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DELETED PRODUCTS WITH HOMOTOPY TYPES OF SPHERES(1)

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1. **Introduction and notation.** The deleted product space X^* of a space X is $X\times X-\Delta$. The principal purpose of this paper is to describe, for each integer $n\geqq 3$, an infinite collection $\mathscr{B}_n$ of finite, contractible, n-dimensional polyhedra whose deleted products have the homotopy type of the n-sphere. It follows from previous work of the author [3] that the triod is the only tree whose deleted product has the homotopy type of the circle. In [5], the author computed the homology groups of the deleted product of a polyhedron in a subcollection $\mathfrak{B}$ of the finite, contractible, 2-dimensional polyhedra, and, in [6], the author described a subcollection $\mathfrak{C}$ of $\mathfrak{B}$ such that the deleted product of each member of $\mathfrak{C}$ has the homotopy type of the 2-sphere. Now $\mathfrak{C}$ is an infinite collection, but there are two members, C and D, of $\mathfrak{C}$ which have the property that any other member of $\mathfrak{C}$ can be constructed by starting with C or D and appending simplexes in a certain specified manner. In [7], the author described, for each $n\geqq 3$, $n+1$ finite, contractible, n-dimensional polyhedra whose deleted products have the homotopy type of the n-sphere, and n of these polyhedra, $C_1^n, C_2^n,\dots, C_n^n$, have the property that any member of $\mathscr{B}_n$ can be constructed by starting with some C_i^n and appending simplexes in a certain specified manner. The importance of spaces whose deleted products have the homotopy type of a sphere was illustrated in [6].

If X and Y are spaces and $f\colon X\to Y$ is a continuous function, then X_f^* is the inverse image of Y^* in the map $f\times f\colon X\times X\to Y\times Y$. In [1], Brahana asks the question: *What maps f are such that there is a homotopy equivalence between X_f^* and X^**? In §2, we give some partial answers to this question, and we use some of these results in §3.

In §3, we examine the effect on X^* of adding a simplex to X in a certain specified manner. These results are used in §4 to describe the members of $\mathscr{B}_n$. §3 is also a first step in determining which homology groups of deleted product spaces are trivial.

In a forthcoming paper, we continue the investigations begun in this paper.

If v is a vertex of a polyhedron A, we let St (v, A) denote the open star of v in A, and if $v_1, v_2,\dots, v_n$ are the vertices of a simplex σ, we denote σ by $\langle v_1, v_2,\dots, v_n\rangle$. We use the circumflex $\hat{v}_i$ to denote that v_i has been omitted, and if $w_1, w_2,\dots, w_n$

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are points, j is an integer $(1 \leqq j \leqq n)$, and $k=1, 2, \ldots, j$; we let $w_1, w_2, \ldots, \{\hat{w}_{i_k}\}, \ldots, w_n$ denote the subset of $w_1, w_2, \ldots, w_n$ obtained by omitting $w_{i_1}, w_{i_2}, \ldots, w_{i_j}$. Thus $\bigcup_{k=1}^{j} (\langle w_1, w_2, \ldots, \{\hat{w}_{i_k}\}, \ldots, w_n\rangle)$ denotes the simplex whose vertices are $w_1, w_2, \ldots, w_n$ with $w_{i_1}, w_{i_2}, \ldots, w_{i_j}$ omitted. We use the group of integers as the coefficient group for the homology groups. If X is a finite polyhedron and A and B are sub-polyhedra of X, let $P(A \times B - \Delta) = \bigcup \{\sigma \times \tau \mid \sigma$ is a simplex of A, τ is a simplex of B, and $\sigma \cap \tau = \varnothing\}$. Hu [2] has shown that X^* and $P(X^*)$ are homotopically equivalent. If X and Y are finite polyhedra and $f\colon X \to Y$ is a simplicial map, let $P(X_f^*) = \bigcup \{\sigma \times \tau \mid \sigma$ and τ are simplexes of X and $f(\sigma) \cap f(\tau) = \varnothing\}$. The author [4] has observed that X_f^* and $P(X_f^*)$ are homotopically equivalent.

2. The space X_f^*.

THEOREM 1. *Let A be a finite, n-dimensional polyhedron, and let B be an m-simplex. Suppose $A \cap B = C$, where C is a simplex of A and a proper face of B. Let v be a vertex of C, and let $v_1, v_2, \ldots, v_p$ denote the vertices of B which are not vertices of A. If $X = A \cup B$ and $f\colon X \to A$ is the simplicial map defined by $f(w) = w$ for each vertex w of A and $f(v_j) = v$ for each $j = 1, 2, \ldots, p$, then $P(X_f^*)$ is homotopically equivalent to $P(A^*)$.*

Proof. We will show that $\eta_f = f \times f \mid P(X_f^*)\colon P(X_f^*) \to P(A^*)$ is a homotopy equivalence. Let $i\colon A \to X$ be the inclusion map, and let $\eta_i = i \times i \mid P(A^*)$. Then $\eta_i\colon P(A^*) \to P(X_f^*)$, and $\eta_f \eta_i$ is the identity.

Let $(x_1, x_2) \in P(X_f^*)$, and let σ_1 and σ_2 be the smallest closed simplexes of X such that $x_i \in \sigma_i$. Since $(x_1, x_2) \in P(X_f^*), f(\sigma_1) \cap f(\sigma_2) = \varnothing$. If, for each $j = 1, 2, \ldots, p$, v_j is not a vertex of either σ_1 or σ_2, then $\eta_i \eta_f(x_1, x_2) = (if(x_1), if(x_2)) = (x_1, x_2)$. Suppose that, for some $j = 1, 2, \ldots, p$, v_j is a vertex of σ_1, and let $v_{k_1}, v_{k_2}, \ldots, v_{k_q}$ denote the subcollection of $v_1, v_2, \ldots, v_p$ consisting of those vertices which are vertices of σ_1. Then, for each $j = 1, 2, \ldots, p$, v_j is not a vertex of σ_2, and hence $\eta_i \eta_f(x_1, x_2) = (if(x_1), x_2)$. Now $if(x_i)$ is either in the face of σ_1 obtained by omitting $v_{k_1}, v_{k_2}, \ldots, v_{k_q}$ or the face of B we get from σ_1 by replacing $v_{k_1}, v_{k_2}, \ldots, v_{k_q}$ by v. Since, in this case, v is not a vertex of σ_2, the line segment joining (x_1, x_2) and $\eta_i \eta_f(x_1, x_2)$ is contained in $P(X_f^*)$. If, for some $j = 1, 2, \ldots, p$, v_j is a vertex of σ_2, then a similar argument shows that the line segment joining (x_1, x_2) and $\eta_i \eta_f(x_1, x_2)$ is contained in $P(X_f^*)$. Therefore we may define $F\colon P(X_f^*) \times I \to P(X_f^*)$ by

$$F(x_1, x_2, t) = t(x_1, x_2) + (1-t)(if(x_1), if(x_2)).$$

It is clear that F is a homotopy between $\eta_i \eta_f$ and the identity.

The proof of the following combinatorial lemma is straightforward and hence it is omitted.

