TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, January 1970

# SOME IMMERSION THEOREMS FOR PROJECTIVE SPACES

BY
A. DUANE RANDALL

1. **Introduction.** In this paper we obtain some results on the classical problem of immersing projective spaces into Euclidean space. Let $\alpha(n)$ denote the number of 1's appearing in the dyadic expansion of $n$. We prove the following

THEOREM 1.1. *$CP^n$ immerses in $R^{4n-5}$ for $n$ odd and $\alpha(n)>2$.*

Applying Theorem 4 of [6] with Theorem 1.1 gives

COROLLARY 1.2. *$CP^n$ has a best possible immersion in $R^{4n-5}$ for $n=2^r+2^s+1$ with $r>s>0$.*

THEOREM 1.3. *$RP^n$ immerses in $R^{2n-7}$ for $n\equiv 4 \bmod 8$ and $\alpha(n)>2$.*

We remark that the proof of (1.3) also shows $RP^n$ does not immerse in $R^{2n-7}$ for $n=2^r+4$ with $r>3$, a result of [7].

THEOREM 1.4. *$RP^n$ immerses in $R^{2n-9}$ for $n\equiv 0 \bmod 8$ and $n$ not a power of 2.*

COROLLARY 1.5. *$RP^n$ immerses in $R^{2n-4\alpha(n)-1}$ for $n=2^r+2^s$ with $r>s>2$.*

It follows from [4] that $RP^n$ does not immerse in $R^{2n-11}$ for $n=2^r+8$ and $r>3$.

THEOREM 1.6. *$RP^n$ immerses in $R^{2n-8}$ for $n\equiv 1 \bmod 4$ and $\alpha(n)>3$.*

Adem and Gitler showed in [4] and [7] that $RP^n$ has a best possible immersion in $R^{2n-4}$ for $n\equiv 1 \bmod 4$ and $\alpha(n)=3$.

These results are interesting only for small values of $\alpha(n)$ due to Milgram's construction of linear immersions in [21]. The method of proof consists of expressing certain obstructions to the lifting of an appropriate map by Adams-Maunder operations and then evaluating these operations in projective space. The author wishes to express his gratitude to his advisor, Professor Emery Thomas, and to the Centro de Investigacion y de Estudios Avanzados del IPN, Mexico.

2. **Preliminaries.** The coefficient group for singular cohomology is understood to be $Z_2$ whenever omitted. We let $\alpha\in H^1(RP^\infty)$ and $\beta\in H^2(CP^\infty)$ denote generators for the cohomology rings. Let $A_k$ denote the vector subspace of the mod 2 Steenrod algebra $A$ consisting of homogeneous elements of degree $k$. If $\alpha(k+s)>\alpha(s)$,

Received by the editors March 24, 1969.

$A_k(\alpha^s)=0$. A standard fact in number theory states that the highest power of 2 dividing a binomial coefficient $\binom{r+s}{s}$ is $2^{\alpha(r)+\alpha(s)-\alpha(r+s)}$. Let $\xi$ and $\eta$ denote the Hopf line bundles over $RP^\infty$ and $CP^\infty$. The Thom complex $T(m\xi)$ is homeomorphic to the stunted projective space $RP^{m+s}/RP^m$ for $m\xi$ based on $RP^s$. $T(r\eta)$ is homeomorphic to $CP^{r+s}/CP^r$ for $r\eta$ based on $CP^s$. The Hopf map $H\colon RP^\infty \to CP^\infty$ gives the real bundle equation $H^*\eta=2\xi$.

$$W(m\xi) = \sum_s \binom{m}{s}\alpha^s.$$

We refer to [15] and [3] for these facts. In [4] Adem and Gitler prove for $n>8$

PROPOSITION 2.1. *$RP^n$ immerses in $R^{n+k}$ iff $(n+k+1)\xi$ has $n+1$ independent nonzero sections iff $(2^{\varphi(n)}-(n+1))\xi$ has $2^{\varphi(n)}-(n+k+1)$ independent nonzero sections.*

3. **Cohomology operations in projective space.** In [3] Adem and Gitler formulate an algorithm for computing a family of stable secondary cohomology operations in complex projective space. Let $\rho(r,s)$ denote the following relation in $A$ for any positive integers $r$ and $s$:

$$(3.1)\qquad (Sq^{2^r}Sq^1)Sq^{2^r s}+\sum_{t=0}^{r-1} Sq^{2^r(s+1)+1-2^t}Sq^{2^t}+sSq^1Sq^{2^r(s+1)} = 0.$$

A straightforward generalization of Theorem 8.2 in [4] is the following

PROPOSITION 3.2 *Let $\Phi(r,s)$ denote any stable secondary operation associated to $\rho(r,s)$. Let $a=2^t$ be such that $a\leqq 2^r(s+1)<2a$. Let $c$ be any integer such that $c<s$ and $\alpha(c+s+1)>\alpha(c)$. For $m=ha+2^{r-1}c$ with $h>0$, $\Phi(r,s)$ is defined on $\beta^m$ and with zero indeterminacy*

$$\Phi(r,s)(\beta^m) = h\binom{2^r c}{2^r(s+1)-a}\beta^{m+2^{r-1}(s+1)}.$$

The proof of (3.2) is essentially given in [3] and [4] and so is omitted.

In [10] Gitler, Mahowald, and Milgram show that many secondary operations defined on the Thom class of a complex vector bundle measure the divisibility by 2 of its Chern classes. Applications of their argument yield the following results.

PROPOSITION 3.3. *Let $\omega$ denote a complex bundle over a complex $X$ such that $c_{2t+1}(\omega)=2x$ for $x$ in $H^{4t+2}(X;Z)$ and $c_{2t+2}(\omega)=2y$ for $y$ in $H^{4t+4}(X;Z)$. A secondary operation $\varphi$ associated to the relation*

$$(Sq^2Sq^1)Sq^{4t+2}+Sq^1Sq^{4t+4}+Sq^{4t+4}Sq^1 = 0$$

*can be chosen independently of $\omega$ so that*

$$Sq^2(U_\omega\cdot x)+U_\omega\cdot y\in\varphi(U_\omega).$$

PROPOSITION 3.4. *Let $\rho$ denote a complex bundle over a complex $X$ such that $c_{4t+2}(\rho)=2x$, $c_{4t+3}(\rho)=2y$, $c_{4t+4}(\rho)=2z$ for classes $x$, $y$, and $z$ in $H^*(X;Z)$. A secondary operation $\Gamma$ associated to the relation*

$$(Sq^4Sq^1)Sq^{8t+4}+Sq^1Sq^{8t+8}+Sq^1(Sq^{8t+6}Sq^2)+Sq^{8t+8}Sq^1 \;=\; 0$$

*can be chosen independently of $\rho$ so that*

$$U_\rho\cdot(z+y\cdot w_2(\rho)+x\cdot w_2^2(\rho))+Sq^4(U_\rho\cdot x)\in\Gamma(U_\rho).$$

PROPOSITION 3.5. *Let $\omega$ denote a complex bundle over a complex $X$ such that $c_1(\omega)=0$, $w_4(\omega)=0$, and $c_{8t}(\omega)=2x$ for $x$ in $H^{16t}(X;Z)$. Let $\Phi$ denote a secondary operation associated to the defining relation*

$$(Sq^8Sq^1)Sq^{16t}+Sq^{16t+8}Sq^1+Sq^{16t+7}Sq^2+Sq^1(Sq^{16t+4}Sq^4) \;=\; 0 \quad \textit{for } t>0.$$

*Then $\Phi$ can be chosen independently of $\omega$ so that $Sq^8(U_\omega\cdot x)\in\Phi(U_\omega)$.*

REMARK. mod 2 reduction of integral classes is understood whenever applicable in the above propositions. The proofs involve direct applications of the argument given in [10]. We give only the proof of (3.5).

