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A CLASS OF NONLINEAR EVOLUTION EQUATIONS IN A BANACH SPACE

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We treat the nonlinear evolution equation

$$(*) \qquad f'(t) = A(t, f(t))f(t)$$

where the unknown function f is from a real number interval into a Banach space X. For suitable real numbers t and vectors x in X, $A(t, x)$ is the infinitesimal generator of a holomorphic semigroup of linear contraction operators in X, and certain regularity requirements are placed on the function $(t, x) \to A(t, x)$.

After proving a local existence, uniqueness, and stability theorem for (*), we consider the case $A(t, x) = H(x)$ and obtain conditions under which there is a strongly continuous semigroup of nonlinear nonexpansive transformations whose infinitesimal generator is an extension of the transformation $Qx = H(x)x$.

We state our main results in §1 and prove them in §2. In §3, we prove some theorems about linear semigroups in a function space which yield examples of our main results and are of some interest in themselves.

1. **The main results.** Let X be a complex Banach space. If $0 < \phi \leqq \pi/2$, then let $S_\phi = \{z \in C : z = 0 \text{ or } |\arg z| \leqq \phi\}$, where C denotes the complex plane. Following [6], we denote by CH (ϕ) the collection of all semigroups $\{T(z) : z \in S_\phi\}$ of linear contraction operators in X which are holomorphic on int (S_ϕ) and strongly continuous on S_ϕ. We denote by GH (ϕ) the collection of all infinitesimal generators of semigroups in CH (ϕ).

Let $[a, b]$ be a closed real number interval, $0 < \phi \leqq \pi/2$, and S a closed set in X. Let A be a function from $[a, b] \times S \to$ GH (ϕ) such that the following conditions are satisfied:

(C_1) The operators $A(t, x)$ all have the same domain D_0.

(C_2) There is a locally bounded nonnegative function K on $[a, b] \times S$ such that

$$\|[I - A(t, y)][I - A(s, x)]^{-1} - I\| \leqq K(s, x)(|s-t| + \|x-y\|)$$

for $a \leqq s, t \leqq b$ and $x, y \in S$, where I denotes the identity transformation on X,

(C_3) $(\exp[\xi A(t, x)])S \subset S$ for $\xi \geqq 0$, $a \leqq t \leqq b$, and $x \in S$, where

$$\{\exp[zA(t, x)]; z \in S_\phi\}$$

is the class CH (ϕ) semigroup generated by $A(t, x)$.

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In connection with the condition (C_2), we mention that the invertibility of $I-A(s, x)$ follows from the fact that $A(s, x) \in \mathrm{GH}(\phi)$, see [6], or [2], which serves also as a general reference for semigroups of operators. The fact that $[I-A(t, y)] \cdot [I-A(s, x)]^{-1}$ is bounded follows from (C_1) and [3, Lemma 2, p. 212].

THEOREM 1. *Suppose $x_0 \in D_0 \cap S$ and $a \leqq t_0 < b$. Then there is a number c in $(t_0, b]$ such that there is a unique continuously differentiable function f from $[t_0, c]$ into $D_0 \cap S$ satisfying $f(t_0) = x_0$ and*

$$(*) \qquad f'(t) = A(t, f(t))f(t) \quad \text{for } t_0 \leqq t \leqq c.$$

Also, if $\varepsilon > 0$, then there exists $\delta > 0$ such that if $x_1 \in D_0 \cap S$, $t_0 < c_1 \leqq c$, $\|x_0 - x_1\| < \delta$, and g is a continuously differentiable function from $[t_0, c_1]$ into $D_0 \cap S$ such that $g(t_0) = x_1$ and $g'(t) = A(t, g(t))g(t)$ for $t_0 \leqq t \leqq c_1$, then $\|g(t) - f(t)\| < \varepsilon$ for $t_0 \leqq t \leqq c_1$.

DEFINITION 1.1. A *semi-inner product* on X means a function $[\cdot, \cdot]$ from $X \times X$ into C such that for each $y \in Y$, $[\cdot, y]$ is a bounded liner functional of norm $\|y\|$, and $[y, y] = \|y\|^2$ (see [5]).

DEFINITION 1.2. If $[\cdot, \cdot]$ is a semi-inner product on X, then a transformation W with domain and range contained in X is said to be *dissipative* (with respect to $[\cdot, \cdot]$) if $\mathrm{Re}\,[Wx - Wy, x - y] \leqq 0$ for $x, y \in D(W)$, the domain of W.

