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DYNAMICAL SYSTEMS WITH AN INVARIANT SPACE OF VECTOR FIELDS

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I. In the theory of differentiable dynamical systems, the natural actions on coset spaces are of particular interest, not least because the general situation is so intractable. Let G be a Lie group and H a closed subgroup such that G/H is compact. A natural action on G/H is a group of transformations of G/H of the form $xH \mapsto a\alpha(x)H$, where a is an element of G and α is an automorphism of G such that $\alpha(H)=H$. The space of right-invariant vector fields on G is carried by the natural projection π, $x \mapsto xH$, onto a space of vector fields on G/H, and this space is invariant under each natural action. Suppose now that H is discrete. Then G is unimodular. A Haar measure on G determines a finite Borel measure on G/H invariant under each action. Analogously, a translation invariant n-form ω on G, where n is the dimension of G, determines an n-form η on G/H, and $T^*\eta$ equals $\pm\eta$ for each natural transformation T.

It is such a phenomenon that we examine in this paper: A dynamical system in which a certain finite-dimensional linear space of vector fields and a certain differential form are (essentially) invariant under the action.

In the next few paragraphs we define several expressions, and then state our main result. The following section is devoted to a proof of this theorem. Additional results are presented in the third section.

Let M be a differentiable manifold and let $\mathscr{X}$ be a finite-dimensional linear space of vector fields on M. We say that $\mathscr{X}$ is *spanning* if, for each p in M, the evaluation at p, $X \mapsto X_p$, maps $\mathscr{X}$ onto the tangent space of M at p. If the dimension of $\mathscr{X}$ equals the dimension of M, in which case these mappings are each one-to-one, then $\mathscr{X}$ is said to be *simply spanning*. If $\mathscr{X}$ is simply spanning, an affine connection ∇ on M is determined by the condition:

$$\nabla_Y X = 0$$

for X in $\mathscr{X}$ and Y any vector field on M. We call ∇ the *connection associated with* $\mathscr{X}$.

Let $\mathscr{T}$ be a group of diffeomorphisms of M. For each T in $\mathscr{T}$, let T_* be the associated mapping of vector fields, and let T^* be the associated mapping of differential forms. A linear space $\mathscr{X}$ of vector fields on M is said to be *invariant under* $\mathscr{T}$, if $T_*(\mathscr{X})=\mathscr{X}$ for each T in $\mathscr{T}$. In this case, the restriction of $\{T_* : T\in\mathscr{T}\}$ to $\mathscr{X}$ is a group of linear transformations of $\mathscr{X}$.

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Let $\mathcal{T}$ be a group of homeomorphisms of a space S. An *eigenfunction for* $\mathcal{T}$ *on* S is a function f on S, with values in the unit circle, such that, for each T in $\mathcal{T}$, there exists a complex number λ_T satisfying $fT=\lambda_T f$.

THEOREM 1. *Let M be a compact differentiable manifold of dimension n, ω an n-form on M, $\mathcal{X}$ a finite-dimensional linear space of vector fields on M, and $\mathcal{T}$ a group of diffeomorphisms of M. Suppose the following statements are true.*

(I) *ω is not identically zero, and $T^*\omega=\pm\omega$ for each T in $\mathcal{T}$.*

(II) *$\mathcal{X}$ is spanning and invariant under $\mathcal{T}$.*

(III) *$\mathcal{T}$ is abelian and each differentiable eigenfunction for $\mathcal{T}$ on M is constant.*

Then the following statements are also true.

(I) *Except for multiplication by a scalar, ω is the only n-form on M satisfying assumption* (I) *above.*

(II) *$\mathcal{X}$ is simply spanning.*

(III) *If $\mathcal{Y}$ is a finite-dimensional linear space of vector fields on M invariant under $\mathcal{T}$, then $\mathcal{Y}\subseteq\mathcal{X}$.*

(IV) *$\mathcal{X}$ is a Lie algebra.*

(V) *The system $(M, \omega, \mathcal{X}, \mathcal{T})$ is isomorphic to $(G/\Gamma, \eta, \mathcal{Y}, \mathcal{S})$ where: G is a simply connected Lie group whose Lie algebra is isomorphic to $\mathcal{X}$; Γ is a discrete subgroup of G; η corresponds to a translation-invariant n-form on G; $\mathcal{Y}$ corresponds to the space of right-invariant vector fields on G; and, each transformation, $x\Gamma\mapsto S(x\Gamma)$, in $\mathcal{S}$ is of the form*

$$x\Gamma\mapsto a\alpha(x)\Gamma,$$

a being in G and α being an automorphism of G such that $\alpha(\Gamma)=\Gamma$.

