that

$$(2.3) \quad e^5 \text{ is attached to } S^3 \text{ by } \eta_3 \text{ in } SU(3).$$

(2.4) e^7 is attached to S^3 by ω' in $Sp(2)$, where ω' is the Blakers-Massey element and a generator of $\pi_6(S^3) \cong Z_{12}$. ($S^n \omega' = 2\omega_{n+3}$ for $n \geq 2$.)

According to James [6], in the suspended complex

$$SG = S^4 \cup e^{n+1} \cup e^{n+4},$$

the cell e^{n+4} is attached by the element j_*x , where

$$j_*: \pi_{n+3}(S^4) \rightarrow \pi_{n+3}(S^4 \cup e^{n+1})$$

is the inclusion and $x \in \pi_{n+3}(S^4)$ is obtained from the characteristic element α for G by the Hopf construction. (This fact also follows from Lemma 2.32 of [12].)

Since the rotation group SO (4) splits as $\text{SO} (4) = S^3 \times \text{SO} (3)$, the characteristic elements are η_3 and ω' for $G = \text{SU} (3)$ and $\text{Sp} (2)$ respectively. So the attaching maps of e^{n+4} in SG are $j_*(\nu_4\eta_7)$ and $j_*(2\nu_4^2)$ respectively. By Proposition 5.9 of [15], $S^2(\nu_4\eta_7) = 0$. Hence we obtain

$$S^3 \text{SU} (3) = S^6 \cup e^8 \vee S^{11}.$$

Similarly, $S^2 \text{Sp} (2) = S^5 \cup e^9 \vee S^{12}$ follows, since $S(2\nu_4^2) = 0$ by Proposition 5.11 of [15]. Thus, we have shown

LEMMA 2.1. (i) $S^n(\text{SU} (3))$ has the homotopy type of $S^{n+3} \cup e^{n+5} \vee S^{n+8}$ for $n \geq 3$.

(ii) $S^n(\text{Sp} (2))$ has the homotopy type of $S^{n+3} \cup e^{n+7} \vee S^{n+10}$ for $n \geq 2$.

This lemma means that the cell e^8 in $\text{SU} (3)/S^3 = S^5 \cup e^8$ is not associated with S^5 . So one has

$$(2.5) \quad \text{SU} (3)/S^3 = S^5 \vee S^8.$$

Similarly,

$$(2.6) \quad \text{Sp} (2)/S^3 = S^7 \vee S^{10}.$$

We will study the cell structure of $G \wedge G$, which evidently has 9 cells (see the diagram where the number indicates the dimension of the cell).

$$K = \text{SU} (3) \wedge \text{SU} (3)$$

$$M = \text{Sp} (2) \wedge \text{Sp} (2)$$

For simplicity we set

$$K = \text{SU} (3) \wedge \text{SU} (3) \quad \text{and} \quad A = K^{(10)} = (S^3 \cup_{\eta_3} e^5) \wedge (S^3 \cup_{\eta_3} e^5).$$

$$M = \text{Sp} (2) \wedge \text{Sp} (2) \quad \text{and} \quad B = (S^3 \cup_{\omega'} e^7) \wedge (S^3 \cup_{\omega'} e^7).$$

Then we have

$$\text{LEMMA 2.2. (i) } S^3(\text{SU} (3) \wedge \text{SU} (3)) = S^3 A \vee (S^{14} \cup e^{16}) \vee (S^{14} \cup e^{16}) \vee S^{19}.$$

$$(ii) S^2(\text{Sp} (2) \wedge \text{Sp} (2)) = S^2 B \vee (S^{15} \cup e^{19}) \vee (S^{15} \cup e^{19}) \vee S^{22}.$$

Proof. By making use of Lemma 2.1, we have

$$\begin{aligned}
 S^3(\mathrm{SU}(3) \wedge \mathrm{SU}(3)) &= (S^3 \wedge \mathrm{SU}(3)) \wedge \mathrm{SU}(3) \\
 &= (S^6 \cup e^8 \vee S^{11}) \wedge \mathrm{SU}(3) \\
 (i) \quad &= (S^6 \cup e^8) \wedge \mathrm{SU}(3) \vee (S^{11} \wedge \mathrm{SU}(3)) \\
 &= (S^3 \cup e^5) \wedge (S^3 \wedge \mathrm{SU}(3)) \vee (S^{11} \wedge \mathrm{SU}(3)) \\
 &= (S^3 \cup e^5) \wedge (S^6 \cup e^8 \vee S^{11}) \vee (S^{14} \cup e^{16} \vee S^{19}) \\
 &= S^3 A \vee (S^{14} \cup e^{16}) \vee (S^{14} \cup e^{16}) \vee S^{19}. \quad \text{Q.E.D.}
 \end{aligned}$$

Similarly for (ii).

3. The cell structure of $\mathrm{SU}(3) \wedge \mathrm{SU}(3)$. First of all we study the structure of $A = K^{(10)}$.

Let C_η be the mapping cone of η_3 and let $1_c: C_\eta \rightarrow C_\eta$ be the identity map. Consider the map

$$1_c \wedge \eta_3: C_\eta \wedge S^4 \rightarrow C_\eta \wedge S^3.$$

Clearly $A = C_\eta \wedge C_\eta$ is the mapping cone of $1_c \wedge \eta_3$. However, according to Lemma 3.5 of [15], (taking the trivial coextension, since $2\eta_6 = 0$), there exists an element η^* of $\pi_9(S^6)$ such that $1_c \wedge \eta_3 = (S^4 p)^*(i_* \eta^*)$ holds in $[S^4 C_\eta, S^3 C_\eta]$, where $i = S^6 \rightarrow S^3 C_\eta$ is the inclusion and $p: C_\eta \rightarrow S^5$ is the shrinking map. Hence we obtain

$$A = (S^6 \cup_{\eta_6} e^8) \cup_{1_c \wedge \eta_3} C(S^7 \cup_{\eta_7} e^9) = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\eta e^{10}$$

where $\eta = i_* \eta^* \vee \eta_8$.

We show that $\eta^* = n\nu_6$ with odd n . Clearly η^* belongs to ${}^2\pi_9(S^6) \cong Z_8 = \{\nu_6\}$. Consider the cohomology of A with Z_2 -coefficient $H^*(A; Z_2)$. Let u and v be the generators of $H^3(C_\eta; Z_2)$ and $H^5(C_\eta; Z_2)$ respectively. Note that $S_q^2 u = v$ holds. Then the generators of $H^6(A; Z_2)$ and $H^{10}(A; Z_2)$ are represented by $u \wedge u$ and $v \wedge v$ respectively. By the Cartan formula we obtain

$$S_q^4(u \wedge u) = S_q^2 u \wedge S_q^2 u = v \wedge v.$$

Hence by Proposition 8.1 of [15] one deduces $\eta^* = n\nu_6$ with odd n . Thus

$$(3.1) \quad A = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\eta e^{10}, \quad \text{where } \eta = i_* n\nu_6 \vee \eta_8 \text{ with } n = \text{odd}.$$

Recall that $K = A \cup e_1^{11} \cup e_1^{13} \cup e_2^{13} \cup e_2^{16}$. Consider the attaching element $\alpha_1 \in \pi_{11}(A)$. Then $S^3 \alpha_1$ is the attaching element of e_1^{14} in $S^3(K^{(11)})$, but it is trivial by (i) of Lemma 2.2. Hence $\alpha_1 = 0$. In fact, one has $S^3: \pi_{10}(A) \cong \pi_{13}(S^3 A)$ since A is 5-connected. Similarly, the attaching element α_2 of e_2^{11} is trivial. Thus

$$K^{(11)} = A \vee S_1^{11} \vee S_2^{11}.$$

Next we study the attaching element β_1 of e_1^{13} :

$$\beta_1 \in \pi_{12}(K^{(11)}) \cong \pi_{12}(A) \oplus \pi_{12}(S_1^{11}) \oplus \pi_{12}(S_2^{11}).$$

For this we need

LEMMA 3.1. $S^3: \pi_{12}(K^{(11)}) \rightarrow \pi_{15}(S^3(K^{(11)}))$ is a monomorphism.

Proof. It is sufficient to show that $S^3: \pi_{12}(A) \rightarrow \pi_{15}(S^3A)$ is a monomorphism. The following diagram is commutative and the two rows are exact.

$$(3.2) \quad \begin{array}{ccccccc} \pi_{13}(A, S^6) & \longrightarrow & \pi_{12}(S^6) & \longrightarrow & \pi_{12}(A) & \longrightarrow & \pi_{12}(A, S^6) \\ \downarrow S^3 & & \downarrow S^3 & & \downarrow S^3 & & \downarrow S^3 \\ \pi_{16}(S^3A, S^9) & \longrightarrow & \pi_{15}(S^9) & \longrightarrow & \pi_{15}(S^3A) & \longrightarrow & \pi_{15}(S^3A, S^9) \end{array}$$

Let $p: (A, S^6) \rightarrow (A/S^6, *)$ be a smashing map. Then we have a commutative diagram:

$$\begin{array}{ccc} \pi_{13}(A, S^6) & \xrightarrow{p_*} & \pi_{13}(A/S^6) \\ \downarrow S^3 & & \downarrow S^3 \\ \pi_{16}(S^3A, S^9) & \xrightarrow{(S^3p)_*} & \pi_{16}(S^3A/S^9) \end{array}$$

Since A/S^6 is 7-connected, we have that $S^3: \pi_{13}(A/S^6) \rightarrow \pi_{16}(S^3A/S^9)$ is an isomorphism. By the Blakers-Massey theorem (e.g. Corollary 3.3 of [10]), we obtain that p_* is epimorphic and that $(S^3p)_*$ is isomorphic. Hence the first S^3 in the diagram (3.2) is an epimorphism. Similarly one can show that the last S^3 in the diagram (3.2) is an isomorphism. The second S^3 is known ([15]) to be monomorphic. Thus by the four lemma (Lemma 5.6.9 of [3]) we obtain the lemma. Q.E.D.