REMARK. Throughout this section, $[\cdot, \cdot]$ will denote a fixed semi-inner product on X, and all results will be independent of the particular semi-inner product used.

THEOREM 2. *Suppose H is a function from S into $\mathrm{GH}(\phi)$ which satisfies conditions (C_1), (C_2), and (C_3); more precisely, the function $(t, x) \to H(x)$ satisfies these conditions. Suppose $D_0 \cap S$ is dense in S and define Q on $D_0 \cap S$ by $Qx = H(x)x$. Suppose Q is dissipative. Then there is a unique strongly continuous semigroup $\{T(t); t \geqq 0\}$ of nonexpansive nonlinear transformations from S into S such that for each x in $D_0 \cap S$, $T(\cdot)x$ is a continuously differentiable function from $[0, \infty)$ into $D_0 \cap S$, and $(d/dt)T(t)x = QT(t)x$ for $t \geqq 0$.*

2. **Proof of the main theorems.** We will call a function B from a number interval $[0, R]$ into $\mathrm{GH}(\phi)$ *regular* if the following conditions are satisfied:

(R_1) $B(t)$ has domain D_0 for $0 \leqq t \leqq R$.

(R_2) There is a positive constant L such that

$$\|[I-B(t)][I-B(s)]^{-1} - I\| \leqq L|t-s|$$

for $0 \leqq s,\ t \leqq R$.

(R_3) $(\exp[\xi B(t)])S \subset S$ for $\xi \geqq 0$ and $0 \leqq t \leqq R$.

We point out that a regular operator function B on $[0, R]$ also satisfies:

(R_4) $\|[I-B(t)][I-B(s)]^{-1}\| \leqq 1 + LR$ for $0 \leqq s,\ t \leqq R$, where L is as in (R_2).

$$(R_5) \qquad \begin{aligned} &\|[I-B(r)][I-B(s)]^{-1} - [I-B(t)][I-B(s)]^{-1}\| \\ &\qquad \leqq \|[I-B(r)][I-B(t)]^{-1} - I\| \cdot \|[I-B(t)][I-B(s)]^{-1}\| \\ &\qquad \leqq |r-t|(1+LR)L. \end{aligned}$$

LEMMA 2.1. *Suppose B is a regular operator function on* $[0, R]$, *and β is a positive nonincreasing function on* $[0, R]$ *with Lipschitz constant L'. Define the operator function A on* $[0, R]$ *by* $A(t)=\beta(t)[B(t)-I]$.

Then A satisfies Tanabe's conditions 1^0 *and* 2^0 *of* [8]. *In particular, let* $0<\phi_1<\phi$ *and define*

$$\Sigma = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \leqq \phi_1+\pi/2\}.$$

Also define

$$M = [\beta(R) \sin (\phi-\phi_1)]^{-1}[(1-\sin \phi_1)/2]^{-1/2},$$
$$K = (L\beta(0)+L')(1+LR)/\beta(R),$$

where L is as in (R_2). *Then A satisfies the conditions*:

(T_1) $\rho(A(t))\supset\Sigma$ *for* $0\leqq t\leqq R$, *and*

$$\|[\lambda I-A(t)]^{-1}\| \leqq M/(|\lambda|+1)$$

for $\lambda \in \Sigma$ *and* $0\leqq t\leqq R$. (*If T is an operator in X, then $\rho(T)$ denotes the resolvent set of T.*)

(T_2) $\|A(r)A(s)^{-1}-A(t)A(s)^{-1}\|\leqq K|r-t|$ *for* $0\leqq r, s, t\leqq R$.

Proof. Let $0\leqq s, t, r\leqq R$. Define

$$\Delta_\phi = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \geqq \phi+\pi/2\}.$$

Then $\beta(t)B(t) \in \mathrm{GH}(\phi)$, and $\Sigma\subset C\backslash\Delta_\phi$, so $\rho(\beta(t)B(t))\supset\Sigma$ and $\|[\lambda I-\beta(t)B(t)]^{-1}\| \leqq 1/d(\lambda, \Delta_\phi)$ for $\lambda \in \Sigma$, see [6]. Also $\lambda I-A(t)=[\lambda+\beta(t)]I-\beta(t)B(t)$, and $\lambda+\beta(t) \in \Sigma$ if $\lambda \in \Sigma$, so $\|[\lambda I-A(t)^{-1}\|\leqq 1/d(\lambda+\beta(t), \Delta_\phi)$ for $\lambda \in \Sigma$. Property (T_1) follows from this and the fact that $d(\lambda+\beta(t), \Delta_\phi)\geqq(|\lambda|+1)/M$ for $\lambda \in \Sigma$.