LEMMA 1. *Let A be a finite, n-dimensional polyhedron, and let B be an m-simplex with vertices $v_0, v_1, \ldots, v_m$. Suppose $1 \leqq p \leqq m$ and $A \cap B = \bigcup_{i=1}^{p} r_i$, where, for each i, r_i is the $(m-1)$-simplex $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle$ of A and B. Then there is exactly* *one $(p-1)$-face $\langle v_1, v_2, \ldots, v_p\rangle$ of B which is not in A, and if $q<p-1$, then every q-face of B is also a simplex of A.*

THEOREM 2. *Let A be a finite, n-dimensional polyhedron, and let B be an m-simplex with vertices $v_0, v_1, \ldots, v_m$. Suppose $1 \leqq p \leqq m$ and $A \cap B = \bigcup_{i=1}^{p} r_i$, where, for each i, r_i is the $(m-1)$-simplex $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle$ of A and B. Let u be the barycenter of $\langle v_1, v_2, \ldots, v_p\rangle$, and let X be the polyhedron consisting of A and*

$$\{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, u\rangle \mid i = 1, 2, \ldots, p\}.$$

If $f\colon X \to A$ is the simplicial map defined by $f(w)=w$ for each vertex w of A and $f(u)=v_0$, then $P(X_f^*)$ is homotopically equivalent to $P(A^*)$.

Proof. We will show that $\eta_f = f\times f \mid P(X_f^*)\colon P(X_f^*) \to P(A^*)$ is a homotopy equivalence. Let $i\colon A \to X$ be the inclusion map, and let $\eta_i = i \times i \mid P(A^*)$. Then $\eta_i\colon P(A^*) \to P(X_f^*)$, and $\eta_f\eta_i$ is the identity.

Let $(x_1, x_2) \in P(X_f^*)$, and let σ_1 and σ_2 be the smallest closed simplexes of X such that $x_i \in \sigma_i$. Since $(x_1, x_2) \in P(X_f^*)$, $f(\sigma_1) \cap f(\sigma_2) = \varnothing$. If u is not a vertex of either σ_1 or σ_2, then $\eta_i\eta_f(x_1, x_2) = (if(x_1), if(x_2)) = (x_1, x_2)$. Suppose u is a vertex of σ_1. Then neither u nor v_0 is a vertex of σ_2, and $\eta_i\eta_f(x_1, x_2) = (if(x_1), x_2)$. Now $if(x_1)$ is either in the face of σ_1 obtained by omitting u or in the simplex obtained from σ_1 by replacing u by v_0. Since v_0 is not a vertex of σ_2, the line segment joining (x_1, x_2) and $\eta_i\eta_f(x_1, x_2)$ is contained in $P(X_f^*)$. If u is a vertex of σ_2, a similar argument shows that the line segment joining (x_1, x_2) and $\eta_i\eta_f(x_1, x_2)$ is contained in $P(X_f^*)$. Therefore we may define $F\colon P(X_f^*) \times I \to P(X_f^*)$ by

$$F(x_1, x_2, t) = t(x_1, x_2) + (1-t)(if(x_1), if(x_2)).$$

It is clear that F is a homotopy between $\eta_i\eta_f$ and the identity.

THEOREM 3. *Let A be a finite, n-dimensional polyhedron, and let B be an m-simplex with vertices $v_0, v_1, \ldots, v_m$. Suppose*

$$A \cap B = \{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle \mid i = 0, 1, \ldots, m\},$$

suppose there is a vertex v of A such that, for each i, $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, v\rangle$ is a simplex of A, and suppose there is a vertex w of A such that $w \neq v$ and $w \neq v_i$ for any i. Let u be the barycenter of B and let X be the polyhedron consisting of A and

$$\{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, u\rangle \mid i = 0, 1, \ldots, m\}.$$

If $f\colon X \to A$ is the simplicial map defined by $f(z)=z$ for each vertex z of A and $f(u)=v$, then $P(X_f^*)$ is not homotopically equivalent to $P(A^*)$.

Proof. Since $w \neq v$ and $w \neq v_i$ for any i,

$$w \times [\{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, v\rangle \mid i = 0, 1, \ldots, m\} \\ \cup \{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, u\rangle \mid i = 0, 1, \ldots, m\}]$$

is contained in $P(X_f^*)$. It is clear that this set carries nontrivial m-dimensional homology classes which are not in $P(A^*)$.

3. **The addition of simplexes.** Throughout this section, we let A denote a finite, n-dimensional polyhedron, and we are concerned with methods of attaching a simplex B to A so that either $(A \cup B)^*$ is homotopically equivalent to A^* or $H_k((A \cup B)^*)$ is isomorphic to $H_k(A^*)$ for certain k.

THEOREM 4. *If $B=\langle v_0, v_1\rangle$ is a 1-simplex such that $A \cap B=\langle v_0\rangle$, where v_0 is a vertex of A, and $X=A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for all $k>n$. If, in addition, $\partial(\operatorname{St}(v_0, A))$ is contractible, then $P(X^*)$ is homotopically equivalent to $P(A^*)$.*

Proof. Let $f\colon X \to A$ be the simplicial map defined by $f(w)=w$ for each vertex w of A and $f(v_1)=v_0$. Then, by Theorem 2, $P(X_f^*)$ is homotopically equivalent to $P(A^*)$. Now

$$P(X^*) = P(X_f^*) \cup (\operatorname{Cl}(\operatorname{St}(v_0, A)) \times \langle v_1\rangle) \cup (\langle v_1\rangle \times \operatorname{Cl}(\operatorname{St}(v_0, A))).$$

Since $P(X_f^*) \cap (\operatorname{Cl}(\operatorname{St}(v_0, A)) \times \langle v_1\rangle)=\partial(\operatorname{St}(v_0, A)) \times \langle v_1\rangle$ and $\dim[\partial(\operatorname{St}(v_0, A))] \leqq n-1$, $H_k(P(X_f^*) \cup (\operatorname{Cl}(\operatorname{St}(v_0, A)) \times \langle v_1\rangle))$ is isomorphic to $H_k(P(X_f^*))$ for all $k>n$. If $\partial(\operatorname{St}(v_0, A))$ is contractible, then $P(X_f^*) \cup (\operatorname{Cl}(\operatorname{St}(v_0, A)) \times \langle v_1\rangle)$ has the homotopy type of $P(X_f^*)$. Now the desired result follows immediately since

$$[P(X_f^*) \cup (\operatorname{Cl}(\operatorname{St}(v_0, A)) \times \langle v_1\rangle)] \cap (\langle v_1\rangle \times \operatorname{Cl}(\operatorname{St}(v_0, A))) = \langle v_1\rangle \times \partial(\operatorname{St}(v_0, A)).$$

THEOREM 5. *If $B=\langle v_0, v_1, v_2\rangle$ is a 2-simplex such that $A \cap B=\langle v_0, v_2\rangle \cup \langle v_0, v_1\rangle$, where $\langle v_0, v_2\rangle$ and $\langle v_0, v_1\rangle$ are simplexes of A, u is the barycenter of $\langle v_1, v_2\rangle$, and X is the polyhedron consisting of A, $\langle v_0, v_2, u\rangle$, and $\langle v_0, v_1, u\rangle$; then $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r>n+1$. Furthermore if*

$$\partial(\operatorname{St}(v_0, A)) \cup \bigcup_{\gamma=1}^{2} [\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_\gamma, A)]$$

and either $\partial(\operatorname{St}(v_0, A))-\operatorname{St}(v_1, A)$ or $\partial(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)$ or $\partial(\operatorname{St}(v_0, A)) -\bigcup_{\gamma=1}^{2} \operatorname{St}(v_\gamma, A)$ are contractible, then X^* is homotopically equivalent to A^*.

