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COUNTEREXAMPLES IN STABLE SEMIGROUPS

BY
L. O'CARROLL

1. **Introduction.** A semigroup S is *stable* if and only if (1) for a and b in S, $Sa \subseteq Sab$ implies $Sa = Sab$, and (2) for c and d in S, $cS \subseteq dcS$ implies $cS = dcS$. A semigroup S is *pseudo-invertible* if and only if some power of every element lies in a subgroup of S.

We can adjoin an identity element 1 to a semigroup not having one. If S is an arbitrary semigroup, S^1 denotes the semigroup $S \cup \{1\}$ if S does *not* have an identity element, and denotes S otherwise [2, p. 4].

Stable semigroups were first investigated by Koch and Wallace [4], who noted that if a semigroup S is stable then S^1 is stable. If S is regular the converse holds. However, the main purpose of this paper is to show that the converse does not hold in general. This possibility was first raised in [1], p. 521, footnote (2). In this paper, Anderson, Hunter and Koch developed the theory of stable semigroups further, and some basic results from [1] and [4] are collected together in Theorem 1.1 below.

Let S be a semigroup. S is called *weakly stable* if S^1 is stable. If S is weakly stable but not stable, it is called *very weakly stable.* A weakly stable semigroup S is called *very weakly stable on the left* [*right*] if it fails to satisfy (1) [(2)] of the definition of a stable semigroup given above.

By investigating the structure of the members of a certain class of semigroups very weakly stable on the left, we produce a general method for constructing counterexamples (Theorem 3.5). Pseudo-invertible semigroups, studied by Munn in [5], are used extensively in this paper. We prove that if S is pseudo-invertible then S is weakly stable (Theorem 2.1). Finally we end the paper with a counterexample to a surmise which arises in the course of our work (Theorem 4.4).

The equivalences of Green [3], defined for any semigroup S, are as follows:

$$a\mathscr{R}b[a\mathscr{L}b, a\mathscr{J}b] \quad \text{if and only if} \quad aS^1 = bS^1[S^1a = S^1b,\ S^1aS^1 = S^1bS^1].$$

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}, \qquad \mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}.$$

For $a \in S$, let $L(a)[R(a), J(a)]$ be the set $S^1a[aS^1, S^1aS^1]$. Let

$$L_a = \{x \in S : L(x) = L(a)\},$$

with analogous definitions for H_a, R_a, D_a, J_a; and let $I(a) = J(a) \backslash J_a$.

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Let ρ denote any of the Green's relations. The ρ-equivalences on S and on S^1 differ only when $S \neq S^1$, and then merely in the ρ-class $\{1\}$ of 1. Thus the principal factors of S and S^1 are the same, except when $S \neq S^1$, and then $\{1\} \cup \{0\}$ is the only principal factor of S^1 not a principal factor of S. Moreover, as we shall see in Theorem 1.1, if S is weakly stable the $\mathscr{D}$- and $\mathscr{J}$-equivalences on S^1 coincide; thus the corresponding equivalences on S coincide also.

We shall treat each principal factor of a semigroup S as if it had a zero. If S has a principal factor K without a zero, which is so only when S has kernel K [2, Lemma 2.39], the theorems and definitions of this paper applying to K are read off "without the zero".

If a semigroup S has a principal series $S = S_0 \supset S_1 \supset \cdots \supset S_n = K \supset S_{n+1} = \square$, then S has kernel K and $\{S_i \backslash S_{i+1} : 0 \leqq i \leqq n\}$ is the set of $\mathscr{J}$-classes of S. For each a in $S_i \backslash S_{i+1}$, $J_a = S_i \backslash S_{i+1}$ and $I(a) \subseteq S_{i+1}$.

Hence the factors $\{S_i / S_{i+1} : 0 \leqq i \leqq n\}$ of the principal series are the principal factors $\{J(a)/I(a) : a \in S\}$ of S.

These facts are used below without further mention.

The notation and terminology of [2] are used throughout. Standard results in the theory of semigroups, which are to be found in [2], are assumed.