COROLLARY 1. *Let M be a compact differentiable manifold, ∇ an affine connection on M associated with a simply spanning space of vector fields, and $\mathcal{T}$ an abelian group of diffeomorphisms of M which preserve the affine structure. If each differentiable eigenfunction for $\mathcal{T}$ on M is constant, then $\mathcal{T}$ preserves no other such affine structure, and $(M, \mathcal{T})$ is isomorphic to a natural action on a coset space G/Γ with Γ discrete.*

II.

LEMMA 1. *Let $\mathcal{E}$ be a finite-dimensional real vector space, $\mathcal{L}$ an abelian group of linear transformations of $\mathcal{E}$, and C a compact subset of $\mathcal{E}$ invariant under $\mathcal{L}$. If C has more than one point, then there exists a differentiable function $f\colon \mathcal{E}\to\mathbf{C}$ such that f is an eigenfunction for $\mathcal{L}$ on C.*

Proof. Let $\mathcal{F}$ be the space of all complex-valued real-linear functionals on $\mathcal{E}$; $\mathcal{F}$ is a complex vector space. For each L in $\mathcal{L}$ define $L^*\colon\mathcal{F}\to\mathcal{F}$ by $L^*f=fL$. Thus, $\mathcal{L}^*=\{L^* : L\in\mathcal{L}\}$ is an abelian group of linear transformations of $\mathcal{F}$. These transformations can be put simultaneously in triangular form. That is, there exists a basis $\{f_1, f_2, \ldots, f_n\}$ of $\mathcal{F}$ such that for each k, the linear span of $\{f_1, \ldots, f_k\}$

is invariant under $\mathscr{L}^*$. Let m be the smallest integer k such that f_k is not constant on C. Set $g=f_m$. Hence, for each L in $\mathscr{L}$, there exists α_L in $\boldsymbol{C}$ such that $L^*g-\alpha_L g$ is constant on C; let the constant be β_L; on C then, $gL=\alpha_L g+\beta_L$.

Let $B=g(C)$. Define $A_L\colon \boldsymbol{C}\to\boldsymbol{C}$ by $A_L z=\alpha_L z+\beta_L$. Since $A_L(B)=B$, $|\alpha_L|=1$. Let K be the convex hull of C; then $A_L(K)=K$. Thus there exists a point γ in K left fixed by each A_L:

$$\gamma = \alpha_L\gamma+\beta_L$$

for L in $\mathscr{L}$. Define $h\colon \mathscr{E}\to\mathscr{E}$ by $h(x)=g(x)-\gamma$. For x in C,

$$h(Lx) = \alpha_L g(x)+\beta_L-\gamma = \alpha_L h(x).$$

Thus $hL=\alpha_L h$ on C for each L.

If $|h|$ is constant on C, let the constant be c (which cannot equal 0) and define f by $f=h/c$.

If $|h|$ is not constant on C, there exists an $\varepsilon>0$ such that

$$f = e^{i\varepsilon|h|}$$

is not constant on C, and this f is an eigenfunction for $\mathscr{L}$ on C, each eigenvalue being 1 in this case.

LEMMA 2. *Given the assumptions of Theorem* 1, *conditions* (I) *and* (II) *are valid.*

Proof. Let $\mathscr{E}$ be the space of all multilinear alternating mappings $\eta\colon \mathscr{X}^n\to\boldsymbol{R}$; here $\mathscr{X}$ is regarded as a real vector space. Thus $\mathscr{E}$ is a finite-dimensional real vector space. For each T in $\mathscr{T}$, define $T'\colon \mathscr{E}\to\mathscr{E}$ by

$$(T'\eta)(X_1,\ldots,X_n) = \eta(T_*X_1,\ldots,T_*X_n).$$

Each T' is linear. For each T, let ε_T in $\{1,-1\}$ be such that $T^*\omega=\varepsilon_T\omega$. Thus