The attaching element β_1 of e_1^{13} is of the form:

$$\beta_1 = \gamma \oplus x_1\eta_{11} \oplus x_2\eta_{11} \quad \text{with } \gamma \in \pi_{12}(A), x_i \in \mathbb{Z}_2.$$

Then $S^3\beta = S^3\gamma \oplus x_1\eta_{14} \oplus x_2\eta_{14}$ is the attaching element of e_1^{16} in S^3K . We know, however, by (i) of Lemma 2.2 that e_1^{16} is attached not to S^3A but just to S_1^{14} . This implies that $S^3\gamma = 0$, $x_1 = 1$ and $x_2 = 0$. Since S^3 is a monomorphism by Lemma 3.1, we obtain that $\beta_1 = \eta_{11}$. Similarly the attaching element β_2 of e_2^{13} is checked to be η_{11} . Thus

$$K^{(13)} = A \vee (S_1^{11} \cup_{\eta_{11}} e_1^{13}) \vee (S_2^{11} \cup_{\eta_{11}} e_2^{13}).$$

The attaching element δ of e^{16} belongs to

$$\pi_{15}(K^{(13)}) \cong \pi_{15}(A) \oplus \pi_{15}(S_1^{11} \cup e_1^{13}) \oplus \pi_{15}(S_2^{11} \cup e_2^{13}).$$

On the other hand, by (2.5), we have that

$$(3.3) \quad \begin{aligned} & (\text{SU}(3)/S^3) \wedge (\text{SU}(3)) \\ &= (S^5 \vee S^9) \wedge \text{SU}(3) \\ &= S^5 \text{SU}(3) \vee S^9 \text{SU}(3) \\ &= (S_1^5 \cup e_1^{10}) \vee S^{10} \vee (S_1^{10} \cup e_1^{13}) \vee S^{16} \quad \text{by Lemma 2.1.} \end{aligned}$$

This implies that δ has no components in $\pi_{15}(S_1^{11} \cup e_1^{13})$. Similarly it is checked that δ has no components in $\pi_{15}(S_2^{11} \cup e_2^{13})$, and hence

$$(3.4) \quad K = A \cup_{\delta} e^{16} \vee (S_1^{11} \cup e_1^{13}) \vee (S_2^{11} \cup e_2^{13}).$$

Then we will show that

$$(3.5) \quad A \cup_\delta e^{16}/S^6 = S_1^8 \times S_2^8 \cup e^{10}.$$

It is easily checked that the homomorphism induced by the inclusion:

$$\pi_{15}(S_1^8 \vee S_2^8) \rightarrow \pi_{15}((S_1^8 \vee S_2^8) \cup e^{10})$$

is an isomorphism, and hence $A/S^6 = (S_1^8 \vee S_2^8) \cup_\varepsilon C(S^9 \vee S^{15})$, where ε belongs to

$$\begin{aligned} [S^9 \vee S^{15}, S_1^8 \vee S_2^8] &\cong \pi_9(S_1^8 \vee S_2^8) \oplus \pi_{15}(S_1^8) \oplus \pi_{15}(S_2^8) \\ &\quad \oplus \pi_{16}(S_1^8 \times S_2^8, S_1^8 \vee S_2^8). \end{aligned}$$

Thus the restriction of ε to S^{15} has the form

$$(\varepsilon|S^{15}) = \alpha_1 \oplus \alpha_2 \oplus b\tau,$$

where $\alpha_i \in \pi_{15}(S_i^8)$, $i=1, 2$, τ is a generator of $\pi_{16}(S_1^8 \times S_2^8, S_1^8 \vee S_2^8)$ and b is some integer.

To prove (3.5) it is sufficient to show $\alpha_i=0$ and $b=1$. Let u_1 and u_2 be the generators of $H^8(A \cup e^{16}/S^6) \cong Z \oplus Z$, then the generator of $H^{16}(A \cup e^{16}/S^6)$ is a product $u_1 \wedge u_2$, since $H^{16}(K) \cong H^8(\text{SU}(3)) \otimes H^8(\text{SU}(3))$ holds. Suitably choosing orientations, we obtain $b=1$.

Consider the map $p: A^{(10)}/S^6 \xrightarrow{p_1} K/S^6 \xrightarrow{p_2} (\text{SU}(3)/S^3) \wedge \text{SU}(3)$, where p_1 is the inclusion and p_2 is the map shrinking $S_2^8 \vee S_2^{11}$ to a point in $K/S^6 = (S_1^8 \vee S_2^8) \cup C(S^9 \vee S^{15}) \vee (S_1^{11} \cup e_1^{13}) \vee (S_2^{11} \cup e_2^{13})$. Then we have that $p_*(\varepsilon|S^{15}) = \alpha_1$, which is the attaching element of e^{16} in $(\text{SU}(3)/S^3) \wedge \text{SU}(3)$. Hence $\alpha_1=0$ by (3.5). Similarly $\alpha_2=0$. Thus $(\varepsilon|S^{15})$ is a universal Whitehead product and hence (3.5) follows.

One can easily check that the cell e^{10} is attached to $S_1^8 \times S_2^8$ by (η_8, η_8) in $A \cup e^{16}/S^6$. Similarly $C(S_1^7 \vee S_2^7)$ is attached to S^6 by (η_6, η_6) in $A^{(8)}$.

Summing up these facts we state

PROPOSITION 3.2. *The smash product $\text{SU}(3) \wedge \text{SU}(3)$ has the cell structure of $K = A \cup e^{16} \vee (S^{11} \cup_{\eta_{11}} e^{13}) \vee (S^{11} \cup_{\eta_{11}} e^{13})$, where*

- (i) $A = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\alpha e^{10}$, $\alpha = i_* n v_6 \vee \eta_8$ with odd n ,
- (ii) $A \cup e^{16}/S^6 = S^8 \times S^8 \cup_\eta e^{10}$, $\eta = (\eta_8, \eta_8)$,
- (iii) $A^{(8)} = S^6 \cup_{\eta_6} e^8 \vee S^8$.

Further we investigate the property of the suspended complex $S(A \cup e^{16})$ of $A \cup e^{16}$.

PROPOSITION 3.3. *Let α be the same as in (i) of Proposition 3.2. Then we have*

$$S(A \cup e^{16}) = SA^{(8)} \cup_\gamma C(S^{10} \vee S^{16}) \quad \text{where } \gamma = S\alpha \vee \beta \circ \eta_{15}$$

with some β of $\pi_{15}(SA^{(8)})$.

The rest of this section is devoted to the proof of this proposition.

Let D be the complex $S(SU(3) \wedge SU(3))$. According to the above Proposition 3.2, the complex $S(A \cup e^{16})$ is a retract of D .

Let us recall (see §2) that $S(SU(3)) = S^4 \cup_{\xi'} C(S^5 \vee S^8)$ with $\xi' = \eta_4 \vee \nu_4 \eta_7$.

Let 1_G be the identity map of $SU(3)$. Then D has the homotopy type of

$$(3.6) \quad \begin{aligned} (S^4 \cup_{\xi'} C(S^5 \vee S^8)) \wedge SU(3) \\ = (S^4 \wedge SU(3)) \cup_{\xi'} C(S^5 \wedge SU(3) \vee S^8 \wedge SU(3)) \end{aligned}$$

with $\xi = 1_G \wedge \eta_4 \vee 1_G \wedge (\nu_4 \eta_7)$.

By the naturality we have that

$$(3.7) \quad 1_G \wedge (\nu_4 \eta_7) = (1_G \wedge \nu_4) \circ (1_G \wedge \eta_7).$$

For simplicity we put $G_n = S^n \wedge SU(3)$. Then we have $G_n = S^{n+3} \cup_{\eta_{n+3}} e^{n+5} \vee S^{n+8}$ for $n \geq 3$ by Lemma 2.1.