**Proof of 3.5.** Consider the following diagram.

$$\begin{tikzcd}
& F \arrow[r, "j"] & E(2n) \arrow[d, "p"] & \\
\Sigma^s T(\omega) \arrow[urr, "g"] \arrow[rr, "T(f)"] & & MSU(n) \arrow[r, "U_n"] & K(Z,2n)
\end{tikzcd}$$

Here $p$ is the principal fibration induced from the universal example for the operation $\Phi$ on integral classes of dimension $2n$ for large $n$ by the Thom class $U_n\colon MSU(n)\to K(Z,2n)$. Now $p^*(U_n\cdot c_{8t})=2e_1$ for some $e_1$ in $H^*(E(2n);Z)$ since $Sq^{16t}U_n=U_n\cdot w_{16t}$. One notes that $j^*e_1 \bmod 2=Sq^1\iota_1\otimes 1\otimes 1\otimes 1$ where

$$F \;=\; K(Z_2,2n+16t-1)\times K(Z_2,2n+16t+7)\times K(Z_2,2n+1)\times K(Z_2,2n)$$

is the fiber of $p$. (See [10].) Similarly,

$$p^*[U_n\cdot(c_2c_{8t+2}+c_3c_{8t+1}+c_2^2c_{8t})] \;=\; 2e_2$$

for some integral class $e_2$ and $j^*e_2 \bmod 2=1\otimes Sq^1\iota_2\otimes 1\otimes 1$. Let $e_3$ and $e_4$ be classes in $H^*(E(2n))$ such that $j^*e_3=1\otimes 1\otimes\iota_3\otimes 1$ and $j^*e_4=1\otimes 1\otimes 1\otimes\iota_4$. We now choose $\Phi$ so that $\Phi$ vanishes on classes of dimension $\leqq 16t-2$. This is possible by [2] and [14]. It follows that

$$Sq^8e_1+e_2+Sq^{16t+7}e_3+Sq^{16t+8}e_4$$

is the representative for this choice of $\Phi$ in $H^*(E(2n))$. Let $f\colon X\to BSU(n)$

classify the bundle $\omega \oplus s$ where $n-s$ is the fiber dimension of $\omega$. $T(f)$ is the natural map induced by $f$ between the Thom complexes. Thus,

$$\begin{aligned}\Sigma^s\Phi(U_\omega) &= \Phi(\Sigma^s U_\omega)\\ &= \bigcup_g g^*(Sq^8e_1+e_2+Sq^{16t+7}e_3+Sq^{16t+8}e_4)\\ &= \bigcup_g g^*(Sq^8e_1+e_2)\end{aligned}$$

where $g$ ranges over all liftings of $T(f)$. Since the Chern classes $c_2(\omega)$, $c_{8t}(\omega)$, $c_{8t+1}(\omega)$, and $c_{8t+2}(\omega)$ are divisible by 2, it follows that $Sq^8(U_\omega\cdot x)\in\Phi(U_\omega)$.

The proof of Theorem 1.4 uses a tertiary cohomology operation which we define here and evaluate in real projective space. Consider the following relations and associated secondary operations for $s>3$:

$$\begin{aligned}&\Phi_1\colon (Sq^2Sq^1)Sq^{2^s}+Sq^{2^s+2}Sq^1 = 0,\\ &\Phi_2\colon (Sq^4Sq^1)Sq^{2^s}+Sq^{2^s+4}Sq^1+Sq^{2^s+3}Sq^2 = 0,\\ &\Phi_3\colon Sq^2Sq^2+Sq^3Sq^1 = 0,\\ &\Phi_4\colon Sq^1Sq^1 = 0.\end{aligned}$$

Let $\Phi$ denote the 4-valued secondary operation $(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$.

PROPOSITION 3.6. *$\Phi_1$ and $\Phi_2$ can be chosen so $(\Phi_1, \Phi_2)$ vanishes on classes having dimension $<2^s$. For these choices the following relation holds stably and with zero indeterminacy among the component operations $\Phi_i$ of $\Phi$.*

$$Sq^6\Phi_1+Sq^4\Phi_2+Sq^{2^s+5}\Phi_3+Sq^{2^s+7}\Phi_4+(\lambda Sq^{2^s+4})Sq^4 = 0 \tag{3.7}$$

*where $\lambda$ is in $Z_2$.*

**Proof.** The functional cohomology operations associated with the defining relations for $\Phi_1$ and $\Phi_2$ vanish on classes having dimension $<2^s$ by [2, Teorema 6.6]. Now $\Phi_1$ and $\Phi_2$ can be chosen trivial on classes in the domain of $\Phi$ having dimension $<2^s$ by the Peterson-Stein formula [2, Teorema 5.2]. Consider the universal example for the operation $\Phi$ on classes of dimension $n$ for large $n$.

$$\begin{tikzcd} & E(n) \arrow[d, "p"] & \\ X \arrow[ur, "g"] \arrow[r, "f"'] & K(Z_2, n) \arrow[r, "Sq^1\iota\times Sq^2\iota\times Sq^{2^s}\iota"] & C \end{tikzcd}$$

The map $p$ is the principal fibration with classifying map $Sq^1\iota\times Sq^2\iota\times Sq^{2^s}\iota$ and $C$ is a product of Eilenberg-MacLane spaces. Let $k_i^n$ in $H^*(E(n))$ be the representative class for $\Phi_i$ for $1\leqq i\leqq 4$. Let an arbitrary class $x$ in $H^n(X)$ in the domain of $\Phi$ be classified by a map $f\colon X \to K(Z_2, n)$. By definition $\Phi(x) = \bigcup_g g^*(k_1^n, k_2^n, k_3^n, k_4^n)$ where the union ranges over all liftings of $f$. The Serre exact sequence applied to the map $p$ gives

$$Sq^6 k_1^n + Sq^4 k_2^n + Sq^{2^s+5} k_3^n + Sq^{2^s+7} k_4^n = \lambda\theta(p^*\iota) \qquad (\lambda \in Z_2)$$

where $\theta$ is a sum of admissible monomials in $A$ each having degree $2^s+8$ and excess $\geqq 2^s$. The Adem relations applied to $Sq^8 Sq^{2^s}$ show that $Sq^{2^s+8}(p^*\iota) = Sq^{2^s+4} Sq^4(p^*\iota)$ so $\theta = Sq^{2^s+4} Sq^4$.

Let $\psi$ be any stable tertiary operation associated to the relation given by (3.7). The indeterminacy subgroup $\mathrm{Indet}^n(X; \psi)$ arises in the following manner. The operation $\psi$ determines a secondary operation $\mathrm{In}(\psi)$ of three variables. (See [20] and [28].) $\mathrm{In}(\psi)$ is defined on those classes $x \in H^n(X)$, $y \in H^{n+1}(X)$, and $z \in H^{n+2^s-1}(X)$ for which

$$\begin{aligned} &Sq^1 x = 0, \qquad Sq^2 y + Sq^3 x = 0, \\ &Sq^4 Sq^1 z + Sq^{2^s+3} y + Sq^{2^s+4} x = 0, \\ &Sq^2 Sq^1 z + Sq^{2^s+2} x = 0. \end{aligned}$$

Then $\mathrm{Indet}^n(X; \psi) = \text{image } \mathrm{In}(\psi) + \lambda Sq^{2^s+4} H^{n+3}(X)$ in $H^{n+2^s+7}(X)$.

PROPOSITION 3.8. *$\psi$ is defined on $\alpha^{2^{s+1}+8}$ in $H^*(RP^\infty)$ and vanishes with zero indeterminacy.*

**Proof.** Clearly $\Phi_3(\beta^{2^s+4}) = 0$ and $\Phi_4(\beta^{2^s+4}) = 0$ in $H^*(CP^\infty)$. Let $g\colon CP^\infty \to QP^\infty$ be a map such that $g^*\gamma = \beta^2$ where $\gamma$ generates $H^*(QP^\infty)$. Now $\Phi_1(\gamma^{2^{s-1}+2}) = 0$ for dimensional reasons so $\Phi_1(\beta^{2^s+4}) = 0$ from naturality and zero indeterminacy. By Proposition 3.2

$$\Phi_2(\beta^{2^s+4}) = \Phi(2, 2^{s-2})(\beta^{2^s+4}) = \binom{8}{4}\beta^{2^s+2^{s-1}+6} = 0.$$

Thus $\psi$ is defined on $\beta^{2^s+4}$ and so on $\alpha^{2^{s+1}+8}$. Clearly $\psi$ vanishes on $\beta^{2^s+4}$ with zero indeterminacy so naturality under the Hopf map gives $0 \in \psi(\alpha^{2^{s+1}+8})$.