$$\omega(T_*X_1,\ldots,T_*X_n) = \varepsilon_T\omega(X_1,\ldots,X_n)T.$$

For each p in M, define $\omega_p\colon \mathscr{X}^n\to\boldsymbol{R}$ by

$$\omega_p(X_1,\ldots,X_n) = (\omega(X_1,\ldots,X_n))(p).$$

Clearly ω_p is in $\mathscr{E}$ and

$$T'\omega_p = \varepsilon_T\omega_{Tp}.$$

Define $L_T=\varepsilon_T T'$. Thus $\mathscr{L}=\{L_T : T\in\mathscr{T}\}$ is an abelian group of linear transformations of $\mathscr{E}$ leaving the set

$$C = \{\omega_p : p\in M\}$$

invariant. The mapping $p\mapsto\omega_p$ of M into $\mathscr{E}$ is differentiable. By Lemma 1 and assumption (III), C has but one point and it is not 0.

Conclusion (I) is valid because we have proved that ω is everywhere nonzero. To verify (II), suppose that X is in $\mathscr{X}$, p is in M, and $X_p=0$. Then

$$\omega_p(X, X_2, \ldots, X_n) = 0$$

for $X_2, \ldots, X_n$ in $\mathscr{X}$. Thus, for each q in M,

$$\omega_q(X, X_2, \ldots, X_n) = 0$$

for $X_2, \ldots, X_n$ in $\mathscr{X}$; $X_q=0$ for each q; $X=0$.

LEMMA 3. *Let $\mathscr{X}$ be simply spanning on the compact differentiable manifold M of dimension n, and invariant under the group $\mathscr{T}$. Then M is orientable and there exists an everywhere nonzero n-form ω on M with the following property: For each T in $\mathscr{T}$, $T^*\omega = \pm\omega$ and the determinant of T_* on $\mathscr{X}$ equals ± 1, according as (for both) T preserves or reverses the orientation of M.*

Proof. Let $\{X_1, X_2, \ldots, X_n\}$ be a basis for $\mathscr{X}$. Let $\{\omega_1, \omega_2, \ldots, \omega_n\}$ be a dual system of 1-forms on M. That is, $\omega_i(X_j)=\delta_{ij}$. Set $\omega=\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_n$. For each T in $\mathscr{T}$, $T^*\omega=d_T\omega$ where d_T is the determinant of T_* on $\mathscr{X}$. The finite nonnegative Borel measure on M corresponding to ω is invariant under $\mathscr{T}$. Hence $|d_T|=1$.

LEMMA 4. *Let $\mathscr{X}$ be simply spanning on the differentiable manifold M. Then there exists a unique affine connection ∇ on M such that $\nabla_Y X=0$ for X in $\mathscr{X}$ and Y any vector field on M. Moreover, a diffeomorphism T of M is affine with respect to ∇ if and only if $T_*(\mathscr{X})=\mathscr{X}$.*

Proof. The connection ∇ is given by

$$\nabla_Y\left(\sum_{k=1}^{n} f_k X_k\right) = \sum_{k=1}^{n} (Yf_k)X_k,$$

where $\{X_1, X_2, \ldots, X_n\}$ is a basis for $\mathscr{X}$. Uniqueness is clear.

Let T be an affine transformation of M. Thus T carries geodesics to geodesics, and the geodesics are just the integral curves for members of $\mathscr{X}$. Hence $T_*(\mathscr{X})=\mathscr{X}$.

Let T be a diffeomorphism such that $T_*(\mathscr{X})=\mathscr{X}$. Let ∇' be the connection on M induced by T. That is,

$$\nabla'_{T_*Y}T_*X = T_*(\nabla_Y X).$$

Clearly, $\nabla'_Y X=0$ for X in $\mathscr{X}$. Hence $\nabla'=\nabla$.

LEMMA 5. *Let $\mathscr{X}$ be a finite-dimensional linear space of vector fields on the compact differentiable manifold M. Let V be a fixed vector field on M and let $\mathscr{T}=\{T_t : t\in \boldsymbol{R}\}$ be the flow on M corresponding to V. For each vector field X on M, let $LX=[V, X]$. Then $\mathscr{X}$ is invariant under $\mathscr{T}$ if and only if $L(\mathscr{X})\subseteq\mathscr{X}$. If $\mathscr{X}$ is so invariant, then*

$$T_{t*} = e^{-tL}$$

on $\mathscr{X}$.