Proof. Let $f\colon X \to A$ be the simplicial map defined by $f(w)=w$ for each vertex w of A and $f(u)=v_0$. Then, by Theorem 2, $P(X_f^*)$ is homotopically equivalent to $P(A^*)$. Now

$$\begin{aligned} P(X^*) = P(X_f^*) &\cup (\langle v_2, u\rangle \times [\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)]) \\ &\cup (\langle v_1, u\rangle \times [\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_1, A)]) \cup (\langle u\rangle \times \operatorname{Cl}(\operatorname{St}(v_0, A))) \\ &\cup ([\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)] \times \langle v_2, u\rangle) \\ &\cup ([\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_1, A)] \times \langle v_1, u\rangle) \cup (\operatorname{St}(v_0, A) \times \langle u\rangle). \end{aligned}$$

Since

$$\begin{aligned} &P(X_f^*) \cap (\langle v_2, u\rangle \times [\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)]) \\ &\quad = (\langle v_2, u\rangle \times [\partial(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)]) \cup (\langle v_2\rangle \times [\operatorname{Cl}(\operatorname{St}(v_0, A))-\operatorname{St}(v_2, A)]), \end{aligned}$$

$X_1 = P(X_f^*) \cup (\langle v_2, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_2, A)])$ has the homotopy type of $P(X_f^*)$. Now

$$\begin{aligned} X_1 &\cap (\langle v_1, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)]) \\ &= (\langle v_1, u\rangle \times [\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)]) \cup (\langle v_1\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)]) \\ &\cup (\langle u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - (\mathrm{St}\,(v_1, A) \cup \mathrm{St}\,(v_2, A))]). \end{aligned}$$

Since $\dim\,[\partial(\mathrm{St}\,(v_0, A))] \leqq n-1$,

$$H_r(X_1 \cap (\langle v_1, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)])) = 0$$

for all $r > n$. Therefore, if

$$X_2 = X_1 \cup (\langle v_1, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)]),$$

then $H_r(X_2)$ is isomorphic to $H_r(X_1)$ for all $r > n+1$. If either $\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)$ or $\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{\gamma=1}^{2} \mathrm{St}\,(v_\gamma, A)$ is contractible, then

$$X_1 \cap (\langle v_1, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)])$$

is contractible, and hence X_2 has the homotopy type of X_1. If neither $\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)$ nor $\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{\gamma=1}^{2} \mathrm{St}\,(v_\gamma, A)$ is contractible, but $\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_2, A)$ is contractible, then we add $(\langle v_1, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)])$ to $P(X_f^*)$ before we add $(\langle v_2, u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_2, A)])$ and essentially repeat the above argument to show that X_2 has the homotopy type of $P(X_f^*)$. Now

$$\begin{aligned} X_2 \cap (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))) &= (\langle u\rangle \times \partial(\mathrm{St}\,(v_0, A))) \\ &\cup (\langle u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_2, A)]) \\ &\cup (\langle u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)]). \end{aligned}$$

Since $H_r(X_2 \cap (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A)))) = 0$ for all $r > n-1$, if

$$X_3 = X_2 \cup (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))),$$

then $H_r(X_3)$ is isomorphic to $H_r(X_2)$ for all $r > n$. Hence $H_r(X_3)$ is isomorphic to $H_r(A^*)$ for all $r > n+1$. If

$$\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{2} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$$

is contractible, then X_3 has the homotopy type of X_2. Therefore if $\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{2} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$ and either $\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_1, A)$ or

$$\partial(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_2, A)$$

or $\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{\gamma=1}^{2} \mathrm{St}\,(v_\gamma, A)$ are contractible, then X_3 is homotopically equivalent to A^*. We essentially repeat the above argument in order to complete the proof of the theorem.

Now in the remainder of this section, we let B denote an m-simplex with vertices $v_0, v_1, \ldots, v_m$. In Theorems 6 and 7, we assume that $2 \leqq m \leqq n$, and, in Theorems 8, 9, and 10, we assume that $3 \leqq m \leqq n$.

THEOREM 6. *If* $1 \leqq p < m$, $A \cap B = \{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle \mid i = 1, 2, \ldots, p\}$, *where, for each* i, $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle$ *is a simplex of* A, u *is the barycenter of*

$$\langle v_1, v_2, \ldots, v_p\rangle,$$

X *is the polyhedron consisting of* A *and* $\{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, u\rangle \mid i = 1, 2, \ldots, p\}$, St (v_1, A) *denotes the empty set when* $v_1 \notin A$, *and*

$$\Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A)\Big]$$
$$\cup \bigcup_{\gamma=p+1}^{m} \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \Big(\bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A) \cup \mathrm{St}\,(v_\gamma, A)\Big)\Big]$$

and $\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$ *are contractible; then* X^* *is homotopically equivalent to* A^*.

Proof. Let $f\colon X \to A$ be the simplicial map defined by $f(w) = w$ for each vertex w of A and $f(u) = v_0$. Then, by Theorem 2, $P(X_f^*)$ is homotopically equivalent to $P(A^*)$. Now $P(X^*)$ can be constructed by starting with $P(X_f^*)$ and adding cells. Below we express $P(X^*)$ as the union of $P(X_f^*)$ and these cells. After this expression we explain the order in which we are going to add cells to $P(X_f^*)$ in order to get $P(X^*)$. Now

$$\begin{aligned}
P(X^*) = P(X_f^*) &\cup \bigcup_{j=1}^{m} \bigcup_{i_1=1}^{\min[p, m-j+1]} \bigcup_{k=2}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} \\
&\qquad \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \\
&\qquad\qquad \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big) \\
&\cup \bigcup_{j=1}^{m} \bigcup_{i_1=1}^{\min[p, m-j+1]} \bigcup_{k=2}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} \\
&\qquad \Big(\Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big] \\
&\qquad\qquad \times \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle\Big).
\end{aligned}$$

In the above and throughout this proof, we assume that St (v_1, A) denotes the empty set when $v_1 \notin A$. Note that this is the case if and only if $p = 1$. In order to explain the order in which we add the cells in the first union, we introduce the following notation. With each cell

$$\bigcup_{k=1}^{j} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle$$
$$\times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big),$$

associate an m-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ as follows: $\alpha_\rho = 1$ if v_ρ is omitted in the simplex $\bigcup_{k=1}^{j} \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle$ and $\alpha_\rho = 0$ otherwise. If $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $(\beta_1, \beta_2, \ldots, \beta_m)$ are distinct m-tuples obtained in this manner, we define

$$(\alpha_1, \alpha_2, \ldots, \alpha_m) < (\beta_1, \beta_2, \ldots, \beta_m)$$

if and only if either

(1) $\sum_{\rho=1}^{m} \alpha_\rho < \sum_{\rho=1}^{m} \beta_\rho$ or

(2) $\sum_{\rho=1}^{m} \alpha_\rho = \sum_{\rho=1}^{m} \beta_\rho$ and, if $r = \min\{s \mid \alpha_s \neq \beta_s\}$, then $\alpha_r > \beta_r$.