One checks that $\mathrm{In}\,\psi$ is defined with zero indeterminacy on

$$H^{2^{s+1}+8}(RP^\infty) \oplus H^{2^{s+1}+9}(RP^\infty) \oplus H^{2^{s+1}+2^s+7}(RP^\infty). \tag{3.9}$$

Let

$$\begin{aligned} &\Gamma_1\colon Sq^4 Sq^{2^s+4} + Sq^6 Sq^{2^s+2} + Sq^{2^s+5} Sq^3 + Sq^{2^s+7} Sq^1 = 0, \\ &\Gamma_2\colon Sq^4 Sq^{2^s+3} + Sq^{2^s+5} Sq^2 = 0, \\ &\Gamma_3\colon Sq^4(Sq^4 Sq^1) + Sq^6(Sq^2 Sq^1) = 0 \end{aligned}$$

denote any stable secondary operations associated to the above relations. Clearly

$\Gamma_1(\beta^{2^s+4})=0$ so it follows that $\Gamma_1(\alpha^{2^{s+1}+8})=0$ from naturality and zero indeterminacy. By [4, Theorem 5.1] there exists an $S$-map $\lambda$ such that

$$\lambda^*\colon H^{2q}(\textstyle\sum^{2d+1} RP^{2p+1}) \to H^{2q}(CP^{p+d+1}/CP^d)$$

is an isomorphism for all $q$ where $p=2^s+2^{s-1}+7$ and $d+1=t2^{s+1}$ for some positive integer $t$. By dimensionality $\Gamma_2(\gamma^{t2^s+2^{s-1}+2})=0$ in $H^*(QP^\infty)$ so naturality gives

$$\lambda^*\Gamma_2(\Sigma^{2d+1}\alpha^{2^{s+1}+9}) = \Gamma_2(\beta^{2^s+d+5}) = 0.$$

The stability of $\Gamma_2$ and zero indeterminacy imply that $\Gamma_2(\alpha^{2^{s+1}+9})=0$. Similarly,

$$\lambda^*\Gamma_3(\Sigma^{2d+1}\alpha^{2^{s+1}+2^s+7}) = \Gamma_3(\beta^{t2^{s+1}+2^s+2^{s-1}+3}) = \varphi\circ Sq^1(\beta^{t2^{s+1}+2^s+2^{s-1}+3}) = 0$$

where $\varphi$ is a secondary operation associated to the relation

$$Sq^6Sq^2+Sq^4Sq^4+Sq^7Sq^1 = 0.$$

Such a $\varphi$ can be chosen so that $\Gamma_3=\varphi\circ Sq^1$ modulo total indeterminacies by [1]. Thus $\Gamma_3(\alpha^{2^{s+1}+2^s+7})=0$ from stability and zero indeterminacy. We conclude that $\mathrm{In}(\psi)$ vanishes with zero indeterminacy on (3.9). Since $Sq^{2^s+4}\alpha^{2^{s+1}+11}=0$, $\mathrm{Indet}^{2^{s+1}+8}(RP^\infty;\psi)=0$ and the proof of 3.8 is complete.

4. **Generating class theorems.** Thomas formulates a "generating class theorem" for lifting a second-order $k$-invariant to the Thom complex and expressing it by means of a secondary operation applied to the Thom class in [Theorem 6.4, 29] and [Theorem 6.5, 30]. The proof of Theorem 1.4 in §5 uses the generating class theorem to express a third-order $k$-invariant by a tertiary operation so we state the following versions to cover that application. Let $B_m$ and $B$ denote $BSO(m)$ and $BSO$, $B\,\mathrm{Spin}\,(m)$ and $B\,\mathrm{Spin}$, or $BO[8](m)$ and $BO[8]$ where $BSO$, $B\,\mathrm{Spin}$, and $BO[8]$ are the 1, 3, and 7-connective coverings of $BO$. In the appendix Postnikov resolutions are constructed for the fiber map $\pi\colon B_m\to B$ through dimensions $\leqq t$ where $\pi^*$ is surjective and $m<t<2m$. Let $T$ and $U$ denote the Thom complex and Thom class of the universal bundle $\xi$ over $B$ and regard $B$ as $B_s$ for large $s$. Following the notation of [28], [29], [30], we let $T_Y$ and $U_Y$ denote the Thom complex and Thom class of $g^*\xi$ where $g\colon Y\to B$ is any map. Consider the following commutative diagram.

(4.1.)

$$\begin{tikzcd}
 & \Omega C \arrow[r, "j"] & E_2 \arrow[d, "p_2"] & \\
K(Z_2,m) \arrow[rr, "i"] & & E_1 \arrow[r, "k_1\times k_2\times\cdots\times k_r"] \arrow[d, "p_1"] & C \\
B_m \arrow[rr, "\pi"] \arrow[uurr, "q_2"] \arrow[urr, "q_1"] & & B_s \arrow[r, "w_{m+1}"] & K(Z_2,m+1)
\end{tikzcd}$$

The classes $k_j\in H^{t_j}(E_1)$ for $t_j\leqq t$ and $1\leqq j\leqq r$ are the second-order $k$-invariants in the resolution for $\pi$. Now $i^*k_1=\alpha_1\iota$ and $i^*k_2=\beta_1\iota$ for elements $\alpha_1$, $\beta_1$ in $A$. Suppose there are relations in $A$

$$\begin{aligned} \alpha_1 Sq^{m+1}+\sum_{j=2}^{n}\alpha_j\theta_j &= 0,\\ \beta_1 Sq^{m+1}+\sum_{j=2}^{n}\beta_j\theta_j &= 0 \end{aligned} \tag{4.2}$$

where $\theta_j U=0$ and degree $(\alpha_j)>0$, degree $(\beta_j)>0$ for $1<j\leqq n$. Let $\Omega$ denote any secondary operation associated to the relations (4.2) and let $C$ represent the coset in $H^{s+t_1}(T_B)\oplus H^{s+t_2}(T_B)$ of the indeterminacy subgroup of $\Omega$ such that $(T\pi)^*C=\Sigma^{s-m}\Omega(U_{B_m})$. Define $K$ to be the coset of the pair $(k_1, k_2)$ in $H^{t_1}(E_1)\oplus H^{t_2}(E_1)$ with respect to the subgroup

$$(\text{kernel } i^* \cap \text{kernel } q_1^* \cap H^{t_1}(E_1)) \oplus (\text{kernel } i^* \cap \text{kernel } q_1^* \cap H^{t_2}(E_1)).$$

With these assumptions the generating class theorem states

PROPOSITION 4.3. *There is a class* $(\hat{k}_1, \hat{k}_2)\in K$ *and a class* $(c_1, c_2)\in H^{t_1}(B)\oplus H^{t_2}(B)$ *such that* $U\cdot(c_1, c_2)\in C$ *and* $U_{E_1}\cdot((\hat{k}_1, \hat{k}_2)+p_1^*(c_1, c_2))\in \Omega(U_{E_1})$.

The proof of (4.3) is essentially given in [27] and [29] and so is omitted. Let $X$ be a complex of dimension $\leqq t$ and $f\colon X\to B$ a map with $f^*w_{m+1}=0$. Thus $f$ classifies a bundle $\rho$ over $X$ and one defines $(k_1, k_2)(\rho)=\bigcup_g (g^*k_1, g^*k_2)$, the union being over all liftings $g\colon X\to E_1$ of $f$. Recall from [28] there are classes $\hat{\alpha}_1$ and $\hat{\beta}_1$ in $A(B)$ such that $\mu(k_1)=\hat{\alpha}_1(\iota\otimes 1)$ in $H^{t_1}(K(Z_2, m)\times E_1, E_1)$, $\mu(k_2)=\hat{\beta}_1(\iota\otimes 1)$ in $H^{t_2}(K(Z_2, m)\times E_1, E_1)$. Thus $(k_1, k_2)(\rho)$ is a coset of $\mathrm{Indet}^{t_1,t_2}(X; K)=(\hat{\alpha}_1, \hat{\beta}_1)\cdot H^*(X)\cap(H^{t_1}(X)\oplus H^{t_2}(X))$.