Let $A_\xi=A(\xi)$, $B_\xi=B(\xi)$, and $\beta_\xi=\beta(\xi)$ for $0\leqq\xi\leqq R$. Then

$$\begin{aligned} A_rA_s^{-1}-A_tA_s^{-1} &= \beta_s^{-1}\beta_r[(I-B_r)(I-B_s)^{-1}-(I-B_t)(I-B_s)^{-1}] \\ &\quad +\beta_s^{-1}(\beta_r-\beta_t)(I-B_t)(I-B_s)^{-1}, \end{aligned}$$

so that (T_2) follows from (R_4) and (R_5).

LEMMA 2.2. *Let F be a function from* $[0, R]$ *into* $[a, b]$ *with Lipschitz constant L_1, and ψ a function from* $[0, R]$ *into S with Lipschitz constant L_2. Define the operator function B on* $[0, R]$ *by*

$$B(t) = A(F(t), \psi(t)).$$

Then B is regular, where we can take the constant L of (R_2) *as*

$$L = (L_1+L_2) \sup_{0\leqq t\leqq R} K(F(t), \psi(t)),$$

see (C_2).

Thus if β is positive, nonincreasing, and Lipschitz continuous on $[0, R]$, *and we define* $A(t)=\beta(t)[B(t)-I]$ *for* $0\leqq t\leqq R$, *then A satisfies* (T_1) *and* (T_2).

If in addition, $\beta(t)-\beta(s)\leqq -(t-s)\beta(0)L$ *for* $0\leqq s\leqq t\leqq R$, *then A also satisfies*

$$(\mathrm{T}_3) \qquad \|A(t)A(s)^{-1}\| \leqq 1 \quad \text{for } 0 \leqq s \leqq t \leqq R.$$

Proof. Only the last statement needs proof, and it follows from the fact that $\beta(t)\beta(s)^{-1}\leqq e^{[\beta(t)-\beta(s)]/\beta(s)}$, and $\|(I-B(t))(I-B(s))^{-1}\|\leqq e^{L(t-s)}$ for $0\leqq s\leqq t\leqq R$.

LEMMA 2.3. *Let the operator function A: $[0, R] \to \mathrm{GH}(\phi)$ be as in Lemma 2.1, and let $x_0 \in D_0$. Then there is a unique continuously differentiable function f from $[0, R]$ into D_0 such that $f(0)=x_0$ and $f'(t)=A(t)f(t)$ for $0 \leqq t \leqq R$.*

Proof. Tanabe establishes much more than this in [8].

LEMMA 2.4. *Let A, x_0, and f be as in Lemma 2.3. If $\Delta=\{t_0, \ldots, t_n\}$ is a partition of $[0, R]$, then let f_Δ be defined on $[0, R]$ by $f_\Delta(0)=x_0$, and*

$$f_\Delta(t) = T_k(t-t_{k-1})f_\Delta(t_{k-1}) \quad \textit{for } t_{k-1} \leqq t \leqq t_k,$$

where $T_k(\xi)=\exp[\xi A(t_k)]$. Then f_Δ converges uniformly to f on $[0, R]$ as the norm of Δ approaches zero.

Proof. Define A_Δ on $[0, R]$ by $A_\Delta(0)=A(t_1)$ and $A_\Delta(t)=A(t_k)$ for $t_{k-1}<t\leqq t_k$. Then $f'_\Delta(t)=A_\Delta(t)f_\Delta(t)$ for $t \in [0, R]\backslash\Delta$.

Let $h_\Delta(t)=f(t)-f_\Delta(t)$ for $0\leqq t\leqq R$. Then

$$\begin{aligned} h'_\Delta(t) &= [A(t)-A_\Delta(t)]f(t)+A_\Delta(t)h_\Delta(t) \\ &= [I-A_\Delta(t)A(t)^{-1}]f'(t)+A_\Delta(t)h_\Delta(t). \end{aligned}$$

By [4, Lemma 1.3, p. 510], $\|h_\Delta(t)\|(d/dt)\|h_\Delta(t)\|=\mathrm{Re}\,[h'_\Delta(t), h_\Delta(t)]$ a.e. on $[0, R]$. Thus $(d/dt)\|h_\Delta(t)\| \leqq K|\Delta|\Lambda$ a.e. on $[0, R]$, where K is as in (T_2),

$$\Lambda = \sup_{0\leqq t\leqq R} \|f'(t)\|,$$

and $|\Delta|$ denotes the norm of Δ. We have used the fact that $A_\Delta(t)$ is dissipative, see [5].