Then, if $(\alpha_1, \alpha_2, \ldots, \alpha_m) < (\beta_1, \beta_2, \ldots, \beta_m)$, we add the cell associated with $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ before we add the cell associated with $(\beta_1, \beta_2, \ldots, \beta_m)$.

Now, if $\sigma_1 \times \tau_1$ and $\sigma_2 \times \tau_2$ are two cells in the second union, then $\tau_1 \times \sigma_1$ and $\tau_2 \times \sigma_2$ are cells in the first union, and we add $\sigma_1 \times \tau_1$ before $\sigma_2 \times \tau_2$ if and only if we added $\tau_1 \times \sigma_1$ before $\tau_2 \times \sigma_2$.

Now we are ready to see what happens when we add these cells. For each $\beta = 1, 2, \ldots, p$,

$$\begin{aligned}
&\Big[P(X_f^*) \cup \bigcup_{\gamma=1}^{\beta-1} \Big(\langle v_1, v_2, \ldots, \hat{v}_\gamma, \ldots, v_m, u\rangle \\
&\qquad\qquad \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq\gamma}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)\Big] \\
&\qquad \cap \Big(\langle v_1, v_2, \ldots, \hat{v}_\beta, \ldots, v_m, u\rangle \\
&\qquad\qquad \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq\beta}^{m} \mathrm{St}\,(v_q, A)\Big]\Big) \\
&\qquad = \Big(\langle v_1, v_2, \ldots, \hat{v}_\beta, \ldots, v_m, u\rangle \times \Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq\beta}^{m} \mathrm{St}\,(v_q, A)\Big]\Big) \\
&\qquad \cup \Big(\langle v_1, v_2, \ldots, \hat{v}_\beta, \ldots, v_m\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq\beta}^{m} \mathrm{St}\,(v_q, A)\Big]\Big) \\
&\qquad \cup \bigcup_{\gamma=1}^{\beta-1} \Big(\langle v_1, v_2, \ldots, \hat{v}_\gamma, \ldots, \hat{v}_\beta, \ldots, v_m, u\rangle \\
&\qquad\qquad \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \mathrm{St}\,(v_q, A)\Big]\Big).
\end{aligned} \tag{6.1}$$

Therefore

$$P(X_f^*) \cup \bigcup_{i_1=1}^{p} \Big(\langle v_1, v_2, \ldots, \hat{v}_{i_1}, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq i_1}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)$$

has the homotopy type of $P(X_f^*)$. Now suppose $2 \leqq \alpha \leqq m-1$ and $i_2 \leqq p$. Let X_1 be the union of $P(X_f^*)$ with all those cells which have been added before

$$\begin{aligned}
E_1 = \bigcup_{k=1}^{\alpha} &\Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \\
&\times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big).
\end{aligned}$$

Then

$$X_1 \cap E_1 = \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \times \Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\tag{6.2} \cup \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\cup \bigcup_{\beta=1}^{\alpha} \Big\{\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle) \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}\,(v_{i_\beta}, A)\Big]\Big\}.$$

Therefore $X_1 \cup E_1$ has the homotopy type of X_1. Now suppose $2 \leqq \alpha \leqq m-1$, $i_2 > p$, and $i_1 = 1$. Let X_2 be the union of $P(X_f^*)$ with all those cells which have been added before

$$E_2 = \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big).$$

Then

$$X_2 \cap E_2 = \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \times \Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\cup \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\cup \bigcup_{\beta=2}^{\alpha} \Big\{\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle) \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}\,(v_{i_\beta}, A)\Big]\Big\}.$$

Therefore $X_2 \cup E_2$ has the homotopy type of X_2. Now suppose $2 \leqq \alpha \leqq m-1$, $i_2 > p$ and $i_1 > 1$. Let X_3 be the union of $P(X_f^*)$ with all those cells which have been added before

$$E_3 = \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big).$$

Then

$$X_3 \cap E_3 = \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \dots, \{\hat{v}_{i_k}\}, \dots, v_m, u\rangle \\ \times \Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\cup \bigcup_{k=1}^{\alpha} \Big(\langle v_1, v_2, \dots, \{\hat{v}_{i_k}\}, \dots, v_m\rangle \\ \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

$$\cup \bigcup_{\beta=2}^{\alpha} \Big\{\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \dots, \{\hat{v}_{i_k}\}, \dots, v_m, u\rangle) \\ \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}\,(v_{i_\beta}, A)\Big]\Big\}$$

$$\cup \bigcup_{j=1}^{i_1-1} \Big\{\bigcup_{k=1}^{\alpha} (\langle v_1, v_2, \dots, \hat{v}_j, \dots, \{\hat{v}_{i_k}\}, \dots, v_m, u\rangle) \\ \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}\,(v_{i_1}, A)\Big]\Big\}.$$

Now if $i_1 < p$ or $i_1 = p$ and there exists β $(p < \beta \leqq m)$ such that $\beta \neq i_k$ for any k, then $X_3 \cap E_3$ is contractible. However if $i_k = p+k-1$ for each $k = 1, 2, \dots, \alpha$ and $i_\alpha = m$, then $X_3 \cap E_3$ is contractible if and only if

$$\Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A)\Big] \\ \cup \bigcup_{\gamma=p+1}^{m} \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \Big(\bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A) \cup \mathrm{St}\,(v_\gamma, A)\Big)\Big]$$

is contractible. Thus it follows that if

$$\Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A)\Big] \\ \cup \bigcup_{\gamma=p+1}^{m} \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \Big(\bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A) \cup \mathrm{St}\,(v_\gamma, A)\Big)\Big]$$

is contractible, then $X_3 \cup E_3$ has the homotopy type of X_3. Now let X_4 be the union of $P(X_f^*)$ with all those cells which have been added before $\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))$. If $p=1$,

$$X_4 \cap (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))) = (\langle u\rangle \times \partial(\mathrm{St}\,(v_0, A))) \\ \cup \bigcup_{\gamma=2}^{m} (\langle u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]),$$

and if $p > 1$,

$$X_4 \cap (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))) = (\langle u\rangle \times \partial(\mathrm{St}\,(v_0, A))) \\ \cup \bigcup_{\gamma=1}^{m} (\langle u\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]).$$

Therefore, if $\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$ is contractible, $X_4 \cup (\langle u\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A)))$ has the homotopy type of X_4.

It follows from the above proof that if

$$\Big[\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A)\Big]$$
$$\cup \bigcup_{\gamma=p+1}^{m} \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \Big(\bigcup_{q=1}^{p} \mathrm{St}\,(v_q, A) \cup \mathrm{St}\,(v_\gamma, A)\Big)\Big]$$

and $\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$ are contractible, then

$$P(X_f^*) \cup \bigcup_{j=1}^{m} \bigcup_{i_1=1}^{\min[p,m-j+1]} \bigcup_{k=2}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k}$$
$$\Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle$$
$$\times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big)$$

is homotopically equivalent to $P(X_f^*)$. We essentially repeat the above argument in order to complete the proof of the theorem.