COROLLARY 4.4. *Suppose the indeterminacy of* $\Omega(U_\rho)=U_\rho\cdot\mathrm{Indet}^{t_1,t_2}(X; K)$. *Suppose also that* $g^*(\hat{k}_1, \hat{k}_2)=g^*(k_1, k_2)$ *for any* $(\hat{k}_1, \hat{k}_2)$ *in* $K$ *and any lifting* $g\colon X\to E_1$ *of* $f$. *Then*

$$U_\rho\cdot((k_1, k_2)(\rho)+f^*(c_1, c_2)) = \Omega(U_\rho)$$

*as cosets in* $H^{s+t_1}(T_X)\oplus H^{s+t_2}(T_X)$.

In diagram (4.1) the map $p_2\colon E_2\to E_1$ is a principal fibration classified by the cohomology vector $(k_1, \ldots, k_r)$ where $C=\times_{i=1}^{r} K(Z_2, t_i)$. Assume now that $K$ consists only of $(k_1, k_2)$ and that $\Omega$ can be chosen so $U_{E_1}\cdot(k_1, k_2)\in\Omega(U_{E_1})$. Let $\iota_j$ denote the fundamental class of $\Omega K(Z_2, t_j)$ in $\Omega C$ for $1\leqq j\leqq r$. Let $k\in H^r(E_2)$ be a third-order $k$-invariant for $\pi$ (so $r\leqq t$) such that $j^*k=\gamma_1\iota_1+\gamma_2\iota_2$ where $\gamma_1$ and $\gamma_2$ are in $A$. Suppose $\Omega'=(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5)$ is a 5-valued stable secondary operation where $\Omega_1$ and $\Omega_2$ are the component operations belonging to $\Omega$ and the degree of $\Omega_i \leqq$ the connectivity of $B$ for $3 \leqq i \leqq 5$. Thus $(k_1, k_2, 0, 0, 0) \in \Omega'(U_{E_1})$. Assume also the following relation holds.

$$\gamma_1\Omega_1+\gamma_2\Omega_2+\gamma_3\Omega_3+\gamma_4\Omega_4+\gamma_5\Omega_5 = 0. \tag{4.5}$$

Let $\psi$ be a tertiary operation associated to relation (4.5). Let $D$ denote the coset of the indeterminacy subgroup of $\psi$ in $H^*(T_B)$ such that $\Sigma^{s-m}\psi(U_{B_m})=(T\pi)^*D$ and let $K'$ be the coset of $k$ with respect to the subgroup kernel $j^* \cap$ kernel $q_2^* \cap H^r(E_2)$. Under these assumptions the generating class theorem states

Proposition 4.6. *There is a class $\hat{k}$ in $K'$ and a class $d$ in $H^r(B)$ such that $U \cdot d \in D$ and $U_{E_2} \cdot (\hat{k}+(p_1 \circ p_2)^*d) \in \psi(U_{E_2})$.*

Proposition 4.6 is easily proved by applying the arguments of [27], [29], and [30]. See also [9]. An application of (4.6) is also given in [23].

## 5. **Proofs of immersion theorems.**

**Proof of Theorem 1.1.** Write $2n=4t+6$ and refer to Postnikov resolution I in the appendix. Let $v\colon CP^n \to B\,\mathrm{Spin}$ classify the stable normal bundle $v$ of $CP^n$ Now

$$\bar{w}_{2j}(CP^n) = \binom{n+j}{j}\beta^j$$

so $w_{4t+2}(v)=0$ and $w_{4t+4}(v)=0$. The indeterminacy of $k_4(v)=Sq^2H^{4t+2}(CP^n)$ $=H^{2n-2}(CP^n)$ so $v$ lifts to $B\,\mathrm{Spin}\,(4t+1)$ iff $v$ lifts to $E_2$ iff $k_2(v)=0$. Let $\Omega$ denote a stable secondary operation associated to the relation

$$(Sq^2Sq^1)Sq^{4t+2}+Sq^1Sq^{4t+4}+Sq^{4t+4}Sq^1 = 0$$

and chosen so that it vanishes on classes of dimension $<4t+2$ (see [2]). Applying the generating class theorem [30, Theorem 6.5] gives $U_v \cdot k_2(v)=\Omega(U_v)$ since the indeterminacy of $k_2(v)=0=$ the indeterminacy of $\Omega(U_v)$. Here $U_v$ is the Thom class of the Thom complex $T(v)$.

The order of $J(\eta)$ in $J(CP^n)$ is the Atiyah-Todd number $M_{n+1}$ by [31]. Set $s=M_{n+1}-(n+1)$. Since $2^n$ divides $M_{n+1}$, it follows that

$$\binom{s}{2^r} = 1 \text{ iff } \binom{n}{2^r} = 0 \quad \text{for } 0 \leqq r < n.$$

Write $s=ha+c$ where $c<n$ and $a$ is the smallest power of 2 greater than $n$. Atiyah-James duality for projective spaces in [5] states that an $S$-dual for $CP^n$ is the space $X=CP^m/CP^{s-1}=T(s\eta)$ for $s\eta$ based on $CP^{m-s}$ where $m=s+n-1$. Identify the generator of $H^{2s}(X)$ with $\beta^s$ under the collapsing map $CP^m \to X$ and the standard embedding $CP^m \to CP^\infty$. Since $\alpha(c+n-1)>\alpha(c)$, Proposition 3.2 states that

$$\Omega(\beta^s) = \Phi(1, 2t+1)(\beta^{ha+c}) = \binom{2c}{4t+4-a}\beta^{s+n-1}.$$

But $\binom{2n-2}{a}=1$ so $\binom{2c}{2n-2-a}=0$ for $\alpha(n)>2$. Thus $k_2(v)=0$ and the result follows by Hirsch [12]. Note for $\alpha(n)=2$ that

$$\binom{2c}{4t+4-a} = \binom{2}{0} = 1$$

which gives a nonimmersion result of [3].

**Proof of Theorem 1.3.** Refer to Postnikov resolution II in the appendix. Write $n=8t+12$ and let $\gamma\colon RP^n \to BSO$ classify the bundle $\gamma=(16t+18)\xi$ over $RP^n$. It suffices to show $\gamma$ lifts to $BSO(8t+5)$ by Proposition 2.1. Note that $w_{8t+6}(\gamma)=0$ and $w_{8t+8}(\gamma)=0$ from §2 so $\gamma$ lifts to $E_1$. The indeterminacy of $k_1^3(\gamma)=H^n(RP^n)$ so $\gamma$ lifts to $BSO(8t+5)$ iff $\gamma$ lifts to $E_3$. $Sq^1k_2^1$ occurs in the defining relation for $k_1^2$ so $0\in k_1^2(\gamma)$. Likewise, $Sq^1k_4^1$ occurs in the defining relation for $k_2^2$ so $0\in k_2^2(\gamma)$. Thus any lifting of $\gamma$ to $E_2$ can be altered through indeterminacies to produce a lifting of $\gamma$ to $E_3$.

We apply the technique of factoring a classifying map for an even multiple of the Hopf bundle over $RP^n$ through complex projective space in order to determine the second-order $k$-invariants for $\gamma$. Set $m=4t+6$ and let $v\colon CP^m \to BSO$ classify the bundle $v=(8t+9)\eta$. We regard $\gamma=v\circ H\colon RP^n \to BSO$ where $H$ is the Hopf map in §2. Trivially $k_1^1(v)=0$ and $k_3^1(v)=0$. Note that both $k_2^1(v)$ and $k_4^1(v)$ have zero indeterminacy. Choose a stable secondary operation $\varphi$ associated to the relation

$$(Sq^2Sq^1)Sq^{8t+6}+Sq^1Sq^{8t+8}+Sq^{8t+8}Sq^1 = 0$$

so that $\varphi$ vanishes on classes of dimension $\leqq 8t+5$. The generating class theorem [29, Theorem 6.4] gives $\varphi(U_v)=U_v\cdot k_2^1(v)$. But $\varphi(U_v)=2^{\alpha(t+1)-1}U_v\cdot\beta^{4t+4}$ by Proposition 3.3. Thus $k_2^1(v)\neq 0$ iff $\alpha(t+1)=1$ iff $n=2^r+4$ for $r\geqq 4$. Let $g\colon CP^m\to E_1$ be any lifting for $v$ and set $f=g\circ H$. Now $f^*(k_1^1, k_2^1, k_3^1)=(0, \alpha^{2^r}, 0)$ for $n=2^r+4$. The indeterminacy subgroup of $(k_1^1, k_2^1, k_3^1)(\gamma)$ is generated by $(\alpha^{2^r-1}, 0, \alpha^{2^r+1})$ and $(0, \alpha^{2^r}, \alpha^{2^r+1})$. So $f$ cannot be altered to produce a lifting of $\gamma$ to $E_2$. That is, $RP^n \not\subseteq R^{2n-7}$ for $n=2^r+4$ and $r>3$.