LEMMA 2.5. *Let A, x_0, and f be as in Lemma 2.3. Then*

$$\|f(t)\| \leqq \|f(0)\| \exp\left[-\int_0^t \beta\right]$$

for $0\leqq t\leqq R$.

If A satisfies (T_3), then $\|f'(t)\| \leqq \|f'(0)\| \exp[-\int_0^t \beta]$ for $0\leqq t\leqq R$.

Proof. Let Δ, f_Δ, and T_k be as in Lemma 2.4. Then $T_k(\xi)=e^{-\xi\beta(t_k)}\exp[\xi\beta(t_k)B(t_k)]$, so that $\|T_k(\xi)\| \leqq e^{-\xi\beta(t_k)}$. Therefore,

$$\|f_\Delta(t)\| \leqq \|f(0)\| \exp\left[-\beta(t_k)(t-t_{k-1})-\sum_{j=1}^{k-1} \beta(t_j)(t_j-t_{j-1})\right]$$

for $t_{k-1}\leqq t\leqq t_k$, and the first conclusion follows.

Define $X_k=T_k(t_k-t_{k-1})$, $A_k=A(t_k)$, and $\beta_k=\beta(t_k)$ for $k=0, 1, \ldots, n$.

If $t_{k-1}<t<t_k$, then

$$\begin{aligned} f'_\Delta(t) &= A_k f_\Delta(t) \\ &= A_k T_k(t-t_{k-1})X_{k-1}\cdots X_1 \\ &= T_k(t-t_{k-1})A_k A_{k-1}^{-1} A_{k-1} X_{k-1}\cdots X_1 \\ &= T_k(t-t_{k-1})A_k A_{k-1}^{-1} X_{k-1}\cdots X_1 A_1 A_0^{-1} A_0 x_0, \end{aligned}$$

and the second conclusion follows.

LEMMA 2.6. *Let A, x_0, and f be as in Lemma 2.3, but add the condition that $x_0 \in S$. Then* $(\exp [\int_0^t \beta])f(t) \in S$ *for* $0 \leqq t \leqq R$.

Proof. Let Δ and f_Δ be as in Lemma 2.4. From (R_3) and the construction of f_Δ, we get

$$\left(\exp\left[(t-t_{k-1})\beta(t_{k-1})+\sum_{j=1}^{k-1}\beta(t_j)(t_j-t_{j-1})\right]\right)f_\Delta(t) \in S$$

for $t_{k-1} \leqq t \leqq t_k$.

2.7. **Proof of Theorem 1.** Choose $\delta>0$ so that $K(t, x)$ (see condition (C_2)) is bounded for $|t-t_0| \leqq \delta$, $\|x-x_0\| \leqq \delta$. Let K_0 be an upper bound for $K(t, x)$ on this set, with $K_0>1$, $(1/\delta)$. Let

$$y_0 = A(t_0, x_0)x_0, \qquad \gamma = 2K_0(1+2\|x_0\|+\|y_0\|),$$
$$c = \min\,[b, t_0+(1/2\gamma)], \qquad R = -\gamma^{-1}\ln(1-\gamma(c-t_0)).$$

We will need the following two inequalities, which follow immediately from the above definitions:

$$R(2\|x_0\|+\|y_0\|) \leqq \delta, \tag{2.7.1}$$

$$c-t_0 \leqq \delta. \tag{2.7.2}$$

Define F from $[0, R]$ onto $[t_0, c]$ by $F(\tau)=t_0+\gamma^{-1}[1-e^{-\gamma\tau}]$. Define β on $[0, R]$ by $\beta(\tau)=e^{-\gamma\tau}$, and define G from $[t_0, c]$ onto $[0, R]$ by $G(t)=-\gamma^{-1}\ln[1-\gamma(t-t_0)]$. Then

$$F(G(t)) = t, \qquad G(F(\tau)) = \tau, \tag{2.7.3}$$

$$G'(t)\beta(G(t)) = 1, \tag{2.7.4}$$

$$\int_0^\tau \beta = F(\tau)-t_0. \tag{2.7.5}$$

Define α on $[0, R]$ by $\alpha(\tau)=\exp\left(\int_0^\tau \beta\right)$.