The multiplication of maps used in this paper is that of composition. Finally we note that "$\subset$" means "*properly* contained in".

1.1 THEOREM. *Let S be a semigroup.* (i) *If S is weakly stable then the $\mathscr{D}$- and $\mathscr{J}$-equivalences on S^1 coincide, and each principal factor of S is either completely 0-simple or null.*

(ii) *S is weakly stable if and only if $S^1L \cap J = L$ and $RS^1 \cap J = R$ for each $\mathscr{R}$-[$\mathscr{L}$-, $\mathscr{J}$-] class $R[L, J]$ of S^1 such that $R \cup L \subseteq J$.*

Proof. (i) This follows from Theorem 1, (i), [4] and the proof of Corollary 2.3, [1].

(ii) $\Rightarrow$ This follows from Corollary 1.1, (1), [1], in view of part (i) of this theorem.

$\Leftarrow$ Consider a, b in S^1 such that $S^1a \subseteq S^1ab$. Then $S^1aS^1 \subseteq S^1abS^1 \subseteq S^1aS^1$, and so $a \in S^1L_{ab} \cap J_{ab} = L_{ab}$. Hence $S^1a = S^1ab$.

The result follows by a dual argument on the right.

In [1], Anderson, Hunter and Koch regard statement (4), p. 528 as a *weakened* form of the stability of S^1. In fact, Theorem 1.1(ii) above shows that statement (5) and hence (4), p. 528 [1], is equivalent to the stability of S^1.

2. The structure of the first counterexample.

2.1. THEOREM. *Let S be a pseudo-invertible semigroup. Then S is weakly stable.*

Proof. Consider $a, b \in S$ such that $S^1a \subseteq S^1ab$. Then $a = xab$ for some $x \in S^1$. Since we wish to prove that $S^1a = S^1ab$, we may assume that both x and b are in S.

S is pseudo-invertible. Hence there exist positive integers n and m such that H_{b^n} and H_{x^m} are subgroups of S. Let $s = x^{mn}$, $t = b^{mn}$; then $s \in H_{x^m}$ and $t \in H_{b^n}$. Let e

and f be the identities of H_s and H_t respectively, and let u and v be the inverses of s and t, respectively, in their particular group $\mathscr{H}$-classes.

Since $a=xab$, it follows that $a=sat$. Hence $ea=af=a$ and $uav=eaf=a$. Moreover, $S^1a\subseteq S^1ab\subseteq S^1at$ and therefore $S^1av\subseteq S^1af=S^1a$. On the other hand $S^1a=S^1uav \subseteq S^1av$, and therefore $S^1a=S^1av$. Hence $S^1at=S^1af=S^1a$. This implies that $S^1a=S^1ab$.

The result now follows by an analogous argument on the right.

Theorem 2.1 explains the similarities between the results of [1], [4] and [5]. The converse result does not hold; the infinite cyclic semigroup is stable, since it is commutative. However it is not pseudo-invertible, since it is without idempotents.

The result of Theorem 2.1 cannot be sharpened, since the counterexample of §3 yields a nonstable pseudo-invertible semigroup (see the proof of Theorem 3.5(i)). However, Propositions 2.10 and 2.11 strengthen the theorem in a partial manner (see Theorem 2.3, Corollary).

Our source of counterexamples are semigroups satisfying the following condition:

2.2. Definition. A semigroup S is said to satisfy the condition (*) if and only if

(*) *S has a principal series and each principal factor is either completely* 0*-simple or null.*

Note that if S satisfies (*), S has a completely simple kernel K.

The next theorem is a particular case of a result due to Munn [5, Theorem 3].

2.3. Theorem. *Let S be a semigroup satisfying* (*). *Then S is pseudo-invertible.*

Proof. Let $S=S_0\supset S_1\supset\cdots\supset S_n=K\supset S_{n+1}=\Box$ be a principal series for S, where S satisfies (*). Since K is a union of groups, being completely simple, every element of K is in a subgroup of S.