THEOREM 7. *If $A \cap B = \langle v_0, v_2, v_3, \ldots, v_m\rangle$, where $\langle v_0, v_2, v_3, \ldots, v_m\rangle$ is a simplex of A, and $X = A \cup B$, then $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$.*

Proof. We use the proof of Theorem 6 with $p=1$ and observe that in this proof i_1 is always 1 since $p=1$ and hence the only time the addition of a cell can change the homotopy type is when we add $\langle v_1\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))$ and $\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) \times \langle v_1\rangle$. (Note that $u = v_1$ since $p=1$.) Using the same terminology as in the proof of Theorem 6, we observed that

$$X_4 \cap (\langle v_1\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))) = (\langle v_1\rangle \times \partial(\mathrm{St}\,(v_0, A)))$$
$$\cup \bigcup_{\gamma=1}^{m} (\langle v_1\rangle \times [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]).$$

Since $\dim\,[\partial(\mathrm{St}\,(v_0, A))] \leqq n-1$, $H_r(X_4 \cup (\langle v_1\rangle \times \mathrm{Cl}\,(\mathrm{St}\,(v_0, A))))$ is isomorphic to $H_r(X_4)$ for all $r > n$. Since the same thing happens when we add $\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) \times \langle v_1\rangle$, $H_r(X^*)$ is isomorphic to $H_r(A^*)$ for all $r > n$.

THEOREM 8. *If $A \cap B = \{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle \mid i = 1, 2, \ldots, m\}$, where, for each i, $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle$ is a simplex of A, u is the barycenter of $\langle v_1, v_2, \ldots, v_m\rangle$, X is the polyhedron consisting of A and $\{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m, u\rangle \mid i = 1, 2, \ldots, m\}$, and*

$$\partial(\mathrm{St}\,(v_0, A)) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \mathrm{St}\,(v_\gamma, A)]$$

and $\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{\gamma=1}^{m} \mathrm{St}\,(v_\gamma, A)$ are contractible; then X^* is homotopically equivalent to A^*.

Proof. The proof is similar to the proof of Theorem 6, and hence we omit some of the details by referring to that proof. Let $f\colon X \to A$ be the simplicial map defined by $f(w)=w$ for each vertex w of A and $f(u)=v_0$. Then, by Theorem 2, $P(X_f^*)$ is homotopically equivalent to $P(A^*)$. If $i_0=0$, then

$$\begin{aligned} P(X^*) = P(X_f^*) &\cup \bigcup_{j=1}^{m} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} \Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \\ &\qquad \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big) \\ &\cup \bigcup_{j=1}^{m} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} \Big(\Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big] \\ &\qquad \times \langle v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle\Big). \end{aligned}$$

We associate an m-tuple with each cell in the first union and define an ordering of m-tuples in exactly the same way that we did in the proof of Theorem 6. Then we add cells according to this ordering the same way we did in this previous proof.

For each $\beta=1, 2, \ldots, m$, we have expression (6.1) in the proof of Theorem 6. If $\beta<m$, then the intersection in this expression is contractible. Therefore

$$P(X_f^*) \cup \bigcup_{i_1=1}^{m-1} \Big(\langle v_1, v_2, \ldots, \hat{v}_{i_1}, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq i_1}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)$$

has the homotopy type of $P(X_f^*)$. If $\partial(\mathrm{St}\,(v_0, A)) - \bigcup_{\gamma=1}^{m} \mathrm{St}\,(v_\gamma, A)$ is contractible, then

$$\begin{aligned} &\Big[P(X_f^*) \cup \bigcup_{\gamma=1}^{m=1} \Big(\langle v_1, v_2, \ldots, \hat{v}_\gamma, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq\gamma}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)\Big] \\ &\qquad \cap \Big(\langle v_1, v_2, \ldots, v_{m-1}, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m-1} \mathrm{St}\,(v_q, A)\Big]\Big) \end{aligned}$$

is contractible, and hence

$$P(X_f^*) \cup \bigcup_{i_1=1}^{m} \Big(\langle v_1, v_2, \ldots, \hat{v}_{i_1}, \ldots, v_m, u\rangle \times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1; q\neq i_1}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)$$

is homotopically equivalent to $P(X_f^*)$. Now suppose $2 \leqq \alpha \leqq m-1$ and let X_1 be the union of $P(X_f^*)$ with all those cells which have been added before

$$\begin{aligned} E_1 = \bigcup_{k=1}^{\alpha} \Big(&\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \\ &\times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big). \end{aligned}$$

Then we have expression (6.2) in the proof of Theorem 6. Therefore $X_1 \cup E_1$ has the homotopy type of X_1, and hence if

$$\begin{aligned} X_2 = P(X_f^*) \cup \bigcup_{j=1}^{m-1} \bigcup_{k=1}^{j} \bigcup_{i_k=i_{k-1}+1}^{m-j+k} &\Big(\langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m, u\rangle \\ &\times \Big[\mathrm{Cl}\,(\mathrm{St}\,(v_0, A)) - \bigcup_{q=1}^{m} \{\mathrm{St}\,(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big) \end{aligned}$$

and $\partial(\mathrm{St}(v_0, A)) - \bigcup_{\gamma=1}^m \mathrm{St}(v_\gamma, A)$ is contractible, then X_2 is homotopically equivalent to $P(X_f^*)$. Now

$$X_2 \cap (\langle u\rangle \times \mathrm{Cl}(\mathrm{St}(v_0, A))) = (\langle u\rangle \times \partial(\mathrm{St}(v_0, A))) \cup \bigcup_{\gamma=1}^{m} (\langle u\rangle \times [\mathrm{Cl}(\mathrm{St}(v_0, A)) - \mathrm{St}(v_\gamma, A)]).$$

Therefore if $\partial(\mathrm{St}(v_0, A)) \cup \bigcup_{\gamma=1}^m [\mathrm{Cl}(\mathrm{St}(v_0, A)) - \mathrm{St}(v_\gamma, A)]$ is contractible, then $X_3 = X_2 \cup (\langle u\rangle \times \mathrm{Cl}(\mathrm{St}(v_0, A)))$ has the homotopy type of X_2. It follows from the above proof that if $\partial(\mathrm{St}(v_0, A)) - \bigcup_{\gamma=1}^m \mathrm{St}(v_\gamma, A)$ and

$$\partial(\mathrm{St}(v_0, A)) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}(\mathrm{St}(v_0, A)) - \mathrm{St}(v_\gamma, A)]$$

are contractible, then X_3 is homotopically equivalent to A^*. We essentially repeat the above in order to complete the proof of the theorem.

THEOREM 9. *If* $A \cap B = \langle v_1, v_2, \ldots, v_m\rangle \cup \langle v_0, v_1\rangle$, *where* $\langle v_1, v_2, \ldots, v_m\rangle$ *and* $\langle v_0, v_1\rangle$ *are simplexes of* A, $X = A \cup B$, *and, for each nonempty subset* F *of* $\{2, 3, \ldots, m\}$,

$$\bigcup_{\gamma=1;\gamma\notin F}^{m} \{A - (\mathrm{St}(v_0, A) \cup \bigcup \{\mathrm{St}(v_p, A) \mid p \in F\} \cup \mathrm{St}(v_\gamma, A))\}$$

is a deformation retract of $A - (\mathrm{St}(v_0, A) \cup \bigcup \{\mathrm{St}(v_p, A) \mid p \in F\})$; *then* X^* *is homotopically equivalent to* A^*.