We intend to solve (*) by first solving

$$g'(\tau) = \beta(\tau)[A(F(\tau), \alpha(\tau)g(\tau))-I]g(\tau), \tag{**}$$

and then making the substitution $f(t)=e^{t-t_0}g(G(t))$. (2.7.3), (2.7.4), and (2.7.5) are the pertinent identities for showing that this yields a solution of (*).

We define inductively the sequence $\{g_n\}$ of functions on $[0, R]$ as follows:

$$g_0(\tau) = x_0, \qquad g_{n+1}(0) = x_0, \qquad g'_{n+1}(\tau) = A_n(\tau)g_{n+1}(\tau),$$

where

$$A_n(\tau) = \beta(\tau)[A(F(\tau), \psi_n(\tau))-I], \qquad \psi_n(\tau) = \alpha(\tau)g_n(\tau).$$

We see that this inductive definition is possible by Lemmas 2.2, 2.3, and 2.6.

We will need the fact that each of the operator functions A_n has property (T_3); in fact this is the reason for our change of variable. Define $B_n(\tau)=A(F(\tau), \psi_n(\tau))$ for $0\leqq\tau\leqq R$, and $n=0, 1, 2, 3, \ldots$. Then each B_n is regular by Lemma 2.2. For each n, let $L^{(n)}$ denote the least constant L that will work in condition (R_2) for B_n. Notice that

$$[\beta(\tau)-\beta(\sigma)]/\beta(0) \leqq -\gamma B(R)(\tau-\sigma) \leqq -(\gamma/2)(\tau-\sigma)$$

for $0\leqq\sigma\leqq\tau\leqq R$. Thus by Lemma 2.2, A_n satisfies (T_3) if $L^{(n)}\leqq(\gamma/2)$. In order to show this we will need

$$|F'(\tau)| = |e^{-\gamma\tau}| \leqq 1, \tag{2.7.6}$$

$$|\alpha'(\tau)| = \left|\exp\left[-\gamma\tau+\int_0^\tau \beta\right]\right| \leqq 1, \tag{2.7.7}$$

$$|F(\tau)-t_0| \leqq \delta. \tag{2.7.8}$$

Thus, we have $L_0\leqq K_0(1+\|x_0\|)<\gamma/2$ since $\|\psi_0(\tau)-x_0\|<R\|x_0\|\leqq\delta$, $\|\psi_0'(\tau)\| \leqq \|x_0\|$, so that A_0 has property (T_3).

Suppose A_n has property (T_3). Then

$$\psi_{n+1}'(\tau) = \alpha(\tau)g_n'(\tau)+\alpha'(\tau)g_n(\tau),$$
$$\|\psi_{n+1}'(\tau)\| \leqq \|g_n'(0)\|+\|x_0\| \leqq 2\|x_0\|+\|y_0\|$$

by Lemma 2.5, and (2.7.7). Therefore,

$$\|\psi_{n+1}(\tau)-x_0\| \leqq \delta \tag{2.7.9}$$

by (2.7.1). Thus $L^{(n+1)}\leqq K_0(1+2\|x_0\|+\|y_0\|)=\gamma/2$ by (2.7.6), (2.7.8), and Lemma 2.2. Thus A_{n+1} also has property (T_3).

Thus, we have

$$\|g_n'(\tau)\| \leqq (\|y_0\|+\|x_0\|)/\alpha(\tau) \tag{2.7.10}$$

for $0<\tau\leqq R$, and $n=0, 1, 2, \ldots$ by Lemma 2.5.

For each $n=1, 2, 3, \ldots$, define h_n on $[0, R]$ by $h_n(\tau)=g_{n+1}(\tau)-g_n(\tau)$. Then

$$\begin{aligned} h_n'(\tau) &= A_n(\tau)g_{n+1}(\tau)-A_{n-1}(\tau)g_n(\tau) \\ &= [A_n(\tau)-A_{n-1}(\tau)]g_{n+1}(\tau)+A_{n-1}(\tau)h_n(\tau) \\ &= [I-A_{n-1}(\tau)A_n(\tau)^{-1}]g_{n+1}'(\tau)+A_{n-1}(\tau)h_n(\tau). \end{aligned}$$

By [4, Lemma 1.3, p. 510], we have

$$\|h_n(\tau)\|(d/d\tau)\|h_n(\tau)\| = \operatorname{Re}\,[h_n'(\tau), h_n(\tau)]$$

a.e. on $[0, R]$, so that $(d/d\tau)\|h_n(\tau)\|\leqq K_0(\|x_0\|+\|y_0\|)\|h_{n-1}(\tau)\|$ a.e. on $[0, R]$. We have used the fact that $A_{n-1}(\tau)$ is dissipative (see [5]), property (C_2), (2.7.8), (2.7.9), and (2.7.10).