Then $1_G \wedge \nu_4$ belongs to $[G_7, G_4]$ and $1_G \wedge \eta_7 \in [G_8, G_7]$, where we have the isomorphisms by (2.1) and (2.2):

$$(3.8) \quad [G_7, G_4] \cong [S^{10} \cup e^{12}, G_4] \oplus \pi_{15}(S^7 \cup e^9) \oplus \pi_{15}(S^{12}).$$

$$(3.9) \quad [G_8, G_7] \cong [S^{11} \cup e^{13}, G_7] \oplus \pi_{16}(S^{10} \cup e^{12}) \oplus \pi_{16}(S^{15}).$$

Let $j: S^7 \rightarrow S^7 \cup_{\eta_7} e^9$ and $k: S^{10} \rightarrow S^{10} \cup_{\eta_{10}} e^{12}$ be injections and consider the cofiber sequence: $S^{10} \xrightarrow{k} S^{10} \cup_{\eta_{10}} e^{12} \rightarrow S^{12}$. Then we have the exact sequence

$$(3.10) \quad \cdots \xleftarrow{\eta_{10}^*} \pi_{10}(G_4) \xleftarrow{k^*} [S^{10} \cup e^{12}, G_4] \xleftarrow{\pi_{12}} \pi_{12}(G_4) \xleftarrow{\cdots},$$

where $\pi_{10}(G_4) \cong \pi_{10}(S^7 \cup e^9)$ by (2.1).

We need the following.

LEMMA 3.4. (i) $\pi_{10}(S^7 \cup e^9) \cong Z_{12}$ and generated by $j_* \omega_7$,

(ii) $\pi_{16}(S^{10} \cup_{\eta_{10}} e^{12}) \cong Z_2$ generated by $k_* \nu_{10}^2$,

(iii) k^* is epimorphic.

Proof. (i) and (ii) follow easily from the exact sequence of the pair $(S^7 \cup e^9, S^7)$ and $(S^{10} \cup e^{12}, S^{10})$ respectively.

(iii) By definition we have

$$\begin{aligned} \eta_{10}^*(j_* \omega_7) &= j_*(\omega_7 \eta_{10}) \\ &= j_*(\nu_7 \eta_{10}) \\ &= 0 \quad \text{by (5.9) of [15].} \end{aligned}$$

So (iii) follows from the exactness of the sequence (3.10). Q.E.D.

Let δ be the attaching element of e^{16} in $A \cup e^{16}$ in Proposition 3.2. Since $S(A \cup e^{16})$ is a retract of $S(SU(3) \wedge SU(3))$ and since dimension of SA is 11, the attaching element $S\delta$ of e^{17} in $S(A \cup e^{16})$ is of the form $S\delta = \beta' \circ k_* \nu_{10}^2 + \beta'' \circ \eta_{15}$ where $\beta' \in [S^{10} \cup e^{12}, G_4]$ and $\beta'' \in \pi_{15}(S^7 \cup e^9)$.

By (i) and (iii) of Lemma 3.4 we have that

$$\beta' \circ k_* \nu_{10}^2 = (k^* \beta') \nu_{10}^2 = (xj_* \omega_7) \circ \nu_{10}^2 = xj_* \nu_7^3$$

with some integer x . Besides we have $\nu_7^3 = \bar{\nu}_7 \circ \eta_{15}$ by Lemma 6.3 of [15]. Namely

$$S\delta = (xj_* \bar{\nu}_7 + \beta'') \eta_{15}.$$

Putting $xj_* \bar{\nu}_7 + \beta'' = \beta \in \pi_{15}(SA^{(8)})$, we obtain the proposition. Q.E.D.

4. The cell structure of $\mathrm{Sp}(2) \wedge \mathrm{Sp}(2)$. Let n be a sufficiently large number. Let C'_n and C_n be the mapping cone of ω_n and $2\omega_n$ respectively. Let $1'_n$ and 1_n be the identity of C'_n and C_n respectively. For simplicity we set

$A_n = C'_n \wedge C'_n$, the mapping cone of $1'_n \wedge \omega_n$.

$B_n = C_n \wedge C_n$, the mapping cone of $1_n \wedge (2\omega_n)$.

Let $j: S^{2n} \rightarrow S^n C'_n$ and $i: S^{2n} \rightarrow S^n C_n$ be injections. Let $(2\omega_{2n+3})^\sim \in \pi_{2n+7}(S^n C'_n)$ (or $(4\omega_{2n+3})^\sim \in \pi_{2n+7}(S^n C_n)$) be the coextension of $2\omega_{2n+3}$ (or $4\omega_{2n+3}$ respectively). Then by Lemma 3.5 of [15] we have that

$$(4.1) \quad A_n = (S^n C'_n \vee S^{2n+4}) \cup_{\alpha_n} e^{2n+8},$$

where $\alpha_n = (j_* \alpha^* + (2\omega_{2n+3})^\sim) \vee \omega_{2n+4}$ with $\alpha^* \in \pi_{2n+7}(S^{2n})$.

$$(4.2) \quad B_n = (S^n C_n \vee S^{2n+4}) \cup_{\beta_n} e^{2n+8},$$

where $\beta_n = (i_* \beta^* + (4\omega_{2n+3})^\sim) \vee 2\omega_{2n+4}$ with $\beta^* \in \pi_{2n+7}(S^{2n})$.

First we show that

$$(4.3) \quad \text{the restriction of } \alpha^* \text{ to } {}^2\pi_{2n+7}(S^{2n}) \text{ is } x\sigma_{2n} \text{ with odd } x.$$

Proof. By Proposition 8.1 of [15] we have

$$S_q^4: H^n(C_n; Z_2) \cong H^{n+4}(C'_n; Z_2)$$

and hence by the Cartan formula we obtain

$$S_q^8: H^{2n}(A_n; Z_2) \cong H^{2n+8}(B_n; Z_2).$$

Therefore, (4.1) follows by Proposition 8.1 of [15]. Q.E.D.

Next we show that

$$(4.4) \quad \text{the restriction of } \beta^* \text{ to } {}^2\pi_{2n+7}(S^{2n}) \text{ is } 4x\sigma_{2n} \text{ with odd } x.$$

Proof. We restrict ourself to the 2-primary components of the homotopy groups. It is easily checked that

$${}^2\pi_{2n+7}(S^n C'_n \vee S^{2n+4}) \cong Z_{16} \oplus Z_4 \oplus Z_8 = \{j_* \sigma_{2n}, (2\omega_{2n+3})^\sim, \nu_{2n+4}\}.$$

$${}^2\pi_{2n+7}(S^n C_n \vee S^{2n+4}) \cong Z_{16} \oplus Z_8 \oplus Z_8 = \{i_* \sigma_{2n}, (\omega_{2n+3})^\sim, \nu_{2n+4}\}.$$

Let F be a map: $B_n \rightarrow A_n$ such that

$F|S^{2n}$ is of mapping degree 1,

$F|S^{2n+4}$ and $F|e^{2n+4}$ are of mapping degree 2,

$F|e^{2n+8}$ is of mapping degree 4.

For the induced homomorphism

$$F_*: {}^{2\pi_{2n+7}}(S^n C_n \vee S^{2n+4}) \rightarrow {}^{2\pi_{2n+7}}(S^n C'_n \vee S^{2n+4})$$

we have that

$$(4.5) \quad F_*(i_*\sigma_{2n}) = j_*\sigma_{2n}, \quad F_*(\omega_{2n+3})^\sim = (2\omega_{2n+3})^\sim, \quad F_*(\nu_{2n+4}) = 2\nu_{2n+4}.$$

Consider the following diagram:

$$\begin{array}{ccccc} {}^{2\pi_{2n+7}}(S^{2n+7}) & \xrightarrow{\beta_n} & {}^{2\pi_{2n+7}}(S^n C_n \vee S^{2n+4}) & \longrightarrow & {}^{2\pi_{2n+7}}(S^n C_n \cup_{\beta_n} e^{2n+8}) \\ \downarrow \times 4 & & \downarrow F_* & & \downarrow \\ {}^{2\pi_{2n+7}}(S^{2n+7}) & \xrightarrow{\alpha_n} & {}^{2\pi_{2n+7}}(S^n C'_n \vee S^{2n+4}) & \longrightarrow & {}^{2\pi_{2n+7}}(S^n C'_n \cup_{\alpha_n} e^{2n+8}) \end{array}$$

Let $y\sigma_{2n}$ be the restriction of β^* to ${}^{2\pi_{2n+7}}(S^{2n})$, where y is some integer. Then by the commutativity of the diagram we have that the 2-primary part of $F_*(\beta_n, {}^{2\pi_{2n+7}})$ is the 2-primary part of $\alpha_n, ({}^{2\pi_{2n+7}})$. Namely,

$$j_*y\sigma_{2n} + 2 \cdot (4\omega_{2n+3})^\sim + 2(2\nu_{2n+4}) = j_*4x\sigma_{2n} + 4 \cdot (2\omega_{2n+3})^\sim + 4\nu_{2n+4}$$

and hence $j_*y\sigma_{2n} = j_*4x\sigma_{2n}$, since $2 \cdot (4\omega_{2n+3})^\sim = 4 \cdot (2\omega_{2n+3})^\sim$. One can easily see that $j_*: \pi_{2n+7}(S^{2n}) \rightarrow \pi_{2n+7}(S^n C'_n \vee S^{2n+4})$ is monomorphic. Therefore

$$y\sigma_{2n} = 4x\sigma_{2n}. \quad \text{Q.E.D.}$$

Now we state the following proposition, which is a little weaker than the case $SU(3) \wedge SU(3)$.