Proof. If $i_0 = 0$, then

$$\begin{aligned}
P(X^*) = {} & P(A^*) \cup \Big(B \times \Big[A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)\Big]\Big) \\
& \cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} \Big(\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \\
& \qquad \times \Big[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big]\Big) \\
& \cup \bigcup_{p=2}^{m} (\langle v_0, v_p\rangle \times [A - (\mathrm{St}(v_0, A) \cup \mathrm{St}(v_p, A))]) \\
& \cup \Big(\Big[A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)\Big] \times B\Big) \\
& \cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} \Big(\Big[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\Big] \\
& \qquad \times \langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle\Big) \\
& \cup \bigcup_{p=2}^{m} ([A - (\mathrm{St}(v_0, A) \cup \mathrm{St}(v_p, A))] \times \langle v_0, v_p\rangle).
\end{aligned}$$

We add the above unions to $P(A^*)$ in the order in which we have listed them. In order to explain the order in which we add the cells in the second union, we associate with each cell in this union an m-tuple and define an ordering of m-tuples in exactly the same way that we did in the proof of Theorem 6. Then we add cells in the second and fifth union according to this ordering in the same way we did in this previous proof. Now

$$P(A^*) \cap \left(B \times \left[A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)\right]\right) = [\langle v_1, v_2, \ldots, v_m\rangle \cup \langle v_0, v_1\rangle] \times \left[A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)\right],$$

and therefore $P(A^*) \cup (B \times [A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)])$ is homotopically equivalent to $P(A^*)$. Now suppose $1 \leqq \alpha \leqq m-2$ and $i_1 = 1$. Let X_1 be the union of $P(A^*)$ with all those cells which have been added before

$$E_1 = \bigcup_{k=1}^{\alpha} \left(\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\right]\right).$$

Then

$$\begin{aligned} X_1 \cap E_1 = {} & \left[\langle v_0\rangle \cup \bigcup_{k=1}^{\alpha} \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle\right] \\ & \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\right] \\ & \cup \bigcup_{\beta=1}^{\alpha} \left\{\bigcup_{k=1}^{\alpha} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle) \right. \\ & \left. \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta}, A)\right]\right\}. \end{aligned}$$

Therefore if

$$\bigcup_{\beta=1}^{\alpha} \left[A - \left(\bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta}, A)\right)\right]$$

is a deformation retract of $A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}$, then $X_1 \cup E_1$ is homotopically equivalent to X_1. Now suppose $1 \leqq \alpha \leqq m-2$ and $i_1 > 1$. Let X_2 be the union of $P(A^*)$ with all those cells which have been added before

$$E_2 = \bigcup_{k=1}^{\alpha} \left(\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\right]\right).$$

Then

$$\begin{aligned} X_2 \cap E_2 = {} & \left[\langle v_0, v_1\rangle \cup \bigcup_{k=1}^{\alpha} \langle v_1, v_2, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle\right] \\ & \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\right] \\ & \cup \bigcup_{\beta=1}^{\alpha} \left\{\bigcup_{k=1}^{\alpha} (\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle) \right. \\ & \left. \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\} \cup \mathrm{St}(v_{i_\beta}, A)\right]\right\}. \end{aligned}$$

Hence $X_2 \cup E_2$ is homotopically equivalent to X_2. Let

$$\begin{aligned} X_3 = {} & P(A^*) \cup \left(B \times \left[A - \bigcup_{q=0}^{m} \mathrm{St}(v_q, A)\right]\right) \\ & \cup \bigcup_{j=1}^{m-2} \bigcup_{k=1}^{j} \bigcup_{i_k = i_{k-1}+1}^{m-j+k} \left(\langle v_0, v_1, \ldots, \{\hat{v}_{i_k}\}, \ldots, v_m\rangle \right. \\ & \left. \times \left[A - \bigcup_{q=0}^{m} \{\mathrm{St}(v_q, A) \mid q \neq i_k \text{ for any } k\}\right]\right). \end{aligned}$$

Then, for each $\beta=2, 3, \dots, m$,

$$\begin{aligned}\Big[X_3 \cup \bigcup_{\delta=2}^{\beta-1} (\langle v_0, v_\delta\rangle \times [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\delta, A))])\Big]\\ \cap (\langle v_0, v_\beta\rangle \times [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\beta, A))])\\ = [\langle v_0\rangle \cup \langle v_\beta\rangle] \times [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\beta, A))]\\ \cup \bigcup_{\varepsilon=1;\varepsilon\neq\beta}^{m} (\langle v_0, v_\beta\rangle \times [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\beta, A) \cup \mathrm{St}\,(v_\varepsilon, A))]).\end{aligned}$$

If, for each $\beta=2, 3, \dots, m$,

$$\bigcup_{\varepsilon=1;\varepsilon\neq\beta}^{m} [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\beta, A) \cup \mathrm{St}\,(v_\varepsilon, A))]$$

is a deformation retract of $A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_\beta, A))$, then

$$X_3 \cup \bigcup_{p=2}^{m} (\langle v_0, v_p\rangle \times [A-(\mathrm{St}\,(v_0, A) \cup \mathrm{St}\,(v_p, A))])$$

is homotopically equivalent to X_3. Now, in order to complete the proof of the theorem, we essentially repeat the above argument.

THEOREM 10. *If* $\lambda_0=0$,

$$A \cap B = \bigcup_{k=1}^{2} \bigcup_{\lambda_k=\lambda_{k-1}+1}^{m-2+k} \{\langle v_0, v_1, \dots, \{\hat{v}_{\lambda_k}\}, \dots, v_m\rangle\},$$

where each $\langle v_0, v_1, \dots, \{\hat{v}_{\lambda_k}\}, \dots, v_m\rangle$ *is a simplex of* A, $X=A \cup B$, *and, for each subset* F *of* $\{1, 2, \dots, m\}$ *consisting of at least* $m-1$ *elements,*

$$\bigcup_{\gamma=0;\gamma\notin F}^{m} \{A-(\bigcup \{\mathrm{St}\,(v_q, A) \mid q \in F\} \cup \mathrm{St}\,(v_\gamma, A))\}$$

is a deformation retract of $A-\bigcup \{\mathrm{St}\,(v_q, A) \mid q \in F\}$; *then* X^* *is homotopically equivalent to* A^*.