Therefore, $\{g_n\}$ converges uniformly to a function g on $[0, R]$. Also

$$\|g(\tau)-g(\sigma)\| \leqq (\|x_0\|+\|y_0\|)|\tau-\sigma|$$

for $0 \leqq \sigma$, $\tau \leqq R$. Let $\psi(\tau) = \alpha(\tau)g(\tau)$, $0 \leqq \tau \leqq R$. Then

$$\|\psi(\tau) - \psi(\sigma)\| \leqq (2\|x_0\| + \|y_0\|)|\tau - \sigma|,$$

and $\|\psi(\tau) - x_0\| \leqq \delta$ for $0 \leqq \tau \leqq R$. Define the operator function A on $[0, R]$ by $A(\tau) = \beta(\tau)[A(F(\tau), \psi(\tau)) - I]$. Then A has properties (T_1), (T_2), and (T_3) (we will not need (T_3)). Define u on $[0, R]$ by $u(0) = x_0$, $u'(\tau) = A(\tau)u(\tau)$.

We wish to show that $u = g$. Let $u_n = u - g_n$. An argument similar to the one used to show that $\{h_n\}$ converges to 0 will show that $\{u_n\}$ converges to 0. Therefore g satisfies (**), and the function f defined on $[t_0, c]$ by $f(t) = e^{t-t_0}g(G(t))$ satisfies (*). Note also that $f(t) = \psi(G(t))$, $\|f(t) - x_0\| \leqq \delta$ for $y_0 \leqq t \leqq c$.

Suppose $x_1 \in D_0 \cap S$, $t_0 < c_1 \leqq c$, and v is a continuously differentiable function from $[t_0, c_1]$ into $D_0 \cap S$ such that $v(t_0) = x_1$ and $v'(t) = A(t, v(t))v(t)$ for $t_0 \leqq t \leqq c_1$. Define w on $[t_0, c_1]$ by $w(t) = f(t) - v(t)$. Then

$$\begin{aligned} w'(t) &= [A(t, f(t)) - A(t, v(t))]f(t) + A(t, v(t))w(t) \\ &= ([I - A(t, v(t))][I - A(t, f(t))]^{-1} - I)(f(t) - f'(t)) + A(t, v(t))w(t), \end{aligned}$$

$$(d/dt)\|w(t)\| \leqq K_0\|w(t)\|(\|f(t)\| + \|f'(t)\|)$$

a.e. on $[t_0, c_1]$. The stability claim, and hence the uniqueness claim, follow from this differential inequality.

2.8. **Proof of Theorem 2.** First we mention that if we prove that for each $x \in D_0 \cap S$, there exists a continuously differentiable function f from $[0, \infty)$ into $D_0 \cap S$ such that $f(0) = x$ and $f'(t) = Qf(t)$ for $t \geqq 0$, then the rest of the theorem follows in routine manner. We define $T_0(t)x = f(t)$ for $x \in D_0 \cap S$ and $t \geqq 0$. The fact that T_0 is nonexpansive on $D_0 \cap S$ follows from the fact that Q is dissipative. Thus each $T_0(t)$ has a unique extension to a nonexpansive transformation $T(t)$ from S into S. $\{T(t); t \geqq 0\}$ is the desired semigroup.

Now we return to the first question. Let $x_0 \in D_0 \cap S$. Then by Theorem 2, there is a number $c > 0$ such that there is a unique continuously differentiable function f from $[0, c]$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leqq t \leqq c$. Let ζ denote the supremum of the set of all such numbers c, and suppose that $\zeta < \infty$. Let f denote the unique continuously differentiable function from $[0, \zeta)$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leqq t < \zeta$.