Let $a\in S\backslash K$. Then $a\in S_i\backslash S_{i+1}$ for some i such that $0\leqq i\leqq n-1$. If S_i/S_{i+1} is null, $a^2\in S_{i+1}$. If S_i/S_{i+1} is completely 0-simple and a is not in a subgroup of S, then $a^2\in S_{i+1}$ [2, Theorem 2.52, (i)]. Continuing in this way, we see that some power of a is in a subgroup of S contained in $S\backslash K$ or some power of a is in K. Since K is a union of groups, the result follows.

Corollary. *Let S be a semigroup satisfying* (*). *Then S is weakly stable.*

Proof. This follows immediately from Theorems 2.1 and 2.3.

The above Corollary is a partial converse to Theorem 1.1(i). Propositions 2.10 and 2.11 partially strengthen the Corollary.

In the next theorem we examine the structure of semigroups which are very weakly stable on the left and which satisfy (*). An example of such a semigroup will be given in §3. First we have some preliminary lemmas.

2.4. Lemma. *Let S be a semigroup which is very weakly stable on the left. Then $S\neq S^1$. Further, there exist $a, b\in S$ with the following properties*:

(i) $Sa\subset Sab$.

(ii) *$J(a)/I(a)$ is null and $J_a=D_a=R_a$. For each $x\in J_a$, $L_x=\{x\}$.*

(iii) *$\langle b\rangle$ is infinite.*

Proof. It is immediate that $S\neq S^1$ and that there exist $a, b\in S$ such that $Sa\subset Sab$.

Now $a\notin Sa$, and therefore D_a is not a regular $\mathscr{D}$-class. By Theorem 1.1, $J(a)/I(a)$ is null and $J_a=D_a$.

Since $S\neq S^1$ and $a\notin Sa$, $L_a=\{a\}$. Green's Lemma [2, Lemma 2.2] then implies that $R_a=D_a$ and that $L_x=\{x\}$ for all $x\in D_a$.

Finally, since $Sa\subset Sab\subseteq Sab^2\subseteq\ldots$, b is of infinite order.

2.5. Lemma. *Let S be a semigroup with kernel K say, where S is very weakly stable on the left. Let a and b be those elements of S described in Lemma 2.4. Then $a\cdot\langle b\rangle\cap K=\square$. In particular $\langle b\rangle\cap K=\square$ and $a\notin K$.*

Proof. By Theorem 1.1(i), K is completely simple. Thus if $ab^r\in K$, for some positive integer r, Sab^r is a minimal left ideal of S. This contradicts the fact that $Sa\subset Sab\subseteq Sab^r$.

The rest of the result follows easily.

2.6. Theorem. *Let S be a semigroup satisfying (*) which is very weakly stable on the left. Let $S=S_0\supset S_1\supset\cdots\supset S_n=K\supset S_{n+1}=\square$ be a principal series for S. Then there exist $u, v\in S$ with the following properties:*

(i) *$Su\subset Suv$.*

(ii) *v generates an infinite group (v), say, where $(v)\subseteq S_j\backslash S_{j+1}$ for some j such that $0\leqq j\leqq n-2$. Let e denote the identity of the group (v).*

(iii) *$u\in S_{j+1}\backslash K$ and $u=ue\in u\cdot(v)\subseteq R_u$. $J(u)/I(u)$ is null.*

Proof. (i) There exist $a, b\in S$ satisfying Lemma 2.4(i) where $\langle b\rangle\cap K=\square$ by Lemma 2.5. The proof of Theorem 2.3 shows that H_{b^m} is a maximal subgroup of S contained in $S\backslash K$, for some positive integer m.

Let $v=b^m$ and let e be the identity of the group H_v. Since $Sa\subset Sab\subseteq Sav$, $Sa=Sae$. Let $u=ae$. Then $Su=Sae=Sa\subset Sav=Saev=Suv$. Hence $Su\subset Suv$.