Proof. First observe that

$$\begin{aligned}P(X^*) = P(A^*) \cup \Big(B \times \Big[A-\bigcup_{q=0}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)\\ \cup \bigcup_{i_1=0}^{m} \Big(\langle v_0, v_1, \dots, \hat{v}_{i_1}, \dots, v_m\rangle \times \Big[A- \bigcup_{q=0;q\neq i_1}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)\\ \cup \bigcup_{i_2=1}^{m} \Big(\langle v_1, v_2, \dots, \hat{v}_{i_2}, \dots, v_m\rangle \times \Big[A- \bigcup_{q=1;q\neq i_2}^{m} \mathrm{St}\,(v_q, A)\Big]\Big)\\ \cup \Big(\Big[A-\bigcup_{q=0}^{m} \mathrm{St}\,(v_q, A)\Big] \times B\Big)\\ \cup \bigcup_{i_1=0}^{m} \Big(\Big[A- \bigcup_{q=0;q\neq i_1}^{m} \mathrm{St}\,(v_q, A)\Big] \times \langle v_0, v_1, \dots, \hat{v}_{i_1}, \dots, v_m\rangle\Big)\\ \cup \bigcup_{i_2=1}^{m} \Big(\Big[A- \bigcup_{q=1;q\neq i_2}^{m} \mathrm{St}\,(v_q, A)\Big] \times \langle v_1, v_2, \dots, \hat{v}_{i_2}, \dots, v_m\rangle\Big).\end{aligned}$$

Now

$$P(A^*) \cap \left(B\times\left[A-\bigcup_{q=0}^{m} \mathrm{St}\,(v_q, A)\right]\right) = (A\cap B)\times\left[A-\bigcup_{q=0}^{m} \mathrm{St}\,(v_q, A)\right],$$

and hence $X_1 = P(A^*) \cup (B\times[A-\bigcup_{q=0}^{m} \mathrm{St}\,(v_q, A)])$ is homotopically equivalent to $P(A^*)$. Also, for each $\alpha=0, 1,\ldots, m$,

$$\begin{aligned}&\left[X_1 \cup \bigcup_{\beta=0}^{\alpha-1}\left(\langle v_0, v_1,\ldots,\hat{v}_\beta,\ldots,v_m\rangle\times\left[A-\bigcup_{q=0;q\neq\beta}^{m}\mathrm{St}\,(v_q, A)\right]\right)\right]\\ &\qquad\cap\left(\langle v_0, v_1,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle\times\left[A-\bigcup_{q=0;q\neq\alpha}^{m}\mathrm{St}\,(v_q, A)\right]\right)\\ &\qquad= [A\cap\langle v_0, v_1,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle]\times\left[A-\bigcup_{q=0;q\neq\alpha}^{m}\mathrm{St}\,(v_q, A)\right]\\ &\qquad\cup\left(\langle v_0, v_1,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle\times\left[A-\bigcup_{q=0}^{m}\mathrm{St}\,(v_q, A)\right]\right).\end{aligned}$$

If $\alpha\neq 0$,

$$X_1 \cup \bigcup_{\beta=0}^{\alpha}\left(\langle v_0, v_1,\ldots,\hat{v}_\beta,\ldots,v_m\rangle\times\left[A-\bigcup_{q=0;q\neq\beta}^{m}\mathrm{St}\,(v_q, A)\right]\right)$$

is homotopically equivalent to

$$X_1 \cup \bigcup_{\beta=0}^{\alpha-1}\left(\langle v_0, v_1,\ldots,\hat{v}_\beta,\ldots,v_m\rangle\times\left[A-\bigcup_{q=0;q\neq\beta}^{m}\mathrm{St}\,(v_q, A)\right]\right).$$

If $A-\bigcup_{q=0}^{m}\mathrm{St}\,(v_q, A)$ is a deformation retract of $A-\bigcup_{q=1}^{m}\mathrm{St}\,(v_q, A)$, then

$$X_1 \cup \left(\langle v_1, v_2,\ldots,v_m\rangle\times\left[A-\bigcup_{q=1}^{m}\mathrm{St}\,(v_q, A)\right]\right)$$

is homotopically equivalent to X_1. Therefore if $A-\bigcup_{q=0}^{m}\mathrm{St}\,(v_q, A)$ is a deformation retract of $A-\bigcup_{q=1}^{m}\mathrm{St}\,(v_q, A)$, then

$$X_2 = X_1\cup\bigcup_{i_1=0}^{m}\left(\langle v_0, v_1,\ldots,\hat{v}_{i_1},\ldots,v_m\rangle\times\left[A-\bigcup_{q=0;q\neq i_1}^{m}\mathrm{St}\,(v_q, A)\right]\right)$$

is homotopically equivalent to A^*. Now, for each $\alpha=1, 2,\ldots, m$,

$$\begin{aligned}&\left[X_2 \cup \bigcup_{\beta=1}^{\alpha-1}\left(\langle v_1, v_2,\ldots,\hat{v}_\beta,\ldots,v_m\rangle\times\left[A-\bigcup_{q=1;q\neq\beta}^{m}\mathrm{St}\,(v_q, A)\right]\right)\right]\\ &\qquad\cap\left(\langle v_1, v_2,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle\times\left[A-\bigcup_{q=1;q\neq\alpha}^{m}\mathrm{St}\,(v_q, A)\right]\right)\\ &\qquad= [A\cap\langle v_1, v_2,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle]\times\left(A-\bigcup_{q=1;q\neq\alpha}^{m}\mathrm{St}\,(v_q, A)\right)\\ &\qquad\cup\left(\langle v_1, v_2,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle\times\left[A-\bigcup_{q=1}^{m}\mathrm{St}\,(v_q, A)\right]\right)\\ &\qquad\cup\left(\langle v_1, v_2,\ldots,\hat{v}_\alpha,\ldots,v_m\rangle\times\left[A-\left(\mathrm{St}\,(v_0, A)\cup\bigcup_{q=1;q\neq\alpha}^{m}\mathrm{St}\,(v_q, A)\right)\right]\right).\end{aligned}$$

Hence, if for each subset F of $\{1, 2, \ldots, m\}$ consisting of at least $m-1$ elements,

$$\bigcup_{\gamma=0;\gamma\notin F}^{m} \{A-(\bigcup \{\mathrm{St}\,(v_q, A) \mid q \in F\} \cup \mathrm{St}\,(v_\gamma, A))\}$$

is a deformation retract of $A-\bigcup \{\mathrm{St}\,(v_q, A) \mid q \in F\}$, then

$$X_2 \cup \bigcup_{i_2=1}^{m} \left(\langle v_1, v_2, \ldots, \hat{v}_{i_2}, \ldots, v_m\rangle \times \left[A - \bigcup_{q=1;q\neq i_2}^{m} \mathrm{St}\,(v_q, A)\right]\right)$$

is homotopically equivalent to A^*. In order to complete the proof of the theorem, we essentially repeat the above argument.

4. **Isotopy types of spheres.** For each integer $n \geqq 3$, let $B^n = \{x \in E^n \mid |x| \leqq 1\}$, and, for each integer m $(1 \leqq m \leqq n)$, let A_m be an m-simplex. Suppose that for each m and n, $A_m \cap B^n$ is an $(m-1)$-face of A_m and $A_m \cap B^n \subset (B^n)^\circ$. Let $C^n_m = B^n \cup A_m$. Also let C^n_n be the polyhedron consisting of three n-simplexes with a common $(n-1)$-face.