PROPOSITION 4.1. *The smash product $Sp(2) \wedge Sp(2)$ has the homotopy type of $M = B \cup (S^{13} \cup e^{17}) \cup (S^{13} \cup e^{17}) \cup e^{20}$, where*

- (i) $B = (S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) \cup_\beta e^{14}$, with $\beta = i_*\sigma'' + (4\omega_9)^\sim \vee 2\omega_{10}$,
 (ii) $M/S^6 = (S^{10} \times S^{10}) \cup_\gamma e^{14} \vee (S^{13} \cup_{2\omega_{13}} e^{17}) \vee (S^{13} \cup_{2\omega_{13}} e^{17})$
 $= (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup_{2\omega_{13}} e^{17}) \vee (S^{13} \cup_{2\omega_{13}} e^{17})$,

where γ is of type $(2\omega_{10}, 2\omega_{10})$ and e^{20} is attached to $S^{10} \vee S^{10}$ by a universal Whitehead product.

$$(iii) \quad M^{(10)} = B^{(10)} = S^6 \cup_{2\omega_6} e^{10} \vee S^{10}.$$

Proof. (i) First we prove (i). Let C be the mapping cone of ω' and 1_C the identity of C . Then $B = C \wedge C$ is a mapping cone of $1_C \wedge \omega': C \wedge S^6 \rightarrow C \wedge S^3$. For any coextension $(4\omega_9)^\sim$ of $4\omega_9$, by Lemma 3.5 of [15], there exists an element σ^* in $\pi_{13}(S^6)$ such that

$$1_C \wedge \omega' = (S^3 p)_*(i_*\sigma^* + (4\omega_9)^\sim)$$

holds in $[C \wedge S^6, C \wedge S^3]$, where p is the shrinking map: $S^3 \cup e^6 \rightarrow S^6$ and i is the inclusion $S^6 \rightarrow C \wedge S^3 = S^6 \cup_{2\omega_6} e^{10}$. Therefore B has the homotopy type of

$$(4.6) \quad B = (S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) \cup_\beta e^{14}$$

where $\beta = i_*\sigma^* + (4\omega_9)^\sim \vee 2\omega_{10}$ and $\sigma^* \in \pi_{13}(S^6) \cong Z_{60}$.

We will prove that

(4.7) $\sigma^* = x\sigma''$ with odd x , where σ'' is a generator of ${}^2\pi_{13}(S^6) \cong Z_4$.

Evidently $\sigma^* \in {}^5\pi_{13}(S^6) \cong Z_{12}$.

Recall that ${}^3\pi_{n+3}(S^n) \cong Z_3$ and ${}^3\pi_{n+7}(S^n) \cong Z_3$ are generated by $\alpha_1(n)$ and $\alpha_2(n)$ respectively for $n \geq 3$ [15]. Let D be the mapping cone of $\alpha_1(6)$. Then it is easily checked that there exists an exact sequence:

$$0 \longrightarrow {}^3\pi_{13}(S^6) \xrightarrow{i_*} {}^3\pi_{13}(D) \xrightarrow{(S^4p)_*} {}^3\pi_{13}(S^{10}) \longrightarrow 0.$$

Since $\alpha_1(6) \circ \alpha_1(9) = 0$, there exists a coextension $[\alpha_1(9)]^\sim$ of $\alpha_1(9)$ in D . By definition $(S^4p)_*[\alpha_1(9)]^\sim = \alpha_1(10)$, which is a generator of ${}^3\pi_{13}(S^{10})$. Besides, by Proposition 1.8 of [15], the element $[\alpha_1(9)]^\sim \circ 3\iota_{13} = 3[\alpha_1(9)]^\sim$ belongs to $-i_*\{\alpha_1(6), \alpha_1(9), 3\iota_{12}\}$. This secondary composition contains an element $-i_*\alpha_2(6)$ by Proposition 4.17 of [14]. So we have

$$3[\alpha_1(9)]^\sim = -i_*\alpha_2(6) \pmod{-i_*\pi_{13}(S^6) \circ 3\iota_{13}}.$$

Thus ${}^3\pi_{13}(D) \cong Z_9$ and it is generated by $[\alpha_1(9)]^\sim$. Therefore, in (4.6) we can choose the coextension $(4\omega_9)^\sim$ so that σ^* belongs to ${}^2\pi_{13}(S^6)$.

Since $S^{n-3}\omega' = 2\omega_n$ for sufficiently large n , we have $S^{2n-6}B = B_n$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccc} \pi_{14}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}, S^6) & \xrightarrow{i_*} & \pi_{13}(S^6) \\ \downarrow S^{2n-6} & & \downarrow S^{2n-6} \\ \pi_{2n+8}(S^n C_n \vee S^{2n+4}, S^{2n}) & \xrightarrow{i_*} & \pi_{2n+7}(S^{2n}) \\ & & \downarrow S^{2n-6} \\ \pi_{13}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) & \xrightarrow{i_*} & \pi_{13}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}, S^6) \\ \downarrow S^{2n-6} & & \downarrow S^{2n-6} \\ \pi_{2n+7}(S^n C_n \vee S^{2n+4}) & \xrightarrow{i_*} & \pi_{2n+7}(S^n C_n \vee S^{2n+4}, S^{2n}) \end{array}$$

By making use of the Blakers-Massey theorem one can easily check that the first and the last S^{2n-6} are isomorphisms. Since the second S^{2n-6} is known to be monomorphic [15] we conclude by the four lemma that the third S^{2n-6} is monomorphic. Since $S^{2n-6}B = B_n$, we obtain that $S^{2n-6}(i_*\sigma^*)$ is $4x(i_*\sigma_{2n})$ by (4.4). On the other hand by Lemma 5.14 of [15], we have $4x(i_*\sigma_{2n}) = S^{2n-6}(i_*x\sigma'')$. As S^{2n-6} is monomorphic, we get that $i_*\sigma^* = x\sigma''$, and hence (4.7) follows.

(ii) We study the structure of $M/S^6 = (B/S^6) \cup (e_1^{13} \cup e_1^{17}) \cup (e_2^{13} \cup e_2^{17}) \cup e^{20}$. By virtue of (2.6) we obtain that

$$\begin{aligned} (\mathrm{Sp}(2)/S^3) \wedge \mathrm{Sp}(2) &= (S^7 \vee S^{10}) \wedge \mathrm{Sp}(2) \\ &= S^7 \mathrm{Sp}(2) \vee S^{10} \mathrm{Sp}(2) \\ &= (S^{10} \cup e^{14}) \vee S^{17} \vee (S^{13} \cup e^{17}) \vee S^{20} \quad \text{by Lemma 2.1.} \end{aligned}$$

So it follows easily that

$$(M/S^6)^{(17)} = (B/S^6) \cup_\gamma C((S_1^{12} \cup e_1^{16}) \vee (S_2^{12} \cup e_2^{16})),$$

where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_i \in [S_i^{12} \cup e_i^{16}, B/S^6]$. On the other hand the double suspended complex $S^2((M/S^6)^{(17)})$ is the 19 skeleton of $(S^2M)/S^8$ and so by Lemma 2.2 it has the homotopy type of $(S^2B)/S^8 \vee (S_1^{15} \cup e_1^{19}) \vee (S_2^{15} \cup e_2^{19}) \vee S^{22}$. Therefore, $S^2\gamma_i = 0$ in $[S^2(S_i^{12} \cup e_i^{16}), S^2(B/S^6)]$. The homomorphism

$$S^2: [S_i^{12} \cup e_i^{16}, B/S^6] \rightarrow [S^2(S_i^{12} \cup e_i^{16}), S^2(B/S^6)]$$

is an isomorphism, since B/S^6 is 9-connected. Thus γ_i is trivial and we obtain $(M/S^6)^{(17)} = (B/S^6) \vee (S_1^{13} \cup e_1^{17}) \vee (S_2^{13} \cup e_2^{17})$.

It is easily checked that e_1^{17} is attached to S_1^{13} by an element $2\omega_{13}$.

The same argument as in the proof of (ii) of Proposition 3.2 gives that the top cell e^{20} in M/S^6 is attached to $S_1^{10} \vee S_2^{10}$ by a universal Whitehead product. Thus $M/S^6 = (S^{10} \times S^{10}) \cup e^{14} \vee (S^{13} \cup e^{17}) \vee (S^{13} \cup e^{17})$. It is easily seen that the attaching element of e^{14} is of type $(2\omega_{10}, 2\omega_{10})$.

One can restate the above fact as follows:

$$M/S^6 = (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup e^{17}) \vee (S^{13} \cup e^{17}).$$

(iii) The proof of (iii) is quite easy. Q.E.D.

For later use we state

COROLLARY 4.2. $SB = (S^7 \cup_{2\omega_7} e^{11} \vee S^{11}) \cup_{S\beta} e^{15}$, where

$$S\beta = i_* S\sigma'' + (4\omega_{10}) \sim \vee 2\omega_{11}.$$

5. The order of $[SU(3) \wedge SU(3), SU(3)]$. According to Arkowitz-Curjel (Lemma 2 of [1]) the order of the group $[SU(3) \wedge SU(3), SU(3)]$ is equal to the number of homotopy classes of multiplications on $SU(3)$. In this section we compute the order of this group.