Let $h\colon CP^{m+1}\to E_1$ be any lifting for the bundle $v$ based on $CP^{m+1}$. The defining relation for $k_3^2$ gives $\beta^3\cdot h^*k_2^1+\beta\cdot h^*k_4^1=0$ in $H^{2m+2}(CP^{m+1})$. So $h^*k_2^1=0$ iff $h^*k_4^1=0$. It follows that $g^*k_2^1=0$ iff $g^*k_4^1=0$ for any lifting $g\colon CP^m\to E_1$ of $v$. Thus $f=g\circ H\colon RP^n\to E_1$ lifts to $E_2$ for $n\neq 2^r+4$.

**Proof of Theorem 1.4.** We consider the case $n\equiv 8 \bmod 16$. The proof for $n\equiv 0 \bmod 16$ is similar and so is omitted. Write $n=16t+8$ and refer to Postnikov resolution III. Let $v\colon CP^m\to B\,\mathrm{Spin}$ classify the bundle $v=(16t+4)\eta$ for $m=8t+4$. Let $\gamma=v\circ H\colon RP^n\to B\,\mathrm{Spin}$ classify the bundle $\gamma=(32t+8)\xi$ over $RP^n$. One checks easily that $\gamma$ lifts to $B\,\mathrm{Spin}\,(16t-1)$ if $v$ lifts to $E_2$. Now

$$w_{16t}(v) = \binom{16t+4}{8t}\beta^{8t} = 0$$

so $v$ lifts to $E_1$. The defining relation for $k_1^2$ gives $Sq^2k_1^1(v)=0$ so $k_1^1(v)=0$. The defining relation for $k_3^2$ gives $Sq^4k_2^1(v)=0$. But $Sq^4\beta^{8t+6}=\beta^{8t+8}$ so that $k_2^1(v)=0$. It follows that $v$ lifts to $E_2$ iff $k_4^1(v)=0$. We proceed to express by a secondary operation a class different from $k_4^1$ but equal to $k_4^1$ under pull-backs to $CP^m$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & & B\,\mathrm{Spin}(16t-1) \arrow[dl, "q"'] \arrow[dr, "q_1"] & \\
\Omega C \arrow[r, "j"] & E \arrow[rr, "r"] \arrow[dr, "p"'] & & E_1 \arrow[dl, "p_1"] \\
CP^m \arrow[ur, "g"] \arrow[rr, "v"] & & B\,\mathrm{Spin} &
\end{tikzcd}$$

Here $p\colon E \to B\,\mathrm{Spin}$ is the principal fibration with classifying map

$$w_{16t} \times W\colon B\,\mathrm{Spin} \to C = K(Z_2, 16t) \times K(Z_2, 16t+8)$$

where $W$ represents the class $w_4w_{16t+4}+w_6w_{16t+2}+w_4^2w_{16t}$ in $H^*(B\,\mathrm{Spin})$. Let $\iota_1$ and $\iota_2$ denote the fundamental classes of the components of $\Omega C$. The following exact sequence holds for $j \leqq 16t+15$ from [26].

$$0 \longrightarrow H^j(E) \xrightarrow{\nu^*} H^j(\Omega C \times B\,\mathrm{Spin}(16t-1)) \xrightarrow{\tau_1} H^{j+1}(B\,\mathrm{Spin}).$$

Regard $k_4^1$ as a class in $H^*(E)$ via $r^*$ and recall that

$$\begin{aligned}\nu^*k_4^1 = {} & Sq^8Sq^1\iota_1 \otimes 1 \otimes 1 + Sq^1\iota_1 \otimes 1 \otimes w_8 + Sq^5\iota_1 \otimes 1 \otimes w_4 \\ & + Sq^1\iota_1 \otimes 1 \otimes w_4^2 + Sq^3\iota_1 \otimes 1 \otimes w_6 + Sq^2\iota_1 \otimes 1 \otimes w_7.\end{aligned}$$

Now

$$\tau_1(1 \otimes Sq^1\iota_2 \otimes 1) = Sq^1W = \tau_1[Sq^1(Sq^4\iota_1 \otimes 1 \otimes w_4+Sq^2\iota_1 \otimes 1 \otimes w_6)].$$

Let $z$ be the unique class in $H^{16t+8}(E)$ for which

$$\nu^*z = Sq^8Sq^1\iota_1 \otimes 1 \otimes 1+Sq^1\iota_1 \otimes 1 \otimes w_8+1 \otimes Sq^1\iota_2 \otimes 1+Sq^1\iota_1 \otimes 1 \otimes w_4^2.$$

Let $y$ be the unique class in $H^{16t+7}(E)$ for which

$$\nu^*y = 1 \otimes \iota_2 \otimes 1+Sq^2\iota_1 \otimes 1 \otimes w_6+Sq^4\iota_1 \otimes 1 \otimes w_4.$$

Since $\nu^*(Sq^1y)=\nu^*(z+k_4^1)$, it follows that $k_4^1=z+Sq^1y$. Choose a stable secondary operation $\Phi$ associated to the relation $(Sq^8Sq^1)Sq^{16t}+Sq^{16t+8}Sq^1+Sq^{16t+7}Sq^2+Sq^1(Sq^{16t+4}Sq^4)=0$ such that $\Phi$ vanishes on classes having dimension $<16t$. Note that $Sq^{16t+4}Sq^4U=U\cdot W$ in $H^*(T_{B\,\mathrm{Spin}})$. By [29, Theorem 6.4] $U_E\cdot(z+k')\in\Phi(U_E)$ for some class $k'$ in $H^{16t+8}(E) \cap \operatorname{kernel} j^* \cap \operatorname{kernel} q^*$. It follows that $\Phi(U_v)=U_v\cdot z(v)=U_v\cdot k_4^1(v)$. Let $\delta$ denote the generator for $H^*(QP^\infty)$ and $\rho$ the Hopf line bundle over $QP^\infty$. We regard $\delta^{8t+2}$ as the Thom class of the bundle

$\zeta=(8t+2)\rho$ based on $QP^l$ for large $l$. The highest power of 2 dividing the Chern class $c_{8t}(\zeta)$ is $2^{\alpha(t)}$ from §2. By Proposition 3.5

$$\Phi(U_\zeta) = 2^{\alpha(t)-1}Sq^8(U_\zeta\cdot\delta^{4t}).$$

But $Sq^8\delta^{12t+2}=\delta^{12t+4}$ so $\Phi(\delta^{8t+2})=0$ iff $\alpha(t)>1$ iff $n\neq 2^r+8$. Naturality under the Hopf map $CP^\infty\to QP^\infty$ shows that $\Phi(\beta^{16t+4})=0$ iff $n\neq 2^r+8$. Thus $k_4^1(v)=0$ for $n\neq 2^r+8$ from identifying $U_v$ with $\beta^{16t+4}$. Since $\gamma$ has a lifting $f\circ H\colon RP^n\to E_2$ where $f\colon CP^m\to E_2$ is a lifting for $v$, clearly $k_3^2(\gamma)=0$ for $n\neq 2^r+8$. One checks indeterminacies and defining relations to show that $\gamma$ lifts to $B\,\mathrm{Spin}(16t-1)$ iff $\gamma$ has a lifting to $E_2$ and $k_3^2(\gamma)=0$. Thus $\gamma$ lifts to $B\,\mathrm{Spin}(16t-1)$ and the result follows from (2.1) for $n\neq 2^r+8$. For $n=2^r+8$ we express the obstruction $k_3^2(\gamma)$ by a tertiary operation.