(ii) and (iii) Lemmas 2.4 and 2.5 now apply with u and v in place of a and b respectively. Hence $u\notin K$ and $v\notin K$. Further, (v) is an infinite group and $J(u)/I(u)$ is null.

It follows that u is not a regular element, but that v is a regular element. Hence $u\notin D_v$. However $u\in J(v)$, since $u=ue$ where $e\in J_v$. By Theorem 1.1(i), $D_v=J_v$. Hence $u\in I(v)$.

Then $(v)\subseteq S_j\backslash S_{j+1}$ for some j such that $0\leqq j\leqq n-2$, and $u\in S_{j+1}\backslash K$.

Finally, let v^{-r} denote the inverse of v^r in the group (v) for each positive integer r. Since $u=ue=uv^r\cdot v^{-r}=uv^{-r}\cdot v^r$, $u=ue\in u\cdot(v)\subseteq R_u$.

Corollary 1. *Let S be a semigroup satisfying (*) which is very weakly stable on the left and let u and v be those elements of S described in Theorem 2.6. Then $J_u=D_u=R_u$ and for integers s and t,*

(i) *$L_{uv^s}=\{uv^s\}$ and* (ii) *$uv^s=uv^t$ implies $s=t$. J_u is infinite.*

Proof. By Theorem 2.6, $Su \subset Suv$. Hence Lemma 2.4 applies with u and v in the place of a and b respectively. It follows that $J_u = D_u = R_u$ and that $L_{uv^s} = \{uv^s\}$ for each integer s, since $uv^s \in R_u$ by Theorem 2.6(iii).

Suppose $uv^s = uv^t$ for some integers s and t. Since $u = ue$ by Theorem 2.6, this implies that $u = uv^{|s-t|}$. However, $Su \subset Suv^k$ for all positive integers k. Hence $s = t$.

This implies that $u \cdot (v)$ is in a natural 1-1 correspondence with the infinite group (v) (see Theorem 2.6(ii)). Hence $u \cdot (v)$ is infinite. However, by Theorem 2.6(iii), $u \cdot (v) \subseteq R_u = J_u$. Therefore J_u is infinite.

COROLLARY 2. *Let S be a semigroup satisfying* (*) *which is very weakly stable on the left and let u and v be those elements of S described in Theorem* 2.6. *For each integer t, let $P(t) = Suv^t$ (where $P(0) = Sue$). Then*

$$\cdots \subset P(-2) \subset P(-1) \subset P(0) = Su \subset P(1) \subset P(2) \subset \cdots.$$

Further $P(t+1) \backslash P(t) = (P(t) \backslash P(t-1)) \cdot v$ for all integers t.

Proof. This follows easily from Theorem 2.6.

The next theorem presents an alternative picture of a semigroup satisfying (*) which is very weakly stable on the left. First we introduce some notation.

In the following lemmas and theorem, S denotes a semigroup satisfying (*) which is very weakly stable on the left. $S = S_0 \supset S_1 \supset \cdots \supset S_n = K \supset S_{n+1} = \square$ is a principal series for S, and u and v are those elements of S described in Theorem 2.6.

By Theorem 1.1(i) $S_i \backslash S_{i+1}$ is a $\mathscr{D}$-class D_i for each i in $0 \leqq i \leqq n$, and each $\mathscr{D}$-class of S is found in this way. We further suppose that $u \in D_q$ for some q such that $1 \leqq q \leqq n-1$ (see Theorem 2.6).

2.7. LEMMA. *Following the above notation, $Su \subset Suv \subseteq S_{q+1}$. In particular, $D_k \cap Su \neq \square$ only if $k \geqq q+1$.*

Proof. By Theorems 1.1(ii) and 2.6, $S^1u \cap D_q = L_u = \{u\}$. Hence $Su \subseteq S_{q+1}$. Since S_{q+1} is an ideal, this implies that $Suv \subseteq S_{q+1}$.