Proof. It is known [3, p. 15] that

$$-L = \lim_{t \to 0} \frac{1}{t}(T_{t*} - I)$$

pointwise, on the space of vector fields, and that L commutes with each T_{t*}.

If $\mathscr{X}$ is invariant under $\mathscr{T}$, the conclusion is clear.

Suppose now that $L(\mathscr{X}) \subseteq \mathscr{X}$. Each space $T_{t*}(\mathscr{X})$ is invariant under L. Let $\mathscr{Y}$ be the linear span of the union of these spaces. Each point in $\mathscr{Y}$ is contained in a finite-dimensional subspace of $\mathscr{Y}$ invariant under L. Define operators K_t on $\mathscr{Y}$ by $K_t = T_{t*}e^{tL}$. It is readily shown that the derivative of the mapping $t \mapsto K_t$ is 0 on $\boldsymbol{R}$. Thus K_t is constant in t and equals the identity transformation on $\mathscr{Y}$.

Proof of Theorem 1. Conclusions (I) and (II) have been verified.

(III) Let $\mathscr{Y}$ be as given. Let $\mathscr{E}$ be the space of linear mappings $f\colon \mathscr{Y} \to \mathscr{X}$. For each T in $\mathscr{T}$, define $L_T\colon \mathscr{E} \to \mathscr{E}$ by

$$(L_T f) = T_* f (T_*)^{-1} = T_* f T_{-*}.$$

Clearly, each L_T is linear, and $\mathscr{L} = \{L_T : T \in \mathscr{T}\}$ is an abelian group of linear transformations of $\mathscr{E}$. For each p in M, define $f_p\colon \mathscr{Y} \to \mathscr{X}$ by requiring that $f_p(Y)$ be that element of $\mathscr{X}$ which agrees with Y at p. That is, $(f_p Y)_p = Y_p$. The mapping $p \mapsto f_p$ is differentiable. Since $T_*(X_p) = (T_* X)_{Tp}$, we have

$$\begin{aligned} [(L_T f_p) Y]_{Tp} &= T_*[(f_p T_{-*} Y)_p] = T_*[(T_{-*} Y)_p] \\ &= Y_{Tp} = (f_{Tp} Y)_{Tp}. \end{aligned}$$

Thus, $L_T f_p = f_{Tp}$. The set $\{f_p : p \in M\}$ is invariant under $\mathscr{L}$ and, by Lemma 1, it must consist of but one point. Therefore, $\mathscr{Y} \subseteq \mathscr{X}$.

(IV) The set $\{[X, Y] : X, Y \in \mathscr{X}\}$ is invariant under $\mathscr{T}$. Its linear span is finite-dimensional and invariant under $\mathscr{T}$. By (III), the set is contained in $\mathscr{X}$.

(V) Let G be a simply connected Lie group with Lie algebra isomorphic to $\mathscr{X}$. It is well known that there exists a transitive action, $(x, p) \mapsto x \cdot p$, of G on M which leaves $\mathscr{X}$ invariant. Since G has dimension n, there exists a discrete uniform subgroup Γ of G and a diffeomorphism $U\colon G/\Gamma \to M$ such that

$$U(xy\Gamma) = x \cdot U(y\Gamma).$$

Let $\mathscr{Y}$ be the image under π_* of the space of right-invariant vector fields on G. Since $\mathscr{Y}$ generates the action of G on G/Γ and $\mathscr{X}$ generates the action of G on M, $U_*(\mathscr{X}) = \mathscr{Y}$. By Lemma 3, ω is invariant under the action of G on M. Hence, $\eta = (U^{-1})^*\omega$ is invariant under the action of G on G/Γ, and η must correspond to a translation-invariant n-form on G.

That the diffeomorphisms $U^{-1}TU$ on G/Γ are of the stated form follows from Theorem 2 of the next section.

Proof of Corollary 1. By Lemma 3, there exists an appropriate n-form on M and Theorem 1 applies. The transformations are affine by Lemma 4, and the structure is unique by (III) of Theorem 1.

III.