We assume now that $n=2^r+8$ for $r>3$. The natural map $BO[8]\to B$ Spin induces a Postnikov resolution for the fiber map $\pi'\colon BO[8](2^r-1)\to BO[8]$ from Postnikov resolution III for the map $\pi$. We denote the $k$-invariants for $\pi'$ also by $k_j^i$ and the spaces in the resolution by $E_i$. Thus $k_2^1$ in $H^*(E_1)$ has the defining relation $Sq^4Sq^1w_{2^r}=0$ in $H^*(BO[8])$ and $k_3^2$ has the defining relation $Sq^6k_1^1+Sq^4k_2^1=0$ in $H^*(E_1)$. Since $Sq^2k_1^1=0$ and $BO[8]$ is 7-connected, the coset $K$ of $(k_1^1, k_2^1)$ defined in §4 contains only $(k_1^1, k_2^1)$. Let $\Phi=(\Phi_1, \Phi_2)$ be the double secondary operation with component operations $\Phi_i$ chosen in (3.6). By Proposition 4.3

$$U_{E_1}\cdot(k_1^1, k_2^1)\in\Phi(U_{E_1}).$$

Let $\psi$ be any tertiary operation associated to the relation (3.7).

By Proposition 4.6

$$U_{E_2}\cdot(\hat{k}_3^2+(p_1\circ p_2)^*P)\in\psi(U_{E_2}).$$

Here $\hat{k}_3^2$ belongs to the coset $K'$ in (4.6) determined by $k_3^2$, and $P$ is a class in $H^{2^r+7}(BO[8])$ such that $U'\cdot P\in\psi(U')$ where $U'$ denotes the Thom class of the universal bundle over $BO[8](2^r-1)$.

Let $h\colon RP^n\to E_2$ be any lifting for the map $\gamma\colon RP^n\to BO[8]$ classifying the bundle $\gamma=(2^{r+1}+8)\xi$. Now $h^*\hat{k}_3^2=h^*k_3^2$ since $k_1^2(\gamma)=0$ and $Sq^2Sq^1h^*k_2^2=Sq^3h^*k_2^2=0$. Clearly $P(\gamma)=0$ so we conclude $U_\gamma\cdot k_3^2(\gamma)\in\psi(U_\gamma)$.

Identify $U_\gamma$ with $\alpha^{2^{r+1}+8}$ in $H^*(RP^\infty)$ and apply Proposition 3.8 to give $k_3^2(\gamma)=0$. Thus $\gamma$ lifts to $BO[8](2^r-1)$ and the result follows by (2.1).

**Proof of Theorem 1.6.** Let $\gamma\colon RP^n\to BSO$ classify the bundle $\gamma=2p\xi=(2^{\phi(n)}-(n+1))\xi$. The argument that $\gamma$ lifts to $BSO(n-8)$ for $n\equiv 5 \bmod 8$ and $\alpha(n)>3$ is similar to the proof of Theorem 1.3 and so is omitted. We consider the case $n\equiv 1 \bmod 8$ and $\alpha(n)>3$. Write $n=8t+9$ and refer to Postnikov resolution IV. By (2.1) it suffices to show $\gamma$ lifts to $BSO(8t+1)$. Let $v\colon CP^m\to BSO$ classify the bundle $v=p\eta$ where $m=4t+4$. One checks easily that $\gamma$ lifts to $BSO(8t+1)$ iff $k_4^2(\gamma)=0$. Clearly $k_4^2(\gamma)=0$ if $v$ lifts to $E_2$. Note that $v$ and hence $\gamma$ lift to $E_1$ by §2.

Let $h: CP^{m+1} \to E_1$ be a lifting for the bundle $v$ based on $CP^{m+1}$. The defining relation for $k_4^2$ gives $\beta \cdot Sq^4(h^* k_2^1) = 0$ in $H^{2m+2}(CP^{m+1})$. But $Sq^4 \beta^{4t+2} = \beta^{4t+4}$ so $h^* k_2^1 = 0$. Thus $k_2^1(v) = 0$ and $v$ lifts to $E_2$ iff $k_5^1(v) = 0$. Consider the following commutative diagram.

$$\begin{tikzcd}
& & BSO(8t+1) \arrow[dl, "q"'] \arrow[dr, "q_1"] & \\
\Omega C \arrow[r, "j"] & E \arrow[rr, "r"] \arrow[dr, "p"] & & E_1 \arrow[dl, "p_1"] \\
CP^m \arrow[ur, "g"] \arrow[rr, "v"] & & BSO &
\end{tikzcd}$$

Here $p: E \to BSO$ is the principal fibration with classifying map

$$w_{8t+2} \times w_{8t+4} \times w_{8t+8} \times W: BSO \to C$$

where $W$ represents the class $w_2 w_{8t+6} + w_3 w_{8t+5} + w_2^2 w_{8t+4}$ in $H^*(BSO)$. The following exact sequence from [26] holds for $j \leqq 8t+15$

$$0 \longrightarrow H^j(E) \xrightarrow{\nu^*} H^j(\Omega C \times BSO(8t+1)) \xrightarrow{\tau_1} H^{j+1}(BSO).$$

Let $\iota_j$ for $1 \leqq j \leqq 4$ denote the fundamental classes of the components of $\Omega C$. Now

$$\begin{aligned}
\nu^* k_5^1 = {} & 1 \otimes 1 \otimes Sq^1 \iota_3 \otimes 1 \otimes 1 \\
& + 1 \otimes Sq^4 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_4 \\
& + 1 \otimes Sq^2 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_2 + Sq^1(1 \otimes Sq^2 \iota_2 \otimes 1 \otimes 1 \otimes w_2).
\end{aligned}$$

Let $y$ be the unique class in $H^{8t+7}(E)$ such that

$$\begin{aligned}
\nu^* y = {} & 1 \otimes 1 \otimes 1 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2 \iota_2 \otimes 1 \otimes 1 \otimes w_2 \\
& + 1 \otimes Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_3.
\end{aligned}$$

Define $z$ in $H^*(E)$ so that

$$\begin{aligned}
\nu^* z = {} & 1 \otimes 1 \otimes Sq^1 \iota_3 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes Sq^1 \iota_4 \otimes 1 \\
& + 1 \otimes Sq^4 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_4 \\
& + 1 \otimes Sq^2 Sq^1 \iota_2 \otimes 1 \otimes 1 \otimes w_2.
\end{aligned}$$

Thus $k_5^1 = z + Sq^1 y$ since $\nu^*(Sq^1 y) = \nu^*(z + k_5^1)$.

By [2] we can select a secondary operation $\Gamma$ associated to the relation

$$(Sq^4 Sq^1) Sq^{8t+4} + Sq^1 Sq^{8t+8} + Sq^1(Sq^{8t+6} Sq^2) + Sq^{8t+8} Sq^1 = 0$$

so that $\lambda Sq^2 x \cdot Sq^3 x \in \Gamma(x)$ for any class $x$ of dimension $8t+3$ in the domain of $\Gamma$

and such that $\lambda \in Z_2$ is independent of $x$. Note that $Sq^{8t+6}Sq^2U = U \cdot W$ in $H^*(T_{BSO})$. The generating class theorem [30, Theorem 6.5] gives the result

$$U_E \cdot (z+k') \in \Gamma(U_E)$$

for some class $k'$ in $H^{8t+8}(E) \cap \operatorname{kernel} j^* \cap \operatorname{kernel} q^*$. It follows that $\Gamma(U_v) = U_v \cdot z(v) = U_v \cdot k_5^1(v)$. One checks from §2 that the highest power of 2 dividing $c_1(v)$ is 2, dividing $c_{4t+2}(v)$ is $2^{\alpha(n)-2}$, and dividing $c_{4t+4}(v)$ is $2^{\alpha(n)-1}$. By Proposition 3.4

$$\Gamma(U_v) = 2^{\alpha(n)-3}U_v \cdot \beta^{4t+4}.$$

Thus $k_5^1(v) = 0$ for $\alpha(n) > 3$ so $v$ lifts to $E_2$ and the proof is complete.

6. **Appendix.** These Postnikov resolutions for the fiber map $\pi: B_m \to B$ are constructed by the techniques of [26]. We refer the reader also to [17] and [8] for the theory and construction of modified Postnikov resolutions. The homotopy groups of the fibers for $\pi$ appear in [13] and [22]. The tower of spaces is displayed only for resolution I. The $k$-invariant $k_j^i$ represents a class in $H^*(E_i)$ whose defining relation is a relation in $H^*(E_{i-1})$ where $E_0 = B$.