If $0 < h < \zeta$, then define f_h on $[0, \zeta - h]$ by $f_h(t) = f(t+h) - f(t)$. Then $f_h'(t) = Qf(t+h) - Qf(t)$, and $(d/dt)\|f_h(t)\| \leqq 0$ a.e. on $[0, \zeta - h)$ since Q is dissipative, so that $\|(1/h)[f(t+h) - f(t)]\| \leqq \|(1/h)(f(h) - f(0))\|$ for $0 \leqq t < \zeta - h$. Therefore $\|f'(t)\| \leqq \|f'(0)\|$ for $0 \leqq t \leqq \zeta$, and thus $x_1 = \lim_{t \to \zeta} f(t)$ exists. Therefore $f([0, \zeta))$ is relatively compact and $K(t, f(t))$ (see (C_2)) is bounded on $[0, \zeta)$. Using this and the fact that $f(t)$ and $f'(t)$ are also bounded on $[0, \zeta)$, we see by examining the argument for Theorem 1 that there is a positive constant η such that for each t in $[0, \zeta)$, there is a unique continuously differentiable function g from $[t, t+\eta]$ into $D_0 \cap S$ such that $g(t) = f(t)$ and $g'(s) = Qg(s)$ for $t \leqq s \leqq t + \eta$. Simply take $\zeta - \eta < t < \zeta$, and use the corresponding function g to extend f beyond ζ.

3. **Semigroups in a function space.** Let E be a set, $B(E)$ the Banach space of bounded complex valued functions on E with supremum norm, and Y a closed real or complex subspace of $B(E)$. We denote by Ω the collection of all positive bounded functions p on E which are bounded away from zero and have the property that $pY \subset Y$.

If Y is complex, we take CH (ϕ) and GH (ϕ) as defined in §1, with $X = Y$. If Y is a real Banach lattice, then CP denotes the collection of all strongly continuous semigroups of linear positive contraction operators in Y, and GP denotes the collection of infinitesimal generators of such semigroups. In either case, G denotes the collection of all infinitesimal generators of strongly continuous semigroups of linear contraction operators in Y.

If $y \in Y$, then $\gamma(y)$ denotes a multiplicative linear functional on $B(E)$ such that $|\langle y, \gamma(y)\rangle| = \|y\|$. We define the semi-inner product $[\cdot, \cdot]$ on $Y \times Y$ by $[x, y] = \langle x, \gamma(y)\rangle \langle y, \gamma(y)\rangle^*$, where * denotes complex conjugation. All reference to a semi-inner product in this section will be to this one just defined. One special property of $[\cdot, \cdot]$ which is useful to us is that $[px, y] = \langle p, \gamma(y)\rangle [x, y]$ for $x, y \in Y$, $p \in B(E)$, $px \in Y$. Also, if Y is a real Banach lattice, then $[\cdot, \cdot]$ has the special properties required in [7]. That is, $[\cdot, y]$ is a positive linear functional if $y \geqq 0$, and $[x, x^+] = \|x^+\|^2$ for each x in Y, where x^+ denotes the positive part of x.

By Definition 1.2, a linear operator A in Y is dissipative if Re $[Ay, y] \leqq 0$ for $y \in D(A)$. Following [6], in case Y is complex, we say that a linear operator A in Y is *ϕ-sectorial* if $e^{i\theta}A$ is dissipative for $|\theta| \leqq \phi$. Following [7], in case Y is a real lattice, we say that a linear operator A in Y is *dispersive* if $[Ax, x^+] \leqq 0$ for all $x \in D(A)$.

LEMMA 3.1. *A linear operator A in Y is in* (G, GP, GH (ϕ)) *if and only if $D(A)$ is dense in Y, the range of $I - A$ is all of Y, and A is* (*dissipative, dispersive, ϕ-sectorial*).

Proof. The proof of this lemma is contained in [5], [7], and [6], respectively. We merely state the lemma here for reference in proving the next theorem, which is a generalization of the author's earlier theorem in [1].

THEOREM 3.1. *Suppose $A \in G$, and $A = A_1 + \cdots + A_n$, where each A_j has domain $D(A)$, and each A_j has a closed extension. If each A_j is* (*dissipative, dispersive, ϕ-sectorial*), *and $p_1, \ldots, p_n \in \Omega$, then $p_1A_1 + \cdots + p_nA_n \in$* (G, GP, GH ($\phi$)).

Proof. $p_1A_1 + \cdots + p_nA_n$ is easily seen to be (dissipative, dispersive, ϕ-sectorial). Thus by Lemma 3.1, we need only show that the range of $I - (p_1A_1 + \cdots + p_nA_n)$ is all of Y.

We will first prove that the range of $I - (p_1A_1 + A_2 + \cdots + A_n)$ is all of Y. By [3, Lemma 2, p. 212], the operator $U_1 = A_1(I - A)^{-1}$ is bounded. Since $F(p_1)Y \subset Y$ for every polynomial F, then $p_1^{(1/m)} \in \Omega$ for every positive integer m by the classical Weierstrass theorem. Choose m so that $\|1 - p_1^{(1/m)}\| < \|U_1\|^{-1}$, and let $r = p_1^{1/m}$. Then

$$I - (rA_1 + A_2 + \cdots + A_n) = I - A + (1 - r)A_1 = (I + (1 - r)U_1)(I - A).$$

Thus the range of $I-(rA_1+A_2+\cdots+A_n)$ is all of Y. Replacing A_1 by rA_1, r^2A_1, etc., we see that the range of $I-(p_1A_1+A_2+\cdots+A_n)$ is all of Y.