Corollary 2. *Let M be a compact differentiable manifold of dimension n, ω an n-form on M, $\mathscr{X}$ a finite-dimensional linear space of vector fields on M. Let there be given a differentiable action of $\boldsymbol{R}^m$ on M. Suppose that:*

(I) *ω is not identically zero, and is invariant under $\boldsymbol{R}^m$;*

(II) *$\mathscr{X}$ is spanning and invariant under $\boldsymbol{R}^m$;*

(III) *Either* (i) *M is simply connected, or* (ii) *the action has a fixed point in M.*

Then there exists a nonconstant differentiable function on M which is invariant.

Proof. Let $\mathscr{T}$ be the group of diffeomorphisms. Let f be a differentiable eigenfunction for $\mathscr{T}$ on M. Choose α_T in $\boldsymbol{R}$ such that

$$fT = e^{i\alpha_T}f.$$

If there exists a fixed point, $fT=f$ for all T, and f is invariant. If M is simply connected, there exists a differentiable function $g\colon M\to\boldsymbol{R}$ such that $f=e^{ig}$; then

$$e^{igT} = e^{i(g+\alpha_T)}$$

and $gT-g$ is constant, hence zero; f is invariant.

Suppose now that there does not exist a nonconstant differentiable invariant function on M. Then Theorem 1 applies. Let $\mathscr{Y}$ be the space of vector fields on M corresponding to the action of $\boldsymbol{R}^m$. Since $\mathscr{Y}$ is invariant under $\mathscr{T}$, $\mathscr{Y}\subseteq\mathscr{X}$. Thus the system $(M,\mathscr{T})$ is isomorphic to $(G/\Gamma,\mathscr{S})$ where $\mathscr{S}$ consists of translations, $x\Gamma\mapsto ax\Gamma$. If $\mathscr{T}$ has a fixed point, this is surely a contradiction. If M is simply connected, then $\Gamma=\{e\}$, since Γ is the fundamental group of G/Γ. In this case $(M,\mathscr{T})$ is isomorphic to (G, A) where G is a compact simply connected group and A is a connected abelian subgroup of G. Let $\bar{A}$ be the closure of A. This is a torus (or just $\{e\}$) and hence not equal to G. But any differentiable function on $G/\bar{A}$ would determine an invariant function on G. This is a contradiction.

Theorem 2. *For $k=1,2$: Let G_k be a simply connected Lie group, Γ_k a discrete subgroup of G_k, and $\mathscr{X}_k$ the space of vector fields on G_k/Γ_k corresponding to the right-invariant vector fields on G_k. Let $T\colon G_1/\Gamma_1\to G_2/\Gamma_2$ be a diffeomorphism and suppose that $T_*(\mathscr{X}_1)=\mathscr{X}_2$. Then there exists a point a in G_2 and an isomorphism $\alpha\colon G_1\to G_2$ such that $\alpha(G_1)=G_2$, $\alpha(\Gamma_1)=\Gamma_2$, and*

$$T(x\Gamma_1) = a\alpha(x)\Gamma_2$$

for x in G_1. Moreover, if a' and α' also have this property, then there exists γ in Γ_2 such that $a'=a\gamma$ and $\alpha'(x)=\gamma^{-1}\alpha(x)\gamma$.

Proof. Let $\mathscr{Y}_k$ be the space of right invariant vector fields on G_k. Let $\pi_k\colon G_k\to G_k/\Gamma_k$ be the natural projection. Then $\mathscr{X}_k=\pi_{k*}(\mathscr{Y}_k)$. Thus $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are isomorphic as Lie algebras, and T_* lifts to an isomorphism $L\colon\mathscr{Y}_1\to\mathscr{Y}_2$ such that $\pi_{2*}L=T_*\pi_{1*}$. Since G_1 and G_2 are simply connected, there exists an isomorphism α of G_1 onto

G_2 such that $L=\alpha_*$. Let e_k be the identity element of G_k. Choose a in G_2 such that $T(e_1\Gamma_1)=a\Gamma_2$. The two mappings of G_1 onto G_2/Γ_2 given by

$$x \mapsto T(x\Gamma_1), \qquad x \mapsto a\alpha(x)\Gamma_2,$$

are equal at e_1 and have equal differentials. Hence, these two mappings are equal. It is clear that $\alpha(\Gamma_1)=\Gamma_2$.