6.1. Postnikov resolution I for the fibration $\pi: B\,\mathrm{Spin}(4t+1) \to B\,\mathrm{Spin}$ for stable spin bundles over complexes of dimension $\leqq 4t+6$ for $t > 1$. $K(n)$ denotes $K(Z_2, n)$.

$$\begin{array}{lll}
B\,\mathrm{Spin}(4t+1) & & \\
\downarrow q_3 & & \\
E_3 & & \\
\downarrow p_3 & & \\
E_2 & \xrightarrow{k_4} & K(4t+4) \\
\downarrow p_2 & & \\
E_1 & \xrightarrow{k_1 \times k_2 \times k_3} & K(4t+3) \times K(4t+4) \times K(4t+5) \\
\downarrow p_1 & & \\
B\,\mathrm{Spin} & \xrightarrow{w_{4t+2} \times w_{4t+4}} & K(4t+2) \times (K(4t+4).
\end{array}$$

Defining relations for $k$-invariants:

$$\begin{aligned}
&k_1\colon Sq^2 w_{4t+2} = 0,\\
&k_2\colon (Sq^2Sq^1)w_{4t+2} + Sq^1 w_{4t+4} = 0,\\
&k_3\colon (Sq^4 + \cdot w_4)w_{4t+2} + tSq^2 w_{4t+4} = 0,\\
&k_4\colon Sq^2 k_1 + Sq^1 k_2 = 0.
\end{aligned}$$

6.2. Postnikov resolution II for the fibration $\pi: BSO(8t+5) \to BSO$ for stable orientable bundles over complexes of dimension $\leqq 8t+13$ for $t>0$. Defining relations for $k$-invariants:

$$k_1^0 = w_{8t+6},$$
$$k_2^0 = w_{8t+8},$$
$$k_1^1\colon (Sq^2+\cdot w_2)w_{8t+6} = 0,$$
$$k_2^1\colon (Sq^2+\cdot w_2)Sq^1w_{8t+6}+Sq^1w_{8t+8} = 0,$$
$$k_3^1\colon (Sq^4+\cdot w_4)w_{8t+6}+Sq^2w_{8t+8} = 0,$$
$$k_4^1\colon (Sq^4+\cdot w_4+\cdot w_2^2)Sq^1w_{8t+8}+Sq^1(w_2\cdot Sq^2)w_{8t+8} = 0,$$
$$t \text{ even } k_5^1\colon (Sq^8+\cdot w_8)w_{8t+6}+w_6\cdot w_{8t+8} = 0,$$
$$t \text{ odd } k_5^1\colon (Sq^8+\cdot w_8)w_{8t+6}+(Sq^6+w_4\cdot Sq^2+w_2\cdot Sq^4)w_{8t+8} = 0,$$
$$k_1^2\colon (Sq^2+\cdot w_2)k_1^1+Sq^1k_2^1 = 0,$$
$$k_2^2\colon Sq^1k_4^1+(Sq^2Sq^3+\cdot w_5)k_2^1+Sq^2(w_2\cdot Sq^1k_2^1) = 0,$$
$$\begin{aligned} k_3^2\colon & (Sq^2+\cdot w_2)k_4^1+(Sq^2Sq^3+w_2\cdot Sq^2Sq^1+\cdot w_5)k_3^1 \\ & +Sq^1(w_2\cdot Sq^2k_3^1)+Sq^4(k_1^1\cdot w_3) \\ & +(Sq^7+Sq^4Sq^2Sq^1+w_4\cdot Sq^2Sq^1+w_6\cdot Sq^1+w_2^2\cdot Sq^2Sq^1+\cdot w_3w_2^2)k_1^1 \\ & +k_1^1\cdot Sq^3w_4+(Sq^4+Sq^3Sq^1)(k_2^1\cdot w_2) \\ & +(Sq^6+w_4\cdot Sq^2+w_3\cdot Sq^2Sq^1+\cdot w_6+\cdot w_2^3+\cdot w_3^2)k_2^1 \\ & +Sq^1(k_2^1\cdot w_2w_3) = 0,\end{aligned}$$
$$k_1^3\colon Sq^1k_2^2+Sq^3(k_1^2\cdot w_2)+(Sq^2Sq^3+w_2\cdot Sq^2Sq^1+w_2^2\cdot Sq^1+\cdot w_5)k_1^2 = 0.$$

6.3. Postnikov resolution III for the fibration $\pi\colon B\,\mathrm{Spin}(16t-1)\to B\,\mathrm{Spin}$ for stable spin bundles over complexes of dimension $\leqq 16t+8$ for $t>0$.

Defining relations for $k$-invariants:

$$k_1^0 = w_{16t},$$
$$k_1^1\colon Sq^2Sq^1w_{16t} = 0,$$
$$k_2^1\colon (Sq^4+\cdot w_4)Sq^1w_{16t} = 0,$$
$$k_3^1\colon (Sq^4+\cdot w_4)Sq^2w_{16t} = 0,$$
$$k_4^1\colon (Sq^8+\cdot w_8)Sq^1w_{16t}+Sq^1(w_4\cdot Sq^4+w_6\cdot Sq^2+\cdot w_4^2)w_{16t} = 0,$$
$$k_1^2\colon Sq^2k_1^1 = 0,$$
$$k_2^2\colon Sq^2Sq^1k_1^1+Sq^1k_1^2 = 0,$$
$$k_3^2\colon (Sq^6+\cdot w_6)k_1^1+(Sq^4+\cdot w_4)k_2^1 = 0,$$
$$k_4^2\colon (Sq^4+Sq^3Sq^1+\cdot w_4)k_3^1+(Sq^6+\cdot w_6)Sq^1k_1^1+(w_4\cdot Sq^1)k_2^1 = 0,$$
$$k_5^2\colon Sq^1k_4^1+(Sq^7+Sq^4Sq^2Sq^1)k_1^1 = 0,$$
$$k_1^3\colon Sq^2k_1^2+Sq^1k_2^2 = 0,$$
$$k_2^3\colon Sq^1k_5^2+Sq^4Sq^1k_2^2 = 0,$$
$$k_1^4\colon Sq^1k_2^3+Sq^2Sq^3k_1^3 = 0.$$

6.4. Postnikov resolution IV for the fibration $\pi\colon BSO(8t+1)\to BSO$ for stable orientable bundles over complexes of dimension $\leqq 8t+9$ for $t>1$.

Defining relations for $k$-invariants:

$k_1^0 = w_{8t+2}\ k_2^0 = w_{8t+4}\ k_3^0 = w_{8t+8},$

$k_1^1$: $(Sq^2 + \cdot w_2)w_{8t+2} = 0,$

$k_2^1$: $(Sq^2 + \cdot w_2)Sq^1 w_{8t+2} + Sq^1 w_{8t+4} = 0,$

$k_3^1$: $(Sq^4 + \cdot w_4)w_{8t+2} + w_2 \cdot w_{8t+4} = 0,$

$k_4^1$: $(Sq^4 + \cdot w_4)w_{8t+4} = 0,$

$k_5^1$: $Sq^1 w_{8t+8} + (Sq^4 + \cdot w_4)Sq^1 w_{8t+4} + (w_2 \cdot Sq^2 Sq^1)w_{8t+4}$
$\quad + Sq^1(w_2 \cdot Sq^2)w_{8t+4} = 0,$

$t$ even $k_6^1$: $(Sq^8 + \cdot w_8)w_{8t+2} + w_2 \cdot w_{8t+8} + (w_4 \cdot Sq^2 + \cdot w_6 + \cdot w_2 w_4)w_{8t+4} = 0,$

$t$ odd $k_6^1$: $(Sq^8 + \cdot w_8)w_{8t+2} + Sq^2 w_{8t+8} + (w_4 \cdot Sq^2 + \cdot w_6 + \cdot w_2 w_4)w_{8t+4} = 0,$

$k_7^1$: $(Sq^4 + \cdot w_4)(Sq^2 + \cdot w_2)w_{8t+4} + Sq^2 w_{8t+8} = 0,$

$k_1^2$: $(Sq^2 + \cdot w_2)k_1^1 + Sq^1 k_2^1 = 0,$

$k_2^2$: $Sq^1 k_5^1 + Sq^1(w_2 \cdot Sq^2)k_2^1 + (Sq^2 Sq^3 + w_2 \cdot Sq^2 Sq^1 + \cdot w_5)k_2^1 = 0,$