Now we consider the operator $A'=A_2+p_1A_1+A_3+\cdots+A_n$ and repeat the previous argument to prove that the range of $I-(p_1A_1+p_2A_2+A_3+\cdots+A_n)$ is all of Y. Repeating this process proves the theorem.

EXAMPLE. Let E denote real Euclidean n-space, and let Y denote any of the subspaces of $B(E)$ in which the Laplacian operator generates a strongly continuous semigroup. The semigroup will then consist of contraction operators and will be in CH (ϕ) if Y is complex, in CP if Y is a real lattice. Let A denote the Laplacian operator in Y, and for each $j=1,\ldots,n$, let A_j denote the restriction of $(\partial^2/\partial s_j^2)$ to the domain of A.

LEMMA 3.2. *Let A be in G with $A=A_1+\cdots+A_n$, where each A_j has domain $D(A)$, each A_j has a closed extension, and each A_j is dissipative. Define the function P from $\Omega^{(n)}$ into G by $P(p)=p_1A_1+\cdots+p_nA_n$.*

Then there is a locally bounded nonnegative function K on $\Omega^{(n)}$ such that

$$\|[I-P(q)][I-P(p)]^{-1}-I\| \leqq \left(\sum \|q_i-p_i\|\right)K(p)$$

for $p, q \in \Omega^{(n)}$.

Proof. If $p, q \in \Omega^{(n)}$, then

$$[I-P(q)][1-P(p)]^{-1}-I = [P(p)-P(q)][I-P(p)]^{-1} = \sum (p_i-q_i)A_i[I-P(p)]^{-1}.$$

There we can take $K(p)=\max_i \|A_i[I-P(p)]\|^{-1}$.

To see that $K(p)$ is locally bounded, notice that

$$A_i(I-P(r))^{-1} = A_i(I-P(p))^{-1}\left(I+\sum (p_i-r_i)A_i(I-P(p))^{-1}\right)^{-1},$$

so that $K(r)\leqq K(p)/(1-K(p)\sum \|p_i-r_i\|)$ for

$$K(p)\sum \|p_i-r_i\| < 1.$$

THEOREM 3.3. *Let B be in* GH (ϕ) *with $B=B_1+\cdots+B_n$, where each B_j has a closed extension, each B_j has domain $D(A)=D_0$, and each B_j is ϕ-sectorial. Let S be a closed set in Y, $[a, b]$ a closed interval, and p a Lipschitz continuous function from $[a, b]\times S$ into $\Omega^{(n)}$. Define the operator function A from $[a, b]\times S$ into* GH (ϕ) *by $A(t, x)=\sum p_i(t, x)A_i$. Then A satisfies conditions* (C_1) *and* (C_2).

Proof. This follows from Lemma 3.2.

There are a variety of ways in which the set S in Theorem 3.2 could be chosen in order that the operator function A will satisfy (C_3), and it seems inappropriate to state any theorems about this. It is not quite so easy to choose H and S so that Q will be dissipative as in Theorem 2, but we will indicate one way in which it can be done.

Let Y be complex, let Y_0 denote the space of real functions in Y, and suppose Y_0 is a lattice. Let $A \in$ GH (ϕ), $A=A_1+\cdots+A_n$ as in Theorem 3.2. Let $D_{00}=D_0 \cap Y_0$,

$A^0 = A|_{D_{00}}$, $A_j^0 = A_j|_{D_{00}}$, and suppose that A^0, A_j^0 satisfy the portion of Theorem 3.1 dealing with positive semigroups. Let Y_{00} denote the nonpositive functions in Y_0, let $S_0 = \bigcap (A_j^0)^{-1} Y_{00}$, and let S denote the closure of S_0. Let $p_1, \ldots, p_n$ be Lipschitz continuous accretive ($-p_j$ dissipative) functions from S into Ω. Define H from S onto GH (ϕ) by $H(x) = \sum p_i(x) A_i$. Then the hypothesis of Theorem 2 is satisfied. This can all be done taking Y, A, A_j as in the example after Theorem 3.1 which dealt with the Laplacian operator.

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