By virtue of (2.2) and Proposition 3.2 we have

$$(5.1) \quad [SU(3) \wedge SU(3), SU(3)] \cong [A \cup e^{16}, SU(3)] \oplus [S^{11} \cup_{\eta_{11}} e^{13}, SU(3)] \\ \oplus [S^{11} \cup_{\eta_{11}} e^{13}, SU(3)].$$

First we compute the order of $[S^{11} \cup_{\eta_{11}} e^{13}, SU(3)]$.

Applying the functor $[_, SU(3)]$ to the cofibration sequence

$$S^{12} \xrightarrow{\eta_{11}} S^{11} \xrightarrow{\eta_{12}} S^{11} \cup e^{13} \xrightarrow{\eta_{12}} S^{13} \xrightarrow{\eta_{12}} S^{12} \longrightarrow \dots,$$

we obtain the exact sequence:

$$\pi_{12}(SU(3)) \xleftarrow{\eta_{11}^*} \pi_{11}(SU(3)) \xleftarrow{\eta_{11}^*} [S^{11} \cup e^{13}, SU(3)] \xleftarrow{\eta_{12}^*} \pi_{13}(SU(3)) \\ \xleftarrow{\eta_{12}^*} \pi_{12}(SU(3)) \xleftarrow{\eta_{12}^*} \dots$$

We quote from [9] that

$$\begin{aligned}\pi_{11}(\mathrm{SU}(3)) &\cong Z_4([\nu_2^2]), & \pi_{12}(\mathrm{SU}(3)) &\cong Z_{60}([\sigma''']), \\ \pi_{13}(\mathrm{SU}(3)) &\cong Z_6(i_*\epsilon'),\end{aligned}$$

where () denotes the generator of the 2-primary component and $[\alpha]$ is such an element that $p_*[\alpha] = \alpha$ for the projection $p: \mathrm{SU}(3) \rightarrow S^5$.

Then we have

LEMMA 5.1. η_{11}^* and η_{12}^* are trivial.

The proof will be given following Lemma 5.2.

As a corollary we have

PROPOSITION 5.2. $\theta([S^{11} \cup e^{13}, \mathrm{SU}(3)]) = 2^3 \cdot 3$.

REMARK. One can show that

$$[S^{11} \cup e^{13}, \mathrm{SU}(3)] = \pi_{11}(\mathrm{SU}(3)) \oplus \pi_{13}(\mathrm{SU}(3))$$

but it is not necessary here.

Consider the complex Stiefel manifold $W_{5,2} = \mathrm{SU}(5)/\mathrm{SU}(3)$, which is known to be a S^7 -bundle over S^9 . And so $W_{5,2} = S^7 \cup_{\eta_7} e^9 \cup e^{16}$. We quote from §7 the following results.

PROPOSITION 7.1.

$$\begin{aligned}{}^2\pi_{15}(W_{5,2}) &\cong Z_4 \oplus Z_2 = \{[\nu_9^2], i_*\bar{\nu}_7\}, \quad 2[\nu_9^2] = i_*\epsilon_7, \\ {}^2\pi_{16}(W_{5,2}) &\cong Z_{16} = \{[2\sigma_9]\}, \quad 8[2\sigma_9] = i_*\mu_7, \\ {}^2\pi_{17}(W_{5,2}) &\cong Z_4 = \{i_*\nu_7\sigma_{10}\}.\end{aligned}$$

In the above an element $[x] \in {}^2\pi_i(W_{5,2})$ is such that $p_*[x] = x \in \pi_i(S^9)$ for the projection $p: W_{5,2} \rightarrow S^9$.

Let $1: S^4 \mathrm{SU}(3) = S^7 \cup_{\eta_7} e^9 \vee S^{12} \rightarrow W^{5,2}$ be a map such that $1|_{S^7 \cup e^9}$ is an identity and $1|_{S^{12}}$ is trivial. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & & \pi_i(S^5) & & & \\ & & & \downarrow p_* \parallel & & & \\ \pi_i(S^3) & \xrightarrow{i_*} & \pi_i(\mathrm{SU}(3)) & \longrightarrow & \pi_i(\mathrm{SU}(3), S^3) & \longrightarrow & \pi_{i-1}(S^3) \xrightarrow{i_*} \pi_{i-1}(\mathrm{SU}(3)) \\ \downarrow S^4 & & \downarrow S^4 & & \downarrow S^4 & & \downarrow S^4 \\ \pi_{i+4}(S^7) & \xrightarrow{i_*} & \pi_{i+4}(S^4 \mathrm{SU}(3)) & \longrightarrow & \pi_{i+4}(S^4 \mathrm{SU}(3), S^7) & \longrightarrow & \pi_{i+3}(S^7) \xrightarrow{i_*} \pi_{i+3}(S^4 \mathrm{SU}(3)) \\ \parallel & & \downarrow 1_* & & \downarrow 1_* & & \parallel \\ \pi_{i+4}(S^7) & \xrightarrow{i_*} & \pi_{i+4}(W_{5,2}) & \longrightarrow & \pi_{i+4}(W_{5,2}, S^7) & \longrightarrow & \pi_{i+3}(S^7) \xrightarrow{i_*} \pi_{i+3}(W_{5,2}) \\ & & & \downarrow p_* \parallel & & & \\ & & & \pi_{i+4}(S^9) & & & \end{array}$$

For simplicity we denote by I the composition of the homomorphism

$$\pi_i(\text{SU}(3)) \xrightarrow{S^4} \pi_{i+4}(S^4 \text{SU}(3)) \xrightarrow{1_*} \pi_{i+4}(W_{5,2}).$$

Then we have

LEMMA 5.2. (i) $I(i_*\epsilon') = 2i_*\nu_7\sigma_{10}$.

(ii) $I([\sigma'']) = 4[2\sigma_9] \bmod i_*\mu_7$.

(iii) $I([\nu_9^2]) = [\nu_9^2] \bmod i_*\bar{\nu}_7$.

Proof. We use the commutativity of the diagram:

- (i) $I(i_*\epsilon') = i_*(S^4\epsilon') = i_*(2\nu_7\sigma_{10})$ by (7.10) of [15].
- (ii) Since $p_*I([\sigma'']) = S^4p_*([\sigma'']) = 8\sigma_9$, we have $I([\sigma'']) = 4[2\sigma_9] \bmod i_*\mu_7$.
- (iii) Similarly, the equality $p_*([\nu_9^2]) = S^4p_*([\nu_9^2]) = \nu_9^2$ implies that

$$I([\nu_9^2]) = [\nu_9^2] \bmod i_*\bar{\nu}_7.$$

Now we turn to the proof of Lemma 5.1.

Proof of Lemma 5.1. It is sufficient to show that $\eta_{11}^*[\nu_5^2] = \eta_{12}^*[\sigma''] = 0$.

Consider the following commutative diagram:

$$\begin{array}{ccccccc} \pi_{12}(\text{SU}(3)) & \xleftarrow{\eta_{11}^*} & \pi_{11}(\text{SU}(3)) & \xleftarrow{[S^{11} \cup e^{13}, \text{SU}(3)]} & \pi_{13}(\text{SU}(3)) & \xleftarrow{\eta_{12}^*} & \pi_{12}(\text{SU}(3)) \\ \downarrow I & & \downarrow I & & \downarrow I & & \downarrow I \\ \pi_{16}(W_{5,2}) & \xleftarrow{\eta_{15}^*} & \pi_{15}(W_{5,2}) & \xleftarrow{[S^{15} \cup e^{17}, W_{5,2}]} & \pi_{17}(W_{5,2}) & \xleftarrow{\eta_{16}^*} & \pi_{16}(W_{5,2}) \end{array}$$

By Lemma 5.2, we have

$$I([\nu_5^2]\eta_{11}) = [\nu_9^2]\eta_{15} \bmod i_*\bar{\nu}_7\eta_{15},$$

where $i_*\bar{\nu}_7\eta_{15}$ is trivial by [9].

As the element $[\nu_5^2]$ one may choose a coextension $\bar{\nu}_8^2$ of ν_8^2 in $S^7 \cup e^9 \subset W_{5,2}$, since $\eta_7 \circ \nu_8^2 = 0$. Then by Proposition 1.8 of [15], $I([\nu_5^2]\eta_{11}) = \bar{\nu}_8^2 \circ \eta_{15}$ belongs to $-i_*\{\eta_7, \nu_8^2, \eta_{14}\}$, which is easily proved to be zero. Thus $I([\nu_5^2]\eta_{14}) = 0$. By Lemma 5.2, the restriction of I on ${}^2\pi_{12}(\text{SU}(3))$ is monomorphic. Therefore, we obtain $[\nu_5^2]\eta_{11} = 0$.

We have $I([\sigma'']\eta_{12}) = 4[2\sigma_9]\eta_{16} = 0 \bmod i_*\eta_7\mu_8$, where $i_*\eta_7\mu_8 = 0$ (see §7). Since I is monomorphic on ${}^2\pi_{13}(\text{SU}(3))$ by Lemma 5.2 we have $[\sigma'']\eta_{12} = 0$. Q.E.D.