2.8. LEMMA. *Let L be an $\mathscr{L}$-class contained in $Su \cap D_k$, where $k \geqq q+1$. Then $L \cdot v$ is an $\mathscr{L}$-class contained in $Suv \cap D_k$.*

Conversely for $k \geqq q+1$, each $\mathscr{L}$-class contained in $Suv \cap D_k$ is of this form.

Proof. Let $r \in L$. By Lemma 2.7, $r = xu$ for some $x \in S$. By Theorem 2.6, $u \mathscr{R} uv$. Hence $xu \mathscr{R} xuv$; that is $r \mathscr{R} rv$. By Green's Lemma [2, Lemma 2.2], $L \cdot v$ is an $\mathscr{L}$-class contained in D_k. Clearly $L \cdot v \subseteq Suv$.

Conversely let L' be an $\mathscr{L}$-class contained in $Suv \cap D_k$, and let $s \in L'$. Then $s = tv$ for some $t \in Su$. By the first part of this lemma, $t \in D_k$. Hence $L_t \cdot v = L'$, by Green's Lemma [2, Lemma 2.2].

2.9. THEOREM. *For each k such that $q+1 \leqq k \leqq n$, let $\{L_\gamma\}_{\Lambda(k)}$ be the set of $\mathscr{L}$-classes contained in $Su \cap D_k$.*

Then $\{L_\gamma \cdot v\}_{\Lambda(k)}$ *is the set of* $\mathscr{L}$*-classes in* $Suv \cap D_k$ *and* $\{L_\gamma\}_{\Lambda(k)} \subseteq \{L_\gamma \cdot v\}_{\Lambda(k)}$ *for each* k *such that* $q+1 \leqq k \leqq n$.

There exists p *such that* $q+1 \leqq p \leqq n$ *and such that* $\Lambda(p)$ *is infinite, where*

$$\{L_\gamma\}_{\Lambda(p)} \subset \{L_\gamma \cdot v\}_{\Lambda(p)}.$$

Proof. This follows easily from Lemmas 2.7, 2.8 and the fact that $Su \subset Suv$.

COROLLARY. *Following the notation of Theorem* 2.9, *we have*:

$$\cdots \subset \{L_\gamma \cdot v^{-1}\}_{\Lambda(p)} \subset \{L_\gamma\}_{\Lambda(p)} \subset \{L_\gamma \cdot v\}_{\Lambda(p)} \subset \cdots,$$

where each of these sets is an (*infinite*) *set of* $\mathscr{L}$*-classes in* D_p.

Proof. This follows easily from Theorem 2.9 and the fact that e, the identity of the group (v), is a right identity for the set Su (see Theorem 2.6).

Thus the simplest possible example of a semigroup satisfying (*) and very weakly stable on the left is a semigroup S having a principal series $S \supset S_1 \supset K \supset \square$ and having elements u and v such that

(i) v generates an infinite group (v), with identity e say, where $S \backslash S_1 = (v)$.

(ii) $u = ue$ and $S_1 \backslash K = u \cdot (v)$.

(iii) $Su \subset Suv \subseteq K$, and K is completely simple.

Finally, we can easily deduce the following partial refinements of Theorem 2.3, Corollary from Theorem 2.6, its Corollaries, and their right duals.

2.10. PROPOSITION. *Let S be a semigroup satisfying* (*) *such that each null principal factor of S is finite. Then S is stable.*

2.11. PROPOSITION. *Let S be a semigroup satisfying* (*) *such that each subgroup of S contained in* $S \backslash K$ *is periodic. Then S is stable.*

It is an easy matter to exhibit stable and pseudo-invertible semigroups not having a principal series.

3. **The first counterexample.** To ensure associativity we work with semigroups of mappings. The elements u and v will now be referred to as α and β respectively. The next theorem tells us how we may choose β.

3.1. THEOREM. *Let* γ *be a map with domain D and range R. Then* γ *is an element of a group of mappings if and only if* (i) $R \subseteq D$ *and* (ii) $\gamma | R$ *is* 1-1 *and onto R.*

Proof. $\Leftarrow$ This essentially is the result of Theorem 2.10(i), [2]. $\Rightarrow$ Let γ be an element of the group of mappings G. Let ι be the identity of G and let γ^{-1} be the inverse of γ in G.