Now suppose that a' and α' are such that

$$a'\alpha'(x)\Gamma_2 = a\alpha(x)\Gamma_2$$

for all x in G_1. Letting $x=e_1$ shows that $a'\Gamma_2=a\Gamma_2$. There exists γ in Γ_2 such that $a'=a\gamma$. Thus

$$\alpha'(x)\Gamma_2 = \gamma^{-1}\alpha(x)\Gamma_2 = \gamma^{-1}\alpha(x)\gamma\Gamma_2.$$

The two isomorphisms, $x \mapsto \alpha'(x)$ and $x \mapsto \gamma^{-1}\alpha(x)\gamma$, are equal on a neighborhood of e_1 in G_1. Thus they are equal on G_1.

EXAMPLE 1. Let C denote the unit circle. There exists a two-dimensional spanning space of vector fields on C and a flow $\mathscr{T}=\{T_t : t \in \boldsymbol{R}\}$ on C such that:

(I) The only invariant Borel measures on C are concentrated on the set $\{1,-1\}$;

(II) $\mathscr{X}$ is invariant under $\mathscr{T}$;

(III) There exist no nonconstant continuous eigenfunctions for $\mathscr{T}$ on C.

To see this, let V denote the vector field $d/d\theta$ on C, with respect to the parametrization $\theta \mapsto e^{i\theta}$. Define the functions x, y, z on C by

$$x(p) = \tfrac{1}{2}(p+\bar{p}), \qquad y(p) = (1/2i)(p-\bar{p}), \qquad z(p) = p.$$

Set $Y=yV$ and let $\mathscr{T}=\{T_t : t \in \boldsymbol{R}\}$ be the flow corresponding to the vector field Y. By considering at which points Y is positive, negative, or zero, one sees that:

$$T_t 1 = 1, \qquad T_t(-1) = -1$$

for all t; and

$$\lim_{t\to\infty} T_t p = -1, \qquad \lim_{t\to-\infty} T_t p = 1$$

for p not in $\{1, -1\}$. Conditions (I) and (III) are thus satisfied.

Define vector fields X_1 and X_2 on C by

$$X_1 = (1+x)V, \qquad X_2 = (1-x)V.$$

It is readily seen that

$$\begin{aligned} Vx &= -y, & Vy &= x \\ [Y, V] &= -xV, & [Y, xV] &= -V \\ [Y, X_1] &= -X_1, & [Y, X_2] &= X_2. \end{aligned}$$

Let $\mathscr{X}$ be the linear span of $\{X_1, X_2\}$. Certainly $\mathscr{X}$ is spanning and (by Lemma 5) invariant under the flow. Thus (II) is satisfied.

Note that Y is real-analytic on C. Thus each T_t is a fractional linear transformation leaving 1 and -1 fixed. Such transformations are of the form

$$p \mapsto (p-a)/(1-ap)$$

for a real and not equal to ± 1. In our case, $-1<a<1$. The system $(C, \mathscr{T})$ is thus isomorphic to a natural action on G/H, where G is the group of all fractional linear transformations, and H is not discrete.

EXAMPLE 2. Let C denote the unit circle. There exists a simply spanning space of vector fields $\mathscr{X}$ on the two-dimensional torus $C \times C$ which is invariant under all translations, but is not a Lie algebra.

To see this, define functions u and v on $C \times C$ by

$$u(p,q) = \tfrac{1}{2}(p+\bar{p}), \qquad v(p,q) = (1/2i)(p-\bar{p}).$$

Define the vector fields U and V on $C \times C$ by $U=\partial/\partial\alpha$ and $V=\partial/\partial\beta$, with respect to the parametrization

$$(\alpha, \beta) \mapsto (e^{i\alpha}, e^{i\beta}).$$

Now define X_1 and X_2 on $C \times C$ by

$$X_1 = uU+vV, \qquad X_2 = vU-uV.$$

One can readily see that

$$\begin{aligned} [U, X_1] &= -X_2, & [U, X_2] &= X_1 \\ [V, X_1] &= 0, & [V, X_2] &= 0 \end{aligned}$$
$$[X_1, X_2] = U.$$

Let $\mathscr{X}$ be the linear span of $\{X_1, X_2\}$. This is simply spanning. It is not a Lie algebra. Moreover, by Lemma 5, $\mathscr{X}$ is invariant under flows corresponding to vector fields of the form $\alpha U+\beta V$. But these are just the flows of translations.

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