Put $L = A \cup e^{16}/S^6 = (A/S^6) \cup e^{16}$. By (2.2), we have

$$[S^9 \vee S^{15}, S^8 \vee S^8] \cong \pi_9(S^8 \vee S^8) \oplus \pi_{15}(S^8 \vee S^8).$$

Let $h: S^8 \rightarrow S^8 \vee S^8$ be the injection to the right S^8 and α be the universal Whitehead product in $\pi_{15}(S^8 \vee S^8)$. Then by Proposition 3.2, we have

$$(5.2) \quad L = (S^8 \vee S^8) \cup_\gamma C(S^9 \vee S^{15}) \quad \text{with } \gamma = h_*\eta_8 \oplus \alpha.$$

Consider the cofiber sequence

$$(5.3) \quad S^6 \rightarrow A \cup e^{16} \rightarrow L \rightarrow S^7 \rightarrow \cdots$$

Applying the functor $[\quad, \text{SU}(3)]$ we get the exact sequence

$$(5.4) \quad \pi_6(\text{SU}(3)) \xleftarrow{i^*} [A \cup e^{16}, \text{SU}(3)] \xleftarrow{\quad} [L, \text{SU}(3)] \xleftarrow{\quad} \pi_7(\text{SU}(3)) \xleftarrow{\quad} \cdots,$$

where $\pi_6(\text{SU}(3)) \cong Z_6$ and $\pi_7(\text{SU}(3)) = 0$ by [9].

So we will calculate the order of $[L, \text{SU}(3)]$.

Consider the following cofiber sequence

$$(5.5) \quad S^9 \vee S^{15} \xrightarrow{\gamma} S^8 \vee S^8 \longrightarrow L \longrightarrow S^{10} \vee S^{16} \longrightarrow S^9 \vee S^9 \longrightarrow \cdots.$$

It follows the exact sequence

$$\begin{aligned} \pi_9(\text{SU}(3)) \oplus \pi_{15}(\text{SU}(3)) &\xleftarrow{\gamma^*} [S^8 \vee S^8, \text{SU}(3)] \xleftarrow{\quad} [L, \text{SU}(3)] \\ &\xleftarrow{\quad} \pi_{10}(\text{SU}(3)) \oplus \pi_{16}(\text{SU}(3)) \xleftarrow{(S\gamma)^*} [S^9 \vee S^9, \text{SU}(3)] \xleftarrow{\quad} \cdots. \end{aligned}$$

PROPOSITION 5.3. γ^* and $(S\gamma)^*$ are trivial.

Proof. Since $\gamma = h_*\eta_8 \oplus \alpha$ and $S\gamma = h_*\eta_9 \oplus S\alpha$, it is sufficient to prove

$$(5.6) \quad \eta_8^*\pi_8(\text{SU}(3)) = \eta_9^*\pi_9(\text{SU}(3)) = 0,$$

$$(5.7) \quad \alpha^*[S^8 \vee S^8, \text{SU}(3)] = (S\alpha)^*[S^9 \vee S^9, \text{SU}(3)] = 0.$$

By [9], $\pi_9(\text{SU}(3)) \cong Z_3$ and hence η_8^* and η_9^* are trivial, since η_8 and η_9 are of order 2. (5.6) follows.

For any element x of $[S^8 \vee S^8, \text{SU}(3)]$, the composition $x \circ \alpha = \alpha^*(x)$ is a Whitehead product in $\text{SU}(3)$ and hence is trivial. Similarly $(S\alpha)^* = 0$. (5.7) follows. Q.E.D.

It follows from Proposition 5.3 that

$$(5.8) \quad \theta([L, \text{SU}(3)]) = \theta(\pi_8(\text{SU}(3))) \cdot \theta(\pi_8(\text{SU}(3))) \cdot \theta(\pi_{10}(\text{SU}(3))) \cdot \theta(\pi_{16}(\text{SU}(3))).$$

Recall [9] that $\pi_8(\text{SU}(3)) \cong Z_{12}$, $\pi_{10}(\text{SU}(3)) \cong Z_{30}$ and $\pi_{16}(\text{SU}(3)) \cong Z_{252} \oplus Z_6$. We obtain

$$\text{COROLLARY 5.4. } \theta([L, \text{SU}(3)]) = 2^8 \cdot 3^6 \cdot 5 \cdot 7.$$

Finally we compute the order of $[A \cup e^{16}, \text{SU}(3)]$. In the sequence (5.4) we have the commutativity:

$$\begin{array}{ccc} \pi_6(\text{SU}(3)) & \xleftarrow{i^*} & [A \cup e^{16}, \text{SU}(3)] \\ \parallel & & \parallel \\ \pi_7(B\text{SU}(3)) & \xleftarrow{i^*} & [S(A \cup e^{16}), B\text{SU}(3)] \end{array}$$

where i also means the inclusion $S^7 \rightarrow S(A \cup e^{16})$.

LEMMA 5.5. i^* is epimorphic.

Proof. Let $j: S^4 \rightarrow B SU(3)$ be the inclusion map. Then it is easily checked that the generator of $\pi_7(B SU(3)) \cong \pi_6(SU(3)) \cong Z_6$ is $j_* S\omega'$. The lemma is equivalent to the fact that $j_* S\omega'$ is extendable to $S(A \cup e^{16})$.

By Proposition 3.3 we have that

$$S(A \cup e^{16}) = SA^{(8)} \cup_{\gamma} C(S^{10} \vee S^{16}) \quad \text{where } \gamma = S\alpha \vee \beta \circ \eta_{15}.$$

Since $\pi_8(B SU(3)) \cong \pi_7(SU(3)) \cong 0$, $j_* S\omega'$ is extended to $SA^{(8)}$. Furthermore $j_* S\omega'$ is extended to $(S(A \cup e^{16}))^{(11)}$, since $j_* S\omega' \circ \alpha = j_* S\omega' \circ n\nu_7$ by (i) of Proposition 3.2 and since $j_* S\omega' \circ (n\nu_7) \in {}^2\pi_{10}(B SU(3)) \cong {}^2\pi_9(SU(3)) = 0$. The attaching element of e^{17} in $S(A \cup e^{16})$ is of the form $\beta \circ \eta_{15}$ by Proposition 3.3. Let ν be the extension of $j_* S\omega'$ to $(S(A \cup e^{16}))^{(11)}$. We must show that $\nu \circ \beta\eta_{15} = 0$. We need the following

$$(5.9) \quad \eta_{15}^* \pi_{15}(B SU(3)) = 0.$$

It is sufficient to show that

$$\eta_{14}^*({}^2\pi_{14}(SU(3))) = 0.$$

In fact, ${}^2\pi_{14}(SU(3)) \cong Z_4 \oplus Z_2 = \{[\nu_5^2] \circ \nu_{11}, i_* \mu'\}$ and hence $\eta_{14}^*({}^2\pi_{14}(SU(3)))$ is generated by $[\nu_5^2] \circ \nu_{11} \cdot \eta_{14}$ and $i_* \mu' \eta_{14}$, where $[\nu_5^2] \nu_{11} \eta_{14} = 0$ by (5.9) of [15] and $i_* \mu' \eta_{14} = 0$ by Lemma 4.2 of [9].

The element $\nu\beta$ belongs to $\pi_{15}(B SU(3))$ and hence $(\nu\beta)\eta_{15} = \eta_{15}^*(\nu\beta) = 0$ by (5.9). Q.E.D.

By (5.4) and Corollary 5.4 we have the following as a corollary of Lemma 5.5.

PROPOSITION 5.6. $\theta([A \cup e^{16}, SU(3)]) = 2^9 \cdot 3^7 \cdot 5 \cdot 7$.

Therefore, by (5.1) and Proposition 5.2, we have obtained that

$$\theta([SU(3) \wedge SU(3), SU(3)]) = 2^{15} \cdot 3^9 \cdot 5 \cdot 7.$$

Thus we have shown

THEOREM 5.7. *There exist precisely $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ distinct homotopy classes of multiplications on $SU(3)$.*

6. The order of $[Sp(2) \wedge Sp(2), Sp(2)]$. Consider the cofiber sequence $(M = Sp(2) \wedge Sp(2))$:

$$(6.1) \quad S^6 \xrightarrow{j} M \longrightarrow M/S^6 \longrightarrow S^7 \longrightarrow SM \longrightarrow S(M/S^6) \longrightarrow S^8 \longrightarrow \dots$$

Applying the functor $[_, Sp(2)]$ we obtain the exact sequence:

$$(6.2) \quad \begin{array}{ccccccc} \pi_6(Sp(2)) & \xleftarrow{j^*} & [M, Sp(2)] & \xleftarrow{} & [M/S^6, Sp(2)] & \xleftarrow{} & \pi_7(Sp(2)) \\ & & \xleftarrow{j^*} & & [SM, Sp(2)] & \xleftarrow{} & [S(M/S^6), Sp(2)] & \xleftarrow{} & \pi_8(Sp(2)) & \xleftarrow{} & \dots \end{array}$$

where $\pi_7(Sp(2)) \cong Z$ and $\pi_6(Sp(2)) \cong \pi_8(Sp(2)) = 0$ ([9]).