Since $\gamma\gamma^{-1} = \iota = \gamma^{-1}\gamma$ and $\gamma\iota = \gamma$, we deduce that D and R are the domain and range of ι respectively, and that $R \subseteq D$. Now $\iota^2 = \iota$, and therefore $\iota | R$ is the identity on R. Hence $\gamma | R$ is 1-1. Since $\gamma = \iota\gamma$, $\gamma | R$ is also *onto* R.

COROLLARY. *Let G be a group of maps. Then each element of G has the same* *domain D and the same range R, where $R \subseteq D$. For $\gamma \in G$ to be of infinite order, it is necessary that R, and hence D, is infinite.*

Proof. We have shown in the proof of Theorem 3.1 that if $\gamma \in G$ has range R and domain D, so has ι the identity of G and $R \subseteq D$. It therefore follows that *all* elements of G have domain D and range R and that $G \subseteq \mathscr{T}_D$, the semigroup of transformations on D.

By Theorem 2.10(ii) [2], G is isomorphic to a subgroup of the symmetric group on R. Therefore if R is finite, G is finite.

The result now follows.

Theorem 3.1 and Corollary lead us to look for our counterexample among the subsemigroups of a transformation semigroup $\mathscr{T}_X$, where X is some infinite set. Such transformation semigroups have the following helpful property:

3.2. Proposition. *Let X be a nonempty set. For each $x \in X$ define the map $\eta_x \in \mathscr{T}_X$ as follows:*

$$\textit{for all} \quad y \in X, \quad y\eta_x = x.$$

Let $N = \{\eta_x : x \in X\}$. Then N is a right zero semigroup, and N is the completely simple kernel of any subsemigroup S of $\mathscr{T}_X$ such that $N \subseteq S$.

Proof. See Questions 6 and 7, p. 6, [2].

The notation introduced in Proposition 3.2 is adhered to below.

3.3. Example. Let X be the set of integers. Let R be the set of integers less than or equal to 1 together with the odd positive integers, and let T be the set of integers less than or equal to 0.

We define α from X onto T and β from X onto R as follows:

$$\alpha: \begin{cases} R\alpha = \{0\}. \\ \text{For each positive integer } n, \ (2n)\alpha = -n. \end{cases}$$

$$\beta: \begin{cases} \text{For } j \in T, \ j\beta = j+1. \\ \text{For } j \in X \backslash T \text{ of the form } 2m \text{ or } 2m-1 \text{ for} \\ \text{some positive integer } m, \ j\beta = 2m+1. \end{cases}$$

Then we note the following:

(A) β is a member of $\mathscr{T}_X$. Further, $X\beta = R \subset X$ and $\beta | R$ is 1-1 and onto R.

(B) α is a member of $\mathscr{T}_X$. Further, $X\alpha = T \subset R$ and $R\alpha = \{p\}$, where p is a fixed element of T.

(C) $X\alpha \subset X\alpha\beta$.

3.4. Lemma. *Given a set X and maps α and β satisfying* (A), (B) *and* (C) *we have*

(i) *β generates an infinite group, which we denote by (β). Let μ be the identity of (β).*

(ii) *$\alpha\mu = \alpha$ and $\{(\beta) \cup \{\alpha\}\} \cdot \alpha = \{\eta_p\} \subseteq N$.*

(iii) *$\{\alpha \cdot (\beta)\}^2 \subseteq N$ and $(\beta) \cdot \alpha \cdot (\beta) \subseteq N$.*

Proof. (i) By Theorem 3.1 and (A), β generates a group (β). If β is of finite order, $\beta^n|X\beta^n$ is the identity map on $X\beta^n$ for some positive integer n. But (C) implies that $T\subset T\beta\subseteq T\beta^n\subseteq X\beta^n$. Hence β is of infinite order.