For each integer $n \geqq 3$, let $\mathscr{A}_n = \{C^n_1, C^n_2, \ldots, C^n_n\}$, and let $\mathscr{B}_n$ be the collection consisting of the members of $\mathscr{A}_n$ and all finite, contractible, n-dimensional polyhedra X with the property that there is a member C^n_i of $\mathscr{A}_n$ such that a homeomorph of X can be constructed out of C^n_i by appending m-simplexes $(1 \leqq m \leqq n)$ in such a way that if the construction is factored $C^n_i = X_1 \to X_2 \to \cdots \to X_e = X$, then X_j is obtained from X_{j-1} by

(1) adding a 1-simplex $\tau = \langle v_0, v_1\rangle$ such that $X_{j-1} \cap \tau = \langle v_0\rangle$, where v_0 is a vertex of X_{j-1} and $\partial(\mathrm{St}\,(v_0, X_{j-1}))$ is contractible,

(2) adding an m-simplex $\tau = \langle v_0, v_1, \ldots, v_m\rangle$, $2 \leqq m \leqq n$, such that $X_{j-1} \cap \tau = \{\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle \mid i = 1, 2, \ldots, p\}$, where $1 \leqq p < m$, $\langle v_0, v_1, \ldots, \hat{v}_i, \ldots, v_m\rangle$ is a simplex of X_{j-1} for each i, and, if $\mathrm{St}\,(v_1, X_{j-1})$ denotes the empty set when $v_1 \notin X_{j-1}$, then

$$\left[\partial(\mathrm{St}\,(v_0, X_{j-1})) - \bigcup_{q=1}^{p} \mathrm{St}\,(v_q, X_{j-1})\right]$$
$$\cup \bigcup_{\gamma=p+1}^{m} \left[\mathrm{Cl}\,(\mathrm{St}\,(v_0, X_{j-1})) - \left(\bigcup_{q=1}^{p} \mathrm{St}\,(v_q, X_{j-1}) \cup \mathrm{St}\,(v_\gamma, X_{j-1})\right)\right]$$

and

$$\partial(\mathrm{St}\,(v_0, X_{j-1})) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, X_{j-1})) - \mathrm{St}\,(v_\gamma, X_{j-1})]$$

are contractible,

(3) adding a 2-simplex $\tau = \langle v_0, v_1, v_2\rangle$ such that $X_{j-1} \cap \tau = \langle v_0, v_2\rangle \cup \langle v_0, v_1\rangle$, where $\langle v_0, v_2\rangle$ and $\langle v_0, v_1\rangle$ are simplexes of X_{j-1} and

$$\partial(\mathrm{St}\,(v_0, X_{j-1})) \cup \bigcup_{\gamma=1}^{2} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, X_{j-1})) - \mathrm{St}\,(v_\gamma, X_{j-1})]$$

and either $\partial(\mathrm{St}\,(v_0, X_{j-1})) - \mathrm{St}\,(v_1, X_{j-1})$ or $\partial(\mathrm{St}\,(v_0, X_{j-1})) - \mathrm{St}\,(v_2, X_{j-1})$ or $\partial(\mathrm{St}\,(v_0, X_{j-1})) - \bigcup_{\gamma=1}^{2} \mathrm{St}\,(v_\gamma, X_{j-1})$ are contractible,

(4) adding an m-simplex $\tau=\langle v_0, v_1, \dots, v_m\rangle$, $3\leqq m\leqq n$, such that $X_{j-1}\cap\tau=\{\langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m\rangle \mid i=1, 2, \dots, m\}$, where $\langle v_0, v_1, \dots, \hat{v}_i, \dots, v_m\rangle$ is a simplex of X_{j-1} for each i and

$$\partial(\mathrm{St}\,(v_0, X_{j-1})) \cup \bigcup_{\gamma=1}^{m} [\mathrm{Cl}\,(\mathrm{St}\,(v_0, X_{j-1}))-\mathrm{St}\,(v_\gamma, X_{j-1})]$$

and $\partial(\mathrm{St}\,(v_0, X_{j-1}))-\bigcup_{\gamma=1}^{m} \mathrm{St}\,(v_\gamma, X_{j-1})$ are contractible,

(5) adding an m-simplex $\tau=\langle v_0, v_1, \dots, v_m\rangle$, $3\leqq m\leqq n$, such that $X_{j-1}\cap\tau=\langle v_1, v_2, \dots, v_m\rangle \cup \langle v_0, v_1\rangle$, where $\langle v_1, v_2, \dots, v_m\rangle$ and $\langle v_0, v_1\rangle$ are simplexes of X_{j-1}, and, for each nonempty subset F of $\{2, 3, \dots, m\}$,

$$\bigcup_{\gamma=1;\gamma\notin F}^{m} \{X_{j-1}-(\mathrm{St}\,(v_0, X_{j-1}) \cup \bigcup \{\mathrm{St}\,(v_p, X_{j-1}) \mid p\in F\} \cup \mathrm{St}\,(v_\gamma, X_{j-1}))\}$$

is a deformation retract of $X_{j-1}-(\mathrm{St}\,(v_0, X_{j-1}) \cup \bigcup \{\mathrm{St}\,(v_p, X_{j-1}) \mid p\in F\})$, or

(6) adding an m-simplex $\tau=\langle v_0, v_1, \dots, v_m\rangle$, $3\leqq m\leqq n$, such that

$$X_{j-1}\cap\tau = \bigcup_{k=1}^{2} \bigcup_{\lambda_k=\lambda_{k-1}+1}^{m-2+k} \{\langle v_0, v_1, \dots, \{\hat{v}_{\lambda_k}\}, \dots, v_m\rangle\},$$

where $\lambda_0=0$, each $\langle v_0, v_1, \dots, \{\hat{v}_{\lambda_k}\}, \dots, v_m\rangle$ is a simplex of X_{j-1}, and, for each subset F of $\{1, 2, \dots, m\}$ consisting of at least $m-1$ elements,

$$\bigcup_{\gamma=0;\gamma\notin F}^{m} \{X_{j-1}-(\bigcup \{\mathrm{St}\,(v_q, X_{j-1}) \mid q\in F\} \cup \mathrm{St}\,(v_\gamma, X_{j-1}))\}$$

is a deformation retract of $X_{j-1}-\bigcup \{\mathrm{St}\,(v_q, X_{j-1}) \mid q\in F\}$.

THEOREM 11. *If $X\in\mathscr{B}_n$, then X^* has the homotopy type of S^n.*

Proof. By Theorem 3 of [7], $(C_n^n)^*$ has the homotopy type of S^n. Also by Theorem 4 of [7], for each $m=1, 2, \dots, n-1$, $(C_m^n)^*$ has the homotopy type of S^n. Therefore $(X_1)^*$ has the homotopy type of S^n. Now suppose $1<j\leqq e$ and $(X_{j-1})^*$ has the homotopy type of S^n. If X_j is obtained from X_{j-1} by (1), then, by Theorem 4, $(X_j)^*$ has the homotopy type of S^n. If X_j is obtained from X_{j-1} by (2), then, by Theorem 6, $(X_j)^*$ has the homotopy type of S^n. If X_j is obtained from X_{j-1} by (3), then, by Theorem 5, $(X_j)^*$ has the homotopy type of S^n. If X_j is obtained from X_{j-1} by (4), then, by Theorem 8, $(X_j)^*$ has the homotopy type of S^n. If X_j is obtained from X_{j-1} by (5), then, by Theorem 9, $(X_j)^*$ has the homotopy type of S^n. Finally if X_j is obtained from X_{j-1} by (6), then, by Theorem 10, $(X_j)^*$ has the homotopy type of S^n. Therefore X^* has the homotopy type of S^n.

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