By (2.2) and Proposition 4.1 we get

$$(6.3) \quad [M/S^6, \text{Sp}(2)] \cong [(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}), \text{Sp}(2)] \\ \oplus [S^{13} \cup_{2\omega_{13}} e^{17}, \text{Sp}(2)] \oplus [S^{13} \cup_{2\omega_{13}} e^{17}, \text{Sp}(2)].$$

First we compute the order of $[S^{13} \cup_{2\omega_{13}} e^{17}, \text{Sp}(2)]$. Consider the sequence:

$$S^{16} \xrightarrow{2\omega_{13}} S^{13} \longrightarrow S^{13} \cup_{2\omega_{13}} e^{17} \longrightarrow S^{17} \xrightarrow{2\omega_{14}} S^{14} \longrightarrow.$$

Then we obtain the exact sequence:

$$\pi_{16}(\text{Sp}(2)) \xleftarrow{(2\omega_{13})^*} \pi_{13}(\text{Sp}(2)) \longleftarrow [S^{13} \cup e^{17}, \text{Sp}(2)] \longleftarrow \pi_{17}(\text{Sp}(2)) \\ \xleftarrow{(2\omega_{14})^*} \pi_{14}(\text{Sp}(2)) \longleftarrow \dots,$$

where $\pi_{13}(\text{Sp}(2)) \cong Z_4 \oplus Z_2 = \{\nu_7\nu_{10}, i_*\eta_3\mu_4\}$, $\pi_{14}(\text{Sp}(2)) \cong Z_{1680}([2\sigma'])$,

$$\pi_{16}(\text{Sp}(2)) \cong Z_2 \oplus Z_2 = \{[\sigma'\eta_{14}]\eta_{15}, [\nu_7]\nu_{10}^2\} \quad \text{and} \quad \pi_{17}(\text{Sp}(2)) \cong Z_{40}([\nu_7]\sigma_{10}).$$

Evidently $(2\omega_{13})^*$ is trivial, since $\pi_{16}(\text{Sp}(2)) \cong Z_2 \oplus Z_2$. On the other hand we get $(2\omega_{14})^*[2\sigma'] = 4[\nu_7]\sigma_{10}$, since for the projection $p: \text{Sp}(2) \rightarrow S^7$ we have

$$\begin{aligned} p_*([2\sigma'] \circ 2\omega_{14}) &= 4\sigma'\nu_{14} \\ &= 4\nu_7\sigma_{10} \quad \text{by (7.19) of [9]} \\ &= p_*(4[\nu_7]\sigma_{10}) \end{aligned}$$

and since $p_*: {}^2\pi_{17}(\text{Sp}(2)) \rightarrow {}^2\pi_{17}(S^7)$ is monomorphic by Lemma 5.2 of [9]. Thus the cokernel of $(2\omega_{14})^*$ is isomorphic to Z_{20} . Therefore we have obtained

PROPOSITION 6.1. $\theta([S^{13} \cup_{2\omega_{13}} e^{17}, \text{Sp}(2)]) = 2^5 \cdot 5$.

For simplicity we put $N = (S^{10} \vee S^{10}) \cup_\xi C(S^{13} \vee S^{19})$, where

$$\begin{aligned} \xi &= (2\omega_{10} \vee 4\omega_{10}) \oplus \delta \in \pi_{13}(S^{10} \vee S^{10}) \oplus \pi_{19}(S^{10} \vee S^{10}) \\ &\cong [S^{13} \vee S^{19}, S^{10} \vee S^{10}] \end{aligned}$$

and δ is a universal Whitehead product by Proposition 4.1. We will compute the order of $[N, \text{Sp}(2)]$.

We have the following cofiber sequence:

$$(6.4) \quad S^{19} \vee S^{13} \xrightarrow[\xi]{} S^{10} \vee S^{10} \longrightarrow N \longrightarrow S^{20} \vee S^{14} \xrightarrow[S\xi]{} S^{11} \vee S^{11} \longrightarrow \dots$$

Therefore we obtain the exact sequence:

$$(6.5) \quad \begin{aligned} \pi_{19}(\text{Sp}(2)) \oplus \pi_{13}(\text{Sp}(2)) &\xleftarrow[\xi^*]{} [S^{10} \vee S^{10}, \text{Sp}(2)] \longleftarrow [N, \text{Sp}(2)] \\ &\longleftarrow \pi_{20}(\text{Sp}(2)) \oplus \pi_{14}(\text{Sp}(2)) \\ &\xleftarrow[(S\xi)^*]{} [S^{11} \vee S^{11}, \text{Sp}(2)] \longleftarrow, \end{aligned}$$

where $\pi_{10}(Sp(2)) \cong Z_{120}([v_7])$, $\pi_{11}(Sp(2)) \cong Z_2 = \{i_* \varepsilon_3\}$, $\pi_{13}(Sp(2)) \cong Z_4 \oplus Z_2 = \{[v_7]v_{10}, i_* \eta_3 \mu_4\}$, $\pi_{14}(Sp(2)) \cong Z_{1680}([2\sigma'])$, $\pi_{19}(Sp(2)) \cong Z_2 \oplus Z_2 = \{i_* \mu_3 \sigma_{12}, i_* \eta_3 \varepsilon_4\}$ and $\pi_{20}(Sp(2)) \cong Z_2 \oplus Z_2 \oplus Z_2 = \{[v_7]\sigma_{10}v_{17}, i_* \eta_3 \mu_4 \sigma_{13}, i_* \bar{\mu}_3\}$ (see [9]).

Evidently $\xi^* = (2\omega_{10} \vee 4\omega_{10})^* \oplus \delta^*$ and $(S\xi)^* = (2\omega_{11} \vee 4\omega_{11})^* \oplus (S\delta)^*$.

PROPOSITION 6.2. (i) $(S\xi)^*$ is trivial.

(ii) The kernel of ξ^* is isomorphic to $Z_{60} \oplus Z_{120}$.

Proof. It is sufficient to prove

(6.6) δ^* and $(S\delta)^*$ are trivial.

(6.7) $(2\omega_{11} \vee 4\omega_{11})^*$ is trivial.

(6.8) The kernel of $(2\omega_{10} \vee 4\omega_{10})^*$ is isomorphic to $Z_{60} \oplus Z_{120}$.

The proof of (6.6) is same as in Proposition 5.3, since δ is a universal Whitehead product in $\pi_{19}(S^{10} \vee S^{10})$. (6.7) is clear since $\pi_{11}(Sp(2)) \cong Z_2$. The image of $(2\omega_{10} \vee 4\omega_{10})^*$ is generated by $(2\omega_{10})^*[v_7] = 2[v_7]v_{10}$. Hence (6.8) follows. Q.E.D.

By (6.5) and Proposition 6.2 we obtain

PROPOSITION 6.3. $\theta([N, Sp(2)]) = 2^{12} \cdot 3^3 \cdot 5^3 \cdot 7$.

Therefore by Proposition 6.1 we obtain

PROPOSITION 6.4. $\theta([M/S^6, Sp(2)]) = 2^{22} \cdot 3^3 \cdot 5^5 \cdot 7$.

In order to enumerate the order of $[M, Sp(2)]$ it is necessary to study the cokernel of the homomorphism in (6.2);

$$j^*: [SM, Sp(2)] \rightarrow \pi_7(Sp(2))$$

where $j: S^7 \rightarrow SM$ is the inclusion.

We have

PROPOSITION 6.5. The cokernel of j^* is isomorphic to Z_{36} .

It follows from this proposition and (6.2) that

$$0 \leftarrow [M, Sp(2)] \leftarrow [M/S^6, Sp(2)] \leftarrow Z_{36} \leftarrow 0$$

and hence

$$(6.9) \quad \theta([M, Sp(2)]) = 2^{20} \cdot 3 \cdot 5^5 \cdot 7.$$

Namely by Lemma 2 of [1] we have shown

THEOREM 6.6. There exist precisely $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ distinct homotopy classes of multiplications on $Sp(2)$.

The rest of this section is devoted to the proof of Proposition 6.5.

We have the following commutative diagram.

$$\begin{array}{ccc} \pi_7(Sp(2)) & \xleftarrow{j^*} & [SM, Sp(2)] \\ \parallel & & \parallel \\ \pi_8(BSp(2)) & \xleftarrow{j^*} & [S^2M, BSp(2)] \end{array}$$

Since $S^2M = S^2B \vee (S^{15} \cup e^{19}) \vee (S^{15} \cup e^{19}) \vee S^{22}$ by Proposition 4.1, we have

$$[S^2M, \text{Sp}(2)] \cong [S^2B, \text{Sp}(2)] \oplus [S^{15} \cup e^{19}, \text{Sp}(2)] \oplus [S^{15} \cup e^{19}, \text{Sp}(2)] \\ \oplus \pi_{22}(\text{Sp}(2)).$$

Let $j: S^7 \rightarrow SB = (S^7 \cup_{2\omega_7} e^{11} \vee S^{11}) \cup_{SB} e^{15}$ be the inclusion, where $S\beta = j_*S\sigma'' + (4\omega_{10})^\sim \vee 2\omega_{11}$. In order to prove Proposition 6.5 it is sufficient to show that (6.7) the cokernel of $j^*: [SB, \text{Sp}(2)] \rightarrow \pi_7(\text{Sp}(2))$ is isomorphic to Z_{36} .

In other words, it is an extension problem of the generator $[12\iota_7]$ of $\pi_7(\text{Sp}(2)) \cong Z$ to SB .

First we remark that we may take the coextension of $12\iota_7$ in $S^3 \cup_{\omega} e^7 \subset \text{Sp}(2)$ as an element $[12\iota_7]$.

We need the following

LEMMA 6.7. $i_*\{\omega', 12\iota_6, 2\omega_6\} = -i_*\alpha_2(3)$.

Proof. It is easily seen that the secondary composition $i_*\{\omega', 12\iota_6, 2\omega_6\}$ is a single element. Since $i_*: {}^2\pi_{10}(S^3) \rightarrow {}^2\pi_{10}(\text{Sp}(2))$ is trivial, we have

$$i_*\{\omega', 12\iota_6, 2\omega_6\} = i_*\{\alpha_1(3), 12\iota_6, 2\alpha_1(6)\},$$

where $\{\alpha_1(3), 3\iota_6, \alpha_1(6)\} = \alpha_2(3)$ by Lemma 13.5 of [15]. Hence

$$i_*\{\omega', 12\iota_6, 2\omega_6\} = -i_*\alpha_2(3).$$

The composition $[12\iota_7] \circ 2\omega_7$ is contained in the secondary composition

$$-i_*\{\omega', 12\iota_6, 2\omega_6\}$$

by Proposition 1.8 of [15]. Hence by Lemma 6.7 we obtain

$$(6.8) \quad [12\iota_7] \circ 2\omega_7 = i_*\alpha_2(3).$$

Since $i_*\alpha_2(3)$ is of order 3, $3([12\iota_7]) = [12\iota_7] \circ 3\iota_7$ is extendable to the complex $S^7 \cup_{2\omega_7} e^{11} \vee S^{11}$. Let ξ be this extension. Consider the composition

$$\xi \circ S\beta = \xi \circ (j_*S\sigma'' + (4\omega_{10})^\sim).$$

LEMMA 6.8. (i) $\xi \circ j_*S\sigma'' = 4[2\sigma'] \bmod \{2i_*\mu' = 8[2\sigma']\}$.

(ii) $\xi \circ (4\omega_{10})^\sim = i_*\alpha_3(3) \bmod \{2i_*\mu' = 8[2\sigma']\}$.

Proof. (i) Since ξ is the extension of $[12\iota_7] \circ 3\iota_7$, we have that

$$\xi \circ j_*S\sigma'' = [12\iota_7] \circ 3\iota_7 \circ S\sigma''.$$

For the projection $p: \text{Sp}(2) \rightarrow S^7$ we have $p_*([12\iota_7] \circ \sigma') = p_*(12\sigma')$ and hence $[12\iota_7] \circ \sigma' = 6[2\sigma'] \bmod \{i_*\mu'\}$. Thus we obtain

$$\xi \circ j_*S\sigma'' = 36[2\sigma'] = 4[2\sigma'] \bmod \{2i_*\mu'\}.$$

(ii) By Proposition 1.7 of [15], the element $\xi \circ (4\omega_{10})^\sim$ is contained in the secondary composition $\{[12\iota_7] \circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2$. The indeterminacy of this is

$\pi_{11}(\text{Sp}(2)) \circ 4\omega_{11} \oplus [12\iota_7] \circ 3\iota_7 \circ S^2\pi_{12}(S^5)$, which is easily checked to be generated by $8[2\sigma'] = 2i_*\mu'$. For any element x of $\{[12\iota_7] \circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2$, we have that $3x \in \{[12\iota_7], 2\nu_7, 4\nu_{10}\}_2 \supset \{[12\iota_7], 2\nu_7, 4\iota_{10}\}_2 \circ \nu_{11}$, which is a subset of $\pi_{11}(\text{SU}(3)) \circ \nu_{11} = 0$. Hence

$$\{[12\iota_7] \circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2 \subset {}^3\pi_{14}(\text{Sp}(2)).$$

So we restrict ourself to the 3-primary components.

We have that

$$\begin{aligned} \{[12\iota_7] \circ 3\iota_7, 2\alpha_1(7), \alpha_1(10)\}_2 &\supset [12\iota_7] \circ \{3\iota_7, 2\alpha_1(7), \alpha_1(10)\}_2 \\ &\quad \text{by Proposition 1.8 of [15]} \\ &\ni [12\iota_7] \circ \alpha_2(7) \\ &\quad \text{by Proposition 4.17 of [14].} \end{aligned}$$

On the other hand, by Proposition 1.7 of [15], we have

$$\begin{aligned} [12\iota_7] \circ \alpha_2(7) &\equiv -i_*\{\alpha_1(3), 12\iota_7, \alpha_2(7)\} \pmod{0} \\ &= -i_*\alpha_3(3) \quad \text{by Lemma 13.5 of [15].} \end{aligned} \quad \text{Q.E.D.}$$

Thus we get that

$$\xi \circ S\beta = 4[2\sigma'] - i_*\alpha_3(3) \pmod{\{2i_*\mu' = 8[2\sigma']\}},$$

which is of order 12. Hence 12ξ is extendable to SB . Namely just the 36 times of $[12\iota_7]$ is extendable to SB . Thus we have proved Proposition 6.5.

7. Computation of some homotopy groups of $W_{5,2}$. Let $W_{5,2} = \text{SU}(5)/\text{SU}(3)$ be the complex Stiefel manifold. For simplicity we set ${}^2\pi_{n+i}(S^n) = \pi_{n+i}^n$ and

$${}^2\pi_i(W_{5,2}) = W_i.$$

We will show the following.

PROPOSITION 7.1.

$$\begin{aligned} {}^2\pi_{15}(W_{5,2}) &\cong Z_4 \oplus Z_2 = \{[\nu_9^3], i_*\bar{\nu}_7\}, \quad \text{where } i_*\varepsilon_7 = 2[\nu_9^3]. \\ {}^2\pi_{16}(W_{5,2}) &\cong Z_{16} = \{[2\sigma_9]\}, \quad \text{where } 8[2\sigma_9] = i_*\mu_7. \\ {}^2\pi_{17}(W_{5,2}) &\cong Z_4 = \{i_*\nu_7\sigma_{10}\}. \end{aligned}$$

Proof. Associated with the bundle $S^7 \xrightarrow{i} W_{5,2} \xrightarrow{p} S^9$ we have the exact sequence:

$$\begin{aligned} \pi_{18}^9 &\xrightarrow{\Delta} \pi_{17}^7 \xrightarrow{i_*} W_{17} \xrightarrow{p_*} \pi_{17}^9 \xrightarrow{\Delta} \pi_{16}^7 \xrightarrow{i_*} W_{16} \xrightarrow{p_*} \pi_{16}^9 \xrightarrow{\Delta} \pi_{15}^7 \\ &\longrightarrow W_{15} \longrightarrow \cdots, \end{aligned}$$

where Δ is the boundary homomorphism.

Recall [15] that $\pi_{18}^9 \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 = \{\sigma_9\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\varepsilon_{10}\}$, $\pi_{17}^9 \cong Z_2 \oplus Z_2 \oplus Z_2 = \{\sigma_9\eta_{16}, \bar{\nu}_9, \varepsilon_9\}$, $\pi_{16}^9 \cong Z_{16} = \{\sigma_9\}$, $\pi_{15}^9 \cong Z_2 = \{\nu_9^3\}$, $\pi_{17}^7 \cong Z_8 \oplus Z_2 = \{\nu_7\sigma_{10}, \eta_7\mu_8\}$, $\pi_{16}^7 \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 = \{\sigma'\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\varepsilon_8\}$, $\pi_{15}^7 \cong Z_2 \oplus Z_2 \oplus Z_2 = \{\sigma'\eta_{14}, \bar{\nu}_7, \varepsilon_7\}$ and $\pi_{14}^7 \cong Z_8 = \{\sigma'\}$.

We also recall the formula (see [9])

$$(7.1) \quad \Delta(S\alpha) = \eta_7 \circ \alpha \quad \text{for any } \alpha \in \pi_i(S^8).$$

By (7.1) one can obtain immediately the following:

$$(7.2) \quad 0 \rightarrow Z_4 = \{\nu_7 \sigma_{10}\} \rightarrow W_{17} \rightarrow 0,$$

$$(7.3) \quad 0 \rightarrow Z_2 = \{\mu_7\} \rightarrow W_{16} \rightarrow Z_8 = \{2\sigma_9\} \rightarrow 0,$$

$$(7.4) \quad 0 \rightarrow Z_2 \oplus Z_2 \rightarrow \{\bar{\nu}_7, \epsilon_7\} \rightarrow W_{15} \rightarrow Z_2 = \{\nu_8^2\} \rightarrow 0.$$

Theorem 2.1 of [9] is useful in determining the extensions (7.3) and (7.4). In fact, the relation $i_*\{\eta_7, 2\sigma_8, 8\iota_{15}\} = i_*\mu_7$ implies $W_{16} \cong Z_{16} = \{[2\sigma_9]\}$ with $8[2\sigma_9] = i_*\mu_7$. The relation $i_*\{\eta_7, \nu_8^2, 2\iota_{14}\} = i_*\epsilon_7$ means $W_{15} \cong Z_4 \oplus Z_2 = \{[\nu_8^2], i_*\bar{\nu}_7\}$ with

$$2[\nu_8^2] = i_*\epsilon_7. \quad \text{Q.E.D.}$$

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