(ii) By the proof of Theorem 3.1, $\mu|X\beta$ is the identity map on $X\beta$. Hence $\alpha\mu=\alpha$, since (B) implies that $X\alpha\subset X\beta$.

Now $X\alpha\subset R$ and by Theorem 3.1, Corollary each element of the group (β) has range R. The rest of (ii) now follows from the fact that $R\alpha=\{p\}$ (see (B)).

(iii) This follows easily from part (ii) of this lemma and Proposition 3.2.

Given a set X, and maps α and β satisfying (A), (B) and (C) let $\mathscr{S}=(\beta)\cup\alpha\cdot(\beta)\cup N$, with composition of maps as multiplication on $\mathscr{S}$.

3.5. Theorem. *Suppose we are given a set X and maps α and β satisfying* (A), (B) *and* (C).

(i) *$\mathscr{S}$ is a semigroup which is very weakly stable on the left.*

(ii) *Let $\mathscr{S}_1=N\cup\alpha\cdot(\beta)$. Then $\mathscr{S}\supset\mathscr{S}_1$, $\mathscr{S}_1\supset N$, and $\mathscr{S}\supset\mathscr{S}_1\supset N\supset\square$ is a principal series for $\mathscr{S}$. $\mathscr{S}\backslash\mathscr{S}_1=(\beta)$, $\mathscr{S}_1\backslash N=\alpha\cdot(\beta)$ and $\mathscr{S}_1/N$ is null.*

Proof. (i) By Lemma 3.4(iii), $\mathscr{S}$ is a subsemigroup of $\mathscr{T}_X$. Since (β) is a group and N a union of groups, by Proposition 3.2, Lemma 3.4(iii) implies that $\mathscr{S}$ is pseudo-invertible. By Theorem 2.1, $\mathscr{S}$ is weakly stable.

By Lemma 3.4(ii), $\mathscr{S}\alpha=\{\eta_y : y\in X\alpha\}$ and therefore $\mathscr{S}\alpha\beta=\{\eta_y : y\in X\alpha\beta\}$. However by (C), $X\alpha\subset X\alpha\beta$. Hence $\mathscr{S}\alpha\subset\mathscr{S}\alpha\beta$.

(ii) By Lemma 2.5, $\alpha\notin N$ and $\beta\notin N$. Now N is the kernel of $\mathscr{S}$, by Proposition 3.2, and Lemma 3.4(ii) and (iii) imply that $\{J(\alpha)\}^2\subseteq K$. Hence $J(\alpha)/I(\alpha)$ is null and α is not a regular element.

By Lemma 3.4(i) β generates the group (β), and in particular β is a regular element. Now $\alpha=\alpha\mu=\alpha\beta\beta^{-1}$, and it follows that $\alpha\in J(\beta)$.

By Theorem 1.1(i) and part (i) of this theorem, $\mathscr{J}=\mathscr{D}$ on $\mathscr{S}$. Hence $\alpha\in I(\beta)$, since α is not a regular element. By the proof of Theorem 2.6(iii) $\alpha\cdot(\beta)\subseteq R_\alpha$, and the result follows.

Since we have exhibited concrete X, α and β satisfying (A), (B) and (C) in Example 3.3, we have constructed our counterexample. Theorem 3.5(ii) shows that $\mathscr{S}$ is in fact the simplest possible counterexample (see the remarks after Theorem 2.9).

4. The second counterexample.

We now exhibit a semigroup T which shows that the hypothesis that S has a principal series is essential in Theorem 2.3, Corollary.

The presentation of this section is similar to that of §3, and wherever a similar proof is required we refer to the appropriate part of §3.

4.1. Example. Let X be the set of integers. Let R be the set of integers less than or equal to 0.

We define α and β in $\mathscr{T}_X$ as follows:

$$\alpha : \begin{cases} \text{for each positive integer } n,\ n\alpha = 1-n, \\ \text{for all nonpositive integers } j,\ j\alpha = 0. \end{cases}$$

$$\beta : \begin{cases} 0\beta = 0,\ 1\beta = 1, \\ \text{for each } j \text{ in } X\backslash\{0, 1\},\ j\beta = j+2. \end{cases}$$

Then we note that

(D) $X\alpha = R$ and $X\alpha \subset X\alpha\beta$.

(E) $\alpha = \beta\alpha\beta$.

(F) $R\beta = R\beta^2$ $(= R\beta^3 = \cdots)$.

(G) $R\beta\alpha = \{r\}$, where r is a fixed element of R.

Given a set X and maps α and β satisfying (D), (E), (F) and (G) let $T = \langle\alpha, \beta\rangle \cup N$, with the composition of maps as multiplication on T.

4.2. Lemma. *T is a semigroup with completely simple kernel N. T is not weakly stable.*

Proof. By Proposition 3.2, N is the completely simple kernel of $\mathscr{T}_X$. It easily follows that T is a semigroup with kernel N.

By (D), α and $\alpha\beta$ have different ranges. Lemma 2.5, [2] then implies that α and $\alpha\beta$ are not $\mathscr{L}$-equivalent in $\mathscr{T}_X$. A fortiori, they are not $\mathscr{L}$-equivalent in T. On the other hand, $T^1\alpha = T^1\beta\alpha\beta \subseteq T^1\alpha\beta$. Hence $T^1\alpha \subset T^1\alpha\beta$.

4.3. Lemma. *α and β satisfy the following properties:*

(i) $\alpha\cdot\langle\beta\rangle \cap N = \Box$ *and* $\langle\beta\rangle \cap N = \Box$.

(ii) $\{\{\alpha\} \cup \alpha\cdot\langle\beta\rangle\}\cdot\alpha \subseteq N$.

(iii) $\langle\beta\rangle$ *is infinite.*

Proof. (i) This follows from the proof of Lemmas 2.5 and 4.2, since $T^1\alpha \subset T^1\alpha\beta$.

(ii) and (iii) See the corresponding parts of Lemma 3.4.

We now show that each principal factor of T is null except for the completely simple kernel N, and that T has a "countable principal series" in a sense made explicit below.

4.4. Theorem. *The principal factors of T are N, $\{\{\alpha\} \cup \alpha\cdot\langle\beta\rangle \cup \langle\beta\rangle\cdot\alpha\} \cup \{0\}$ and $\{\beta^r\} \cup \{0\}$ for each positive integer r, where these are all distinct. All are null except N, which is completely simple.*

Proof. See the proof of Theorem 3.5(ii).

Let $T_\alpha = N \cup \{\alpha\} \cup \alpha\cdot\langle\beta\rangle \cup \langle\beta\rangle\cdot\alpha$ and for each positive integer r, let $T_r = T_\alpha \cup \{\beta^n : n \geqq r\}$. Then T_α and each T_r is an ideal of T and

$$T = T_1 \supset T_2 \supset \cdots \supset T_n \supset T_{n+1} \supset \cdots \supset T_\alpha \supset N.$$

Further, N is maximal in T_α and T_{r+1} is maximal in T_r for each positive integer r. In this sense, T possesses a "countable principal series".

As we have seen in Lemma 4.2, T is not weakly stable. A fortiori, T is not pseudo-invertible; indeed no power of β lies in a subgroup of T.

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Bibliography

1. L. W. Anderson, R. P. Hunter and R. J. Koch, *Some results on stability in semigroups*, Trans. Amer. Math. Soc. **117** (1965), 521–529.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 1, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

3. J. A. Green, *On the structure of semigroups*, Ann. of Math. **54** (1951), 163–172.

4. R. J. Koch and A. D. Wallace, *Stability in semigroups*, Duke Math. J. **24** (1957), 193–196.

5. W. D. Munn, *Pseudo-inverses in semigroups*, Proc. Cambridge Philos. Soc. **57** (1961), 247–250.

Queen's University,
Belfast, Northern Ireland