$k_3^2$: $k_4^1 \cdot w_2 + (Sq^6 + \cdot w_6)k_1^1 + (Sq^4 + Sq^3 Sq^1 + w_2 \cdot Sq^2 + w_3 \cdot Sq^1 + \cdot w_4 + \cdot w_2^2)k_3^1$
$\quad + Sq^2(k_2^1 \cdot w_3) + (w_2 \cdot Sq^3 + w_2 \cdot Sq^2 Sq^1 + w_4 \cdot Sq^1 + w_2^2 \cdot Sq^1)k_2^1 = 0,$

$k_4^2$: $(Sq^2 + \cdot w_2)Sq^1 k_4^1 + Sq^2 Sq^1(k_3^1 \cdot w_2)$
$\quad + (Sq^2 Sq^3 + w_2 \cdot Sq^3 + w_2^2 \cdot Sq^1 + \cdot w_2 w_3)k_3^1$
$\quad + (Sq^6 + w_2 \cdot Sq^4 + w_4 \cdot Sq^2 + \cdot w_6 + \cdot w_3^2)k_2^1 + Sq^3(k_2^1 \cdot w_3)$
$\quad + Sq^4(k_1^1 \cdot w_3) + (w_2^2 \cdot Sq^1)(k_1^1 \cdot w_2)$
$\quad + (w_2 \cdot Sq^2 Sq^3 + Sq^7 + Sq^4 Sq^2 Sq^1 + w_4 \cdot Sq^2 Sq^1$
$\qquad\qquad + w_6 \cdot Sq^1 + \cdot w_7 + \cdot w_3 w_4)k_1^1 = 0,$

$k_1^3$: $Sq^1 k_2^2 + Sq^1(w_2 \cdot Sq^2)k_1^2 + (Sq^2 Sq^3 + w_2 \cdot Sq^2 Sq^1 + \cdot w_5)k_1^2 = 0.$

6.5. Postnikov resolution V for the fibration $\pi$: $B\,\mathrm{Spin}\,(16t+7) \to B\,\mathrm{Spin}$ for stable spin bundles over complexes of dimension $\leqq 16t+16$ for $t>0$.

Defining relations for $k$-invariants:

$k_1^0 = w_{16t+8}, \quad k_2^0 = w_{16t+16},$

$k_1^1$: $Sq^2 Sq^1 w_{16t+8} = 0,$

$k_2^1$: $(Sq^4 + \cdot w_4)Sq^1 w_{16t+8} = 0,$

$k_3^1$: $(Sq^8 + \cdot w_8)w_{16t+8} = 0,$

$k_4^1$: $(Sq^4 + \cdot w_4)Sq^2 w_{16t+8} = 0,$

$k_5^1$: $(Sq^8 + \cdot w_8)Sq^1 w_{16t+8} + Sq^1 w_{16t+16}$
$\quad + Sq^1(w_4 \cdot Sq^4 + w_6 \cdot Sq^2 + \cdot w_4^2)w_{16t+8} = 0,$

$k_1^2$: $Sq^2 k_1^1 = 0,$

$k_2^2$: $Sq^2 Sq^1 k_1^1 + Sq^1 k_2^1 = 0,$

$k_3^2$: $(Sq^6 + \cdot w_6)k_1^1 + (Sq^4 + \cdot w_4)k_2^1 = 0,$

$k_4^2$: $(Sq^4 + Sq^3 Sq^1 + \cdot w_4)k_4^1 + Sq^5 k_2^1 + (Sq^7 + Sq^6 Sq^1 + \cdot w_7)k_1^1 = 0,$

$k_5^2$: $Sq^1 k_5^1 + Sq^4 Sq^1 k_2^1 + Sq^7 k_1^1 = 0,$

$k_1^3$: $Sq^2 k_1^2 + Sq^1 k_2^2 = 0,$

$k_2^3$: $Sq^1 k_5^2 + Sq^2 Sq^3 k_3^2 = 0,$

$k_1^4$: $Sq^1 k_2^3 + Sq^2 Sq^3 k_1^3 = 0.$

## REFERENCES


1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104. MR **25** #4530.

2. J. Adem, *Sobre operaciones cohomologicas secundarias*, Bol. Soc. Mat. Mexicana (2) **7** (1962), 95–110. MR **29** #5243.

3. J. Adem and S. Gitler, *Secondary characteristic classes and the immersion problem*, Bol. Soc. Mat. Mexicana (2) **8** (1963), 53–78. MR **29** #5255.

4. ———, *Non-immersion theorems for real projective spaces*, Bol. Soc. Mat. Mexicana (2) **9** (1964), 37–50. MR **32** #461.

5. M. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310. MR **24** #A1727.

6. S. Feder, *Non-immersion theorems for complex and quaternionic projective spaces*, Bol. Soc. Mat. Mexicana (2) **11** (1966), 62–67. MR **38** #721.

7. S. Gitler, *The projective Stiefel manifolds*. II: *Applications*, Topology **7** (1968), 47–53. MR **36** #3373b.

8. S. Gitler and M. Mahowald, *Geometric dimension of real stable vector bundles*, Bol. Soc. Mat. Mexicana (2) **11** (1966), 85–107. MR **37** #6922.

9. ———, *The immersion of manifolds*, Bull. Amer. Math. Soc. **73** (1967), 696–700. MR **35** #4940.

10. S. Gitler, M. Mahowald and R. J. Milgram, *Secondary cohomology operations and complex vector bundles*, Proc. Amer. Math. Soc. **22** (1969), 223–229.

11. ———, *The nonimmersion problem for $RP^n$ and higher-order cohomology operations*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 432–437. MR **37** #3581.

12. M. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276. MR **22** #9980.

13. C. S. Hoo and M. Mahowald, *Some homotopy groups of Stiefel manifolds*, Bull. Amer. Math. Soc. **71** (1965), 661–667. MR **31** #1675.

14. A. Hughes and E. Thomas, *A note on certain secondary cohomology operations*, Bol. Soc. Mat. Mexicana (1) **13** (1968), 1–17.

15. D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966. MR **37** #4821.

16. I. M. James, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc. **69** (1963), 231–238. MR **26** #1900.

17. M. Mahowald, *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc. **110** (1964), 315–349. MR **28** #620.

18. ———, *Some Whitehead products in $S^n$*, Topology **4** (1965), 17–26. MR **31** #2724.

19. M. Mahowald and F. Peterson, *Secondary cohomology operations on the Thom class*, Topology **2** (1963), 367–377. MR **28** #612.

20. C. R. F. Maunder, *Cohomology operations of the Nth kind*, Proc. London Math. Soc. (31) **13** (1963), 125–154. MR **35** #2279.

21. R. J. Milgram, *Immersing projective spaces*, Ann. of Math. (2) **85** (1967), 473–482. MR **35** #2293.

22. G. Paechter, *The groups $\pi_r(V^{n,m})$*. I, Quart. J. Math. Oxford Ser. (2) **7** (1956), 249–268. MR **24** #A1725.

23. A. D. Randall, *Some immersions of manifolds* (to appear)

24. B. Sanderson, *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. (3) **14** (1964), 137–153. MR **29** #2814.

25. J.-P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198–232. MR **15**, 643.

26. E. Thomas, *Seminar on fiber spaces*, Lecture Notes in Math., no. 13, Springer-Verlag, Berlin, 1966. MR **34** #3582.

27. E. Thomas, *Postnikov invariants and higher order cohomology operations*, Ann. of Math. (2) **85** (1967), 184–217. MR **35** #1029.

28. ———, *Real and complex vector fields on manifolds*, J. Math. Mech. **16** (1967), 1183–1205. MR **35** #1030.

29. ———, *The index of a tangent* 2-*field*, Comment. Math. Helv. **42** (1967), 86–110. MR **35** #6158.

30. ———, *The span of a manifold*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 225–244. MR **38** #2804.

31. G. Walker and J. F. Adams, *On complex Stiefel manifolds*, Proc. Cambridge Philos. Soc. **61** (1965), 81–103. MR **30** #1516.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA