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ON THE COUNTING FUNCTION FOR THE a-VALUES OF A MEROMORPHIC FUNCTION

BY
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Introduction. If $f(z)$ is a nonconstant meromorphic function in $|z|<\infty$, we let $n(r, a)$ denote the number of roots counting multiplicities of the equation $f(z)=a$ in $|z|\leqq r$. Our principal result is an "unintegrated" analogue for $n(r, a)$ of the theorem which asserts that the Valiron deficient values of $f(z)$ have inner capacity zero. Our result contains both an exceptional set of a-values and an exceptional set of r-values. We also obtain a result on $\sup_a n(r, a)$ having an exceptional set of a-values which bears on a question of Hayman and Stewart. We show by examples that all the exceptional sets in our results are in general nonempty. One of our examples also shows that the exceptional set of r-values in Ahlfors' theory of covering surfaces is in general nonempty.

1. **Terminology and notation.** We assume the reader is familiar with such standard notation of Nevanlinna theory as $m(r, a)$, $N(r, a)$, and $T(r,f)$, as well as with the definitions of Nevanlinna and Valiron deficient values.

We let Σ denote the Riemann sphere. If $f(z)$ is meromorphic in $|z|<\infty$ then the mean covering number of the map $f\colon |z|\leqq r\to\Sigma$ is defined by

$$S(r) = \int_\Sigma n(r, a)\, dm(a),$$

where m denotes normalized area measure on Σ. In general the mean covering number of any domain $D\subset\Sigma$ is

$$S(r, D) = \frac{1}{m(D)}\int_D n(r, a)\, dm(a).$$

It is of fundamental importance that the spherical characteristic of $f(z)$ defined by

$$T_0(r) = \int_0^r \frac{S(t)}{t}\, dt$$

has the property that

$$|T_0(r)-T(r,f)| = 0(1) \qquad (r\to\infty). \tag{1.1}$$

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Because of (1.1), $T_0(r)$ and $T(r, f)$ can be used interchangeably for many purposes. In this paper we shall use the spherical characteristic and for notational convenience denote it by $T(r)$.

If $E \subset [1, \infty)$, the logarithmic measure of E is defined by $m_l(E) = \int_E dt/t$. For $r > 1$ we denote $E \cap [1, r)$ by E^r. The upper (lower) logarithmic density of E is defined to be

$$\limsup_{r \to \infty} (\inf) \frac{m_l(E^r)}{\log r}.$$

If $\lim_{r \to \infty} (m_l(E^r)/\log r)$ exists, it is called the logarithmic density of E.

Many of our inequalities hold only for sufficiently large values of the variable, denoted by $r > r_0$ or $n > n_0$. It is not intended that r_0 and n_0 have the same value each time they occur.

2. **Discussion of results.** Throughout this paper our concern is with the functional $n(r, a)$. We show in our principal result (Theorem 2) that if $f(z)$ is a nonconstant meromorphic function in $|z| < \infty$, then there exists a set A_2 in the complex plane having inner capacity zero and there exists $E_2 \subset [1, \infty)$ having logarithmic density zero such that $\lim_{r \to \infty; r \notin E_2} (n(r, a)/S(r)) = 1$ for all $a \notin A_2$. Given $\varepsilon > 0$, we show that the above limit is uniform off a set A_3 such that $\operatorname{cap}\{a : a \in A_3 \text{ and } |a| \leqq 1\} < \varepsilon$ and $\operatorname{cap}\{a^{-1} : a \in A_3 \text{ and } |a| > 1\} < \varepsilon$. We thus prove in Theorem 3 that $\sup_{a \notin A_3} n(r, a) < (1+\varepsilon)S(r)$ for all r in a set of logarithmic density 1.

It is of interest to compare Theorem 3 with the following result of Hayman and Stewart [2].

THEOREM. *If $f(z)$ is meromorphic in $|z| < \infty$ and $\varepsilon > 0$, then there exists a set of r-values having positive lower logarithmic density on which*

$$\sup_{a \in \Sigma} n(r, a) < (1+\varepsilon)eS(r).$$

We observe that Theorem 3 applies to a larger set of r-values than does the Hayman-Stewart result and also has a smaller upper bound; however Theorem 3 involves an exceptional set of a-values. Hayman and Stewart ask whether the factor e can be removed from the bound in their result. Theorem 3 answers this question affirmatively provided we omit from consideration a small set of a-values.

In §4 we are concerned with the existence of both the exceptional sets of a-values and the exceptional sets of r-values in Theorems 2 and 3. Example 1 demonstrates the existence of an entire function $f(z)$ and a set $\tilde{E}$ having positive lower logarithmic density such that $\liminf_{r \to \infty; r \in \tilde{E}} (n(r, 0)/S(r)) > 1$; thus in this example for sufficiently small values of ε the exceptional set A_3 of Theorem 3 is nonempty. Since f is an open mapping we observe that for such values of ε the set A_3 necessarily contains a nonempty open set and hence has positive capacity. Example 1 also shows that $\{a : \limsup_{r \to \infty; r \notin E_2} (n(r, a)/S(r)) > 1\} \subset A_2$ is in general nonempty. It is known [4] that there exists an entire function $f(z)$ such that $T(r)/r$ is bounded away from 0 and ∞ and such that $f(z)$ has uncountably many Valiron deficient values. We show by very elementary methods that for this function every Valiron deficient value belongs to A_2 and hence A_2 is uncountable.

Example 2 demonstrates the existence of an entire function $f(z)$, a disk D in the plane, and a sequence $r_n \to \infty$ such that $n(r_n, a) > 16S(r_n)/15$ for all $a \in D$ and all r_n. It follows that for this function the exceptional sets of r-values in Theorems 2 and 3 are unbounded. In addition $f(z)$ has the property that there exists an arc L in the plane and a number $\beta < 1$ such that $n(r_n, a) < \beta S(r_n)$ for all $a \in L$ and all r_n.

It is of interest to consider Example 2 in connection with two inequalities obtained by Ahlfors [1] in his theory of covering surfaces. Ahlfors showed that if D' is any domain on the sphere and if $\varepsilon > 0$, then there exists a set E_0 of finite logarithmic measure such that

$$(2.1) \qquad |S(r) - S(r, D')| = O(S(r)^{1/2+\epsilon}) \qquad (r \to \infty)$$

for all $r \notin E_0$. He also showed that for any set $a_1, \ldots, a_q$ of distinct points on the sphere

$$(2.2) \qquad \sum_{\nu=1}^{q} (S(r) - \bar{n}(r, a_\nu)) \leqq 2S(r) + O(S(r)^{1/2+\epsilon}) \qquad (r \to \infty)$$

for all $r \notin E_0$. (Here $\bar{n}(r, a_\nu)$ denotes the number of distinct roots of $f(z) = a_\nu$ in $|z| \leqq r$.)

Example 2 shows that the Ahlfors exceptional set E_0 is in general nonempty. Certainly (2.1) does not hold for the disk D and the sequence r_n. Hence $r_n \in E_0$ for $n > n_0$. By choosing $3/(1-\beta)$ distinct points a_ν in L we see that (2.2) also does not hold on the sequence r_n. In fact it is clear that on the sequence r_n, (2.2) does not hold with 2 replaced by any constant independent of q.

3. **Results on $n(r, a)/S(r)$.** If $M > 0$, let $\Delta_M = \{a : |a| \leqq M\}$. Suppose $f(z)$ is a nonconstant meromorphic function in $|z| < \infty$ and $\delta > 0$. Let $\tilde{A}_R$ denote the set of all complex numbers $a \in \Delta_M$ for which there exists $r \geqq R$ such that $N(r, a) < T(r) - T(r)^{1/2+\delta}$. It is a standard result [3, p. 280] that $\lim_{R\to\infty} \operatorname{cap} \tilde{A}_R = 0$. Except for the case $f(0) = \infty$ which can easily be handled separately, we have from Nevanlinna's first fundamental theorem for every $r > 0$ and every complex number $a \neq f(0)$

$$N(r, a) < T(r) - \log |f(0) - a| + \log^+ |a| + \log^+ |f(0)| + 2 \log 2.$$

It follows that if $\varepsilon > 0$, there exists $A \subset \Delta_M$ with cap $A < \varepsilon$ and there exists $R_0 = R_0(M, \varepsilon, \delta)$ such that

$$(3.1) \qquad |N(r, a) - T(r)| < T(r)^{1/2+\delta}$$

for all $r > R_0$ and all $a \in \Delta_M - A$. We use this fact to prove Theorem 1. In what follows we let R_0 (unlike r_0) have the same value each time it occurs.

THEOREM 1. *Suppose $f(z)$ is a transcendental meromorphic function in the plane. Suppose $\varepsilon>0$, $M>0$, and $0<\delta<1/4$. Then there exists $E_1\subset[1,\infty)$ depending only on δ such that $\int_{E_1}(dx/x(\log x)^{1/2+\delta})<\infty$, there exists $A_1\subset\Delta_M$ with* cap $A_1<\varepsilon$, *and there exists $R_1=R_1(M,\varepsilon,\delta)$ such that $r\in[R_1,\infty)-E_1$ implies*

$$(1-1/\log S(r))^2S(r) < n(r,a) < (1+1/\log S(r))^2S(r) \tag{3.2}$$

for all $a\in\Delta_M-A_1$. If $\liminf_{r\to\infty}(\log S(r)/\log\log r)>1$, *then E_1 in fact has finite logarithmic measure.*

THEOREM 2. *Suppose $f(z)$ is a nonconstant meromorphic function in $|z|<\infty$. If $0<\delta<1/4$, there exists $E_2\subset[1,\infty)$ such that $\int_{E_2}(dx/x(\log x)^{1/2+\delta})<\infty$ and there exists a set A_2 in the complex plane having inner capacity zero such that*

$$\lim_{r\to\infty;r\notin E_2}\frac{n(r,a)}{S(r)} = 1 \quad \textit{for all } a\notin A_2.$$

If $\liminf_{r\to\infty}(\log S(r)/\log\log r)>1$, *then E_2 has finite logarithmic measure.*

THEOREM 3. *Suppose $f(z)$ is a nonconstant meromorphic function in $|z|<\infty$. If $\varepsilon>0$ and $0<\delta<1/4$, then there exists $E_3\subset[1,\infty)$ such that $\int_{E_3}(dx/(x(\log x)^{1/2+\delta}))<\infty$ and there exists a set A_3 in the complex plane for which*

$$\text{cap}\,\{a : a\in A_3 \textit{ and } |a| \leqq 1\} < \varepsilon \quad \textit{and} \quad \text{cap}\,\{a^{-1} : a\in A_3 \textit{ and } |a| > 1\} < \varepsilon$$

such that $\sup_{a\notin A_3} n(r,a)<(1+\varepsilon)S(r)$ *for all $r\notin E_3$. If* $\liminf_{r\to\infty}(\log S(r)/\log\log r)$ >1 *then E_3 has finite logarithmic measure.*

Proof of Theorem 1. $S(r)$ is a continuous, strictly increasing function which is unbounded because f is transcendental. Suppose $t_0>1$ is such that $S(t_0)>e$. For $r>t_0$ define $\tilde r$ by the equation

$$S(\tilde r) = S(r)(1+1/\log S(r)).$$

Clearly $\tilde r$ is well defined and $\tilde r>r$. Let

$$E_\alpha = \left\{r > t_0 : \log\frac{\tilde r}{r} < \frac{(\log r)^{1/2+\delta}}{S(\tilde r)^{1/2-2\delta}}\right\} \quad \text{and} \quad E_\beta = \{r > t_0 : \tilde r > r^2\}.$$

Suppose for some values of a and r we have $n(r,a)\geqq(1+1/\log S(r))^2S(r)$; thus

$$N(\tilde r,a)-N(r,a) \geqq \int_r^{\tilde r}\frac{n(r,a)}{t}\,dt \geqq \left(1+\frac{1}{\log S(r)}\right)^2 S(r)\log\frac{\tilde r}{r}.$$

We then have

$$\begin{aligned} |T(\tilde r)-N(\tilde r,a)|+|T(r)-N(r,a)| &\geqq |N(\tilde r,a)-N(r,a)-(T(\tilde r)-T(r))| \\ &\geqq \left(\left(1+\frac{1}{\log S(r)}\right)^2 S(r)-S(\tilde r)\right)\log\frac{\tilde r}{r} \\ &= \frac{S(\tilde r)\log(\tilde r/r)}{\log S(r)}. \end{aligned} \tag{3.3}$$

Suppose $r \notin E_\alpha \cup E_\beta$. Then for $r > r_0$,

$$S(\tilde{r})^{1/2-\delta} \log \frac{\tilde{r}}{r} \geqq S(\tilde{r})^{\delta}(\log r)^{1/2+\delta} > 8(\log S(r))(\log r)^{1/2+\delta}.$$

Hence

$$\frac{S(\tilde{r}) \log (\tilde{r}/r)}{\log S(r)} > 8(S(\tilde{r}) \log r)^{1/2+\delta}.$$

Because $r \notin E_\beta$ and $r > t_0$ we have $\log r \geqq \log \tilde{r}/2$. Hence for $r > r_0$,

$$\begin{aligned} S(\tilde{r}) \log (\tilde{r}/r)/\log S(r) &> 8(1/2)^{1/2+\delta}(S(\tilde{r}) \log \tilde{r})^{1/2+\delta} \\ &> 4(T(\tilde{r})-T(1))^{1/2+\delta} > 2T(\tilde{r})^{1/2+\delta}. \end{aligned} \tag{3.4}$$

However if $a \in \Delta_M - A$ and $r > R_0$, then by (3.1)

$$|T(\tilde{r})-N(\tilde{r}, a)|+|T(r)-N(r, a)| < 2T(\tilde{r})^{1/2+\delta}. \tag{3.5}$$

Hence by (3.3), (3.4), and (3.5), if $a \in \Delta_M - A$ and $r > \max(r_0, R_0)$, $r \notin E_\alpha \cup E_\beta$, then $n(r, a) < (1+1/\log S(r))^2 S(r)$.

We now show this same conclusion holds on the set E_β, namely if $a \in \Delta_M - A$, $r \in E_\beta$, and $r > \max(r_0, R_0)$, then $n(r, a) < (1+1/\log S(r))^2 S(r)$. Since $r \in E_\beta$ we have $\log \tilde{r} - \log r > \log \tilde{r}/2$. Hence for $r > r_0$,

$$\frac{(\log \tilde{r})^{1/2+\delta}}{\log (\tilde{r}/r)} < \frac{2(\log \tilde{r})^{1/2+\delta}}{\log \tilde{r}} < 1 < \frac{S(\tilde{r})^{1/2-\delta}}{4 \log S(r)}.$$

Consequently

$$(\log \tilde{r})^{1/2+\delta} < \frac{(\log (\tilde{r}/r))S(\tilde{r})^{1/2-\delta}}{4 \log S(r)}.$$

This implies for $r > r_0$

$$2T(\tilde{r})^{1/2+\delta} < 4(T(\tilde{r})-T(1))^{1/2+\delta} < \frac{(\log (\tilde{r}/r))(S(\tilde{r}))}{\log S(r)}. \tag{3.6}$$

Thus if $a \in \Delta_M - A$, $r \in E_\beta$, and $r > \max(r_0, R_0)$, then by (3.3), (3.5), and (3.6) we cannot have $n(r, a) \geqq (1+1/\log S(r))^2 S(r)$.

We now show $\int_{E_\alpha} (dx/x(\log x)^{1/2+\delta}) < \infty$. Without loss of generality we assume E_α is unbounded. Suppose $r_1 \in E_\alpha$; thus $S(r_1) > S(t_0) > e$. Define for $n \geqq 2$

$$r_n = \inf\{r \in E_\alpha : r > \tilde{r}_{n-1}\}.$$

Then

$$S(r_n) \geqq S(\tilde{r}_{n-1}) = S(r_{n-1})(1+1/\log S(r_{n-1})).$$

Hence for $n \geqq 2$

$$S(r_n) - S(r_{n-1}) \geqq S(r_{n-1})/\log S(r_{n-1}) > e.$$

Thus $r_n \to \infty$. Certainly $E_\alpha - [1, r_1) \subset \bigcup_{n=1}^{\infty} [r_n, \tilde{r}_n]$. For $n \geqq n_0$,

$$\log S(\tilde{r}_n) - \log S(r_n) = \log\left(1 + \frac{1}{\log S(r_n)}\right) > \frac{1}{2 \log S(r_n)}.$$

Because the intervals $(r_i, \tilde{r}_i)$ are disjoint, for $n \geqq n_0 + 1$,

$$\log S(r_n) \geqq \log S(\tilde{r}_{n-1}) > \frac{1}{2} \sum_{i=n_0}^{n-1} \frac{1}{\log S(r_i)} > \frac{n - n_0}{2 \log S(r_n)}. \tag{3.7}$$

Hence $\sum_{n=1}^{\infty} 1/S(r_n)^k < \infty$ for any $k > 0$.

Let $I_n = [r_n, \tilde{r}_n]$ and $I = \bigcup_{n=1}^{\infty} I_n$. Then since $r_n \in \bar{E}_\alpha$,

$$\int_{r_n}^{\tilde{r}_n} \frac{dx}{x(\log x)^{1/2+\delta}} \leqq \frac{1}{(\log r_n)^{1/2+\delta}} \log \frac{\tilde{r}_n}{r_n} \leqq \frac{1}{S(\tilde{r}_n)^{1/2-2\delta}}.$$

Therefore

$$\int_I \frac{dx}{x(\log x)^{1/2+\delta}} \leqq \sum_{n=1}^{\infty} \frac{1}{S(\tilde{r}_n)^{1/2-2\delta}} \leqq \sum_{n=1}^{\infty} \frac{1}{S(r_n)^{1/2-2\delta}} < \infty.$$

Thus also

$$\int_{E_\alpha} \frac{dx}{x(\log x)^{1/2+\delta}} < \infty.$$

We now suppose that there exists $\eta > 0$ such that $S(r) > (\log r)^{1+\eta}$ for $r > r_0$. Then there exists $\delta > 0$ and $\gamma > 0$ such that $r > r_0$ implies $(\log r)^{1/2+\delta}/S(r)^{1/2-2\delta} < 1/S(r)^\gamma$. We carry out the above discussion for such a $\delta > 0$. Then for $n > n_0$ we have

$$\int_{r_n}^{\tilde{r}_n} \frac{dx}{x} = \log \frac{\tilde{r}_n}{r_n} \leqq \frac{(\log r_n)^{1/2+\delta}}{S(\tilde{r}_n)^{1/2-2\delta}}$$
$$\leqq \frac{(\log r_n)^{1/2+\delta}}{S(r_n)^{1/2-2\delta}} < \frac{1}{S(r_n)^\gamma}.$$

Hence $\int_I dx/x < \infty$ and therefore $\int_{E_\alpha} dx/x < \infty$.

We have thus shown that if $r > \max(r_0, R_0)$ and $r \notin E_\alpha$, then

$$n(r, a) < (1 + 1/\log S(r))^2 S(r)$$

for $a \in \Delta_M - A$ where cap $A < \varepsilon$. Furthermore $\int_{E_\alpha} (dx/x(\log x)^{1/2+\delta}) < \infty$ and if $\liminf_{r\to\infty} \log S(r)/\log\log r > 1$, then, for an appropriate choice of δ, E_α has finite logarithmic measure.

We now consider the left inequality in (3.2). As before we let $t_0 > 1$ be such that $S(t_0) > e$. For $r > t_0$, we define $\tilde{r} < r$ by the equation $S(\tilde{r}) = S(r)(1 - 1/\log S(r))$.

There is no difficulty in showing that $\tilde{}$ is a well-defined, strictly increasing, unbounded, continuous function on (t_0, ∞). We may suppose $\tilde{t}_0 > 1$. Let

$$E'_\alpha = \left\{ r > t_0 : \log \frac{r}{\tilde{r}} < \frac{(\log \tilde{r})^{1/2+\delta}}{S(r)^{1/2-2\delta}} \right\} \quad \text{and} \quad E'_\beta = \{ r > t_0 : r > \tilde{r}^2 \}.$$

If $n(r, a) \leqq (1-1/\log S(r))^2 S(r)$, then

$$
\begin{aligned}
|T(r)-N(r, a)|+|T(\tilde{r})-N(\tilde{r}, a)| &\geqq |T(r)-T(\tilde{r})-(N(r, a)-N(\tilde{r}, a))| \\
&\geqq \left(S(\tilde{r})-\left(1-\frac{1}{\log S(r)}\right)^2 S(r)\right) \log \frac{r}{\tilde{r}} \qquad (3.8)\\
&= \frac{S(\tilde{r}) \log (r/\tilde{r})}{\log S(r)}.
\end{aligned}
$$

If $r \notin E'_\alpha \cup E'_\beta$, then for $r > r_0$,

$$S(r)^{1/2-\delta} \log (r/\tilde{r}) > 16(\log S(r))(\log \tilde{r})^{1/2+\delta}.$$

Since $r \notin E'_\beta$ and $r > t_0$ we have $\log \tilde{r} \geqq \log r/2$. Hence for $r > r_0$,

$$S(r) \log (r/\tilde{r})/\log S(r) > 8(S(r) \log r)^{1/2+\delta} > 4T(r)^{1/2+\delta}.$$

Therefore

$$\frac{S(\tilde{r}) \log (r/\tilde{r})}{\log S(r)} > 2T(r)^{1/2+\delta}. \qquad (3.9)$$

If $\tilde{r} > R_0$ and $a \in \Delta_M - A$, then

$$|T(\tilde{r})-N(\tilde{r}, a)|+|T(r)-N(r, a)| < 2T(r)^{1/2+\delta}. \qquad (3.10)$$

Thus if $a \in \Delta_M - A$, $r \notin E'_\alpha \cup E'_\beta$, and $\tilde{r} > \max (r_0, R_0)$, we have from (3.8), (3.9), and (3.10) that $n(r, a) > (1-1/\log S(r))^2 S(r)$.

Suppose $r \in E'_\beta$. Then for $r > r_0$

$$\frac{(\log r)^{1/2+\delta}}{\log (r/\tilde{r})} < \frac{2(\log r)^{1/2+\delta}}{\log r} < 1 < \frac{S(r)^{1/2-\delta}}{8 \log S(r)}.$$

Therefore for $r > r_0$,

$$
\begin{aligned}
2T(r)^{1/2+\delta} &< 4(S(r) \log r)^{1/2+\delta} \\
&< \frac{S(r) \log (r/\tilde{r})}{2 \log S(r)} < \frac{S(\tilde{r}) \log (r/\tilde{r})}{\log S(r)}. \qquad (3.11)
\end{aligned}
$$

Thus if $a \in \Delta_M - A$, $r \in E'_\beta$, and $\tilde{r} > \max (r_0, R_0)$, we have from (3.8), (3.10), and (3.11) that $n(r, a) > (1-1/\log S(r))^2 S(r)$.

We have thus shown that $n(r, a) > (1-1/\log S(r))^2 S(r)$ for all $a \in \Delta_M - A$ and all $r \notin E'_\alpha$ and such $\tilde{r} > \max (r_0, R_0)$. We now show $\int_{E'_\alpha} (dx/x(\log x)^{1/2+\delta}) < \infty$. We recall t_0 is such that $S(t_0) > e$ and $\tilde{t}_0 > 1$. We define t_n for $n \geqq 1$ by $\tilde{t}_n = t_{n-1}$. Thus $S(t_{n-1}) = S(\tilde{t}_n) = S(t_n)(1-1/\log S(t_n))$. Hence $S(t_n) - S(t_{n-1}) = S(t_n)/\log S(t_n) > e$. Consequently $t_n \to \infty$. We have for $n \geqq 1$

$$\log S(t_n) - \log S(t_{n-1}) = -\log \left(1-\frac{1}{\log S(t_n)}\right) > \frac{1}{\log S(t_n)}.$$

Therefore

$$\log S(t_n) \geqq \sum_{i=1}^{n} \frac{1}{\log S(t_i)} \geqq \frac{n}{\log S(t_n)}.$$

This implies that for any $k > 0$, $\sum_{n=1}^{\infty} 1/S(t_n)^k < \infty$.

Let J denote the set of integers n such that $(t_{n-1}, t_n] \cap E'_\alpha \neq \varnothing$. Without loss of generality we may assume J is unbounded. For $n \in J$, let

$$r_n = \sup \{t \in (t_{n-1}, t_n] : t \in E'_\alpha\}.$$

Certainly $E'_\alpha \subset \bigcup_{n \in J} [\tilde{r}_n, r_n]$. For $n \in J$, let $I'_n = [\tilde{r}_n, r_n]$ and let $I' = \bigcup_{n \in J} I'_n$. If $n \in J$, then

$$\int_{\tilde{r}_n}^{r_n} \frac{dx}{x(\log x)^{1/2+\delta}} \leqq \frac{\log (r_n/\tilde{r}_n)}{(\log \tilde{r}_n)^{1/2+\delta}}.$$

Since $r_n \in \bar{E}'_\alpha$,

$$\frac{\log (r_n/\tilde{r}_n)}{(\log \tilde{r}_n)^{1/2+\delta}} \leqq \frac{1}{S(r_n)^{1/2-2\delta}}.$$

$S(r_n) > S(t_{n-1})$ implies $\sum_{n \in J} [1/S(r_n)^{1/2-2\delta}] < \infty$. Hence

$$\int_{I'} \frac{dx}{x(\log x)^{1/2+\delta}} \leqq \sum_{n \in J} \frac{1}{S(r_n)^{1/2-2\delta}} < \infty.$$

Therefore certainly

$$\int_{E'_\alpha} \frac{dx}{x(\log x)^{1/2+\delta}} < \infty.$$

Finally if $S(r) > (\log r)^{1+\eta}$ for $r > r_0$ and some $\eta > 0$, then as before for some $\delta > 0$ and $\gamma > 0$

$$\frac{(\log r)^{1/2+\delta}}{S(r)^{1/2-2\delta}} < \frac{1}{S(r)^\gamma}$$

for $r > r_0$. For such a $\delta > 0$ and all $n > n_0$,

$$\int_{\tilde{r}_n}^{r_n} \frac{dx}{x} = \log \frac{r_n}{\tilde{r}_n} \leqq \frac{(\log \tilde{r}_n)^{1/2+\delta}}{S(r_n)^{1/2-2\delta}}$$
$$< \frac{(\log r_n)^{1/2+\delta}}{S(r_n)^{1/2-2\delta}} < \frac{1}{S(r_n)^\gamma}.$$

Hence

$$\int_{I'} \frac{dx}{x} \leqq \sum_{n \in J} \frac{1}{S(r_n)^\gamma} < \infty.$$

Since $E'_\alpha \subset I'$, the proof of Theorem 1 is finished upon setting $E_1 = E_\alpha \cup E'_\alpha$ and $A_1 = A$.

We observe that E_1 may be regarded intuitively as the set where $S(r)$ is increasing very rapidly. It is trivial to verify that

$$\int_{E_1} \frac{dx}{x(\log x)^{1/2+\delta}} < \infty$$

implies E_1 has logarithmic density zero.

We also remark that the method of proof of Theorem 1 cannot give a result having an exceptional set of r-values with finite logarithmic measure without some growth condition on $S(r)$. Consider the two functions $g(x)$ and $h_1(x)$ defined on $(0, \infty)$ as follows:

$$\begin{aligned} g(x) &= 0 && -\infty < \log x < 2, \\ &= (n+1)2^{2^{n+1}} && 2^{2^n} \leqq \log x < 2^{2^{n+1}} && n = 0, 1, 2, \dots; \\ h_1(x) &= 2g(x) && 2^{2^n}-1 \leqq \log x < 2^{2^n} && n = 0, 1, 2, \dots, \\ &= g(x) && \text{otherwise}. \end{aligned}$$

Clearly $h_1(x) \geqq 2g(x)$ on a set of infinite logarithmic measure. Since $h_1(x) \geqq g(x) > \log x$ for $\log x \geqq 2$ we easily verify that

$$\int_0^r \frac{h_1(x)}{x}\,dx > \frac{(\log r)^2}{2} \qquad \text{for } r > r_0.$$

A direct computation shows that

$$\int_0^r (h_1(x)-g(x))\,\frac{dx}{x} = o(\log r \log\log r) \qquad (r \to \infty).$$

Therefore, for any $\delta > 0$,

$$\int_0^r (h_1(x)-g(x))\,\frac{dx}{x} \leqq \left(\int_0^r \frac{h_1(x)}{x}\,dx\right)^{1/2+\delta} \qquad \text{for } r > r_0.$$

We note $h_1(x)/\log x \to \infty$ as $x \to \infty$. Certainly h_1 can be redefined to become a strictly increasing continuous function h such that g and h still have these properties. It follows that if we only assume $S(r)/\log r \to \infty$ as $r \to \infty$ then the information contained in (3.1) is not sufficient to imply that if $a \in \Delta_M - A$ then $n(r, a)$ and $S(r)$ are asymptotic off some set of finite logarithmic measure.

Proof of Theorem 2. Since the result is trivial for rational functions, we concern ourselves only with transcendental functions. Let E_2 be the set E_1 of Theorem 1. Theorem 1 implies that $\lim_{r\to\infty; r\notin E_2} n(r, a)/S(r) = 1$ for all $a \in \Delta_M$ except for at most a set of capacity ε. Since this is true for every $\varepsilon > 0$, we conclude the set of all $a \in \Delta_M$ for which $\lim_{r\to\infty; r\notin E_2} n(r, a)/S(r)$ does not exist and equal 1 has capacity zero. Theorem 2 now follows from the fact that the inner capacity of an arbitrary set is the supremum of the capacities of its compact subsets.

Proof of Theorem 3. Again we need only consider transcendental functions. We apply Theorem 1 with $M = 1$ to conclude that there exists R such that $r > R$ and $r \notin E_1$ implies $n(r, a) < (1+\varepsilon)S(r)$ for all $a \in \Delta_1 - A_1$ where cap $A_1 < \varepsilon$. We also apply Theorem 1 to $g(z) = 1/f(z)$. We attach the obvious meanings to $n(r, a, f)$, $n(r, a, g)$, $S(r, f)$, and $S(r, g)$. It is elementary that $S(r, f) = S(r, g)$; we denote the common value by $S(r)$. The functions f and g clearly have the same exceptional set E_1. By Theorem 1 applied to $g(z)$, there exists R' such that $r > R'$ and $r \notin E_1$ implies $n(r, a, g) < (1+\varepsilon)S(r)$ for all $a \in \Delta_1 - A'$ where cap $A' < \varepsilon$. Let $\bar{R} = \max(R, R')$. Since $n(r, a, g) = n(r, a^{-1}, f)$, the result follows upon setting $E_3 = E_1 \cup [1, \bar{R}]$.

4. **Examples.**

EXAMPLE 1. *Suppose $0<\varepsilon<1$. Let p be an integer such that $2p/(p+1)>2-\varepsilon$ and let k be an integer such that $p<1+2^{k-4}$. Let J be the set of all positive integers congruent to s* mod $2k$ *where $s=k+1, k+2, \ldots, 2k-1$,* or 0. *Define*

$$f(z) = \prod_{n\in J}\left(1-\frac{z}{(2p)^n}\right)^{p^n}.$$

For this $f(z)$ there exists $\tilde{E}\subset[1,\infty)$ having positive lower logarithmic density such that

$$\liminf_{r\to\infty;r\in\tilde{E}}\frac{n(r,0)}{S(r)} > 2-\varepsilon.$$

Before proving the above assertion we remark that by familiar considerations $f(z)$ has order $\log p/\log 2p$.

Select $N\equiv 0 \bmod 2k$ and define

$$f_1(z) = \prod_{n\in J;n<N}\left(1-\frac{z}{(2p)^n}\right)^{p^n}, \qquad f_2(z) = \left(1-\frac{z}{(2p)^N}\right)^{p^N}$$

and

$$f_3(z) = \prod_{n\in J;n>N}\left(1-\frac{z}{(2p)^n}\right)^{p^n}.$$

To simplify notation we do not indicate the dependence of f_1, f_2, and f_3 on N.

We consider the behavior of $f=f_1f_2f_3$ on $|z|=r$ for $r\in B_N$ where

$$B_N = \{r : (2p)^N < r < 2(2p)^N\}.$$

We shall be concerned both with

$$\frac{d}{d\theta}\arg f_j(re^{i\theta}) = \operatorname{Re}\frac{re^{i\theta}f_j'(re^{i\theta})}{f_j(re^{i\theta})}$$

and with $\max_{-\pi\leqq\theta\leqq\pi}|f_j(re^{i\theta})|$.

We have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{zf_1'(z)}{f_1(z)}+\operatorname{Re}\frac{zf_2'(z)}{f_2(z)}+\operatorname{Re}\frac{zf_3'(z)}{f_3(z)}. \tag{4.1}$$

Consider $r\in B_N$. On $|z|=r$

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} = \sum_{n\in J;n<N}p^n\operatorname{Re}\frac{z}{z-(2p)^n}.$$

Since $r>(2p)^n$ for all $n<N$, elementary considerations show $\operatorname{Re}(re^{i\theta}/(re^{i\theta}-(2p)^n)) >1/2$ for all $\theta\in[-\pi,\pi]$. Hence for $r\in B_N$ we have for all $z=re^{i\theta}$,

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} \geqq \frac{1}{2}\sum_{n=N-k+1}^{N-1}p^n = \frac{1}{2}(p-1)^{-1}(1-p^{1-k})p^N. \tag{4.2}$$

Similarly for $r\in B_N$ and all $z=re^{i\theta}$,

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} > \frac{1}{2}p^N. \tag{4.3}$$

We now consider

$$\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} = \sum_{n\in J;n>N} p^n \operatorname{Re}\frac{z}{z-(2p)^n}. \tag{4.4}$$

For $r\in\beta_N$ and $n > N$ we have $(2p)^n > r$. Hence for all $\theta\in[-\pi,\pi]$,

$$\operatorname{Re}\frac{re^{i\theta}}{re^{i\theta}-(2p)^n} \geqq \frac{r^2-(2p)^n r}{(r-(2p)^n)^2} = \frac{-r}{(2p)^n-r}.$$

Thus for $r\in B_N$ and all $z=re^{i\theta}$,

$$\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} \geqq -\sum_{n\in J;n>N} p^n\frac{r}{(2p)^n-r}.$$

If $n\in J$ and $n>N$ then $n\geqq N+k+1$. For such n, $r\in B_N$ implies $r<2(2p)^n(2p)^{-k-1}$. Hence for such n and r

$$\frac{1}{(2p)^n-r} \leqq \frac{1}{(2p)^n(1-2(2p)^{-k-1})} \leqq \frac{1+(2p)^{-k}}{(2p)^n}.$$

Thus for $r\in B_N$ and all $z=re^{i\theta}$,

$$\begin{aligned}\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} &\geqq (-r)(1+(2p)^{-k})\sum_{n\in J;n>N} 2^{-n}\\ &\geqq -r(1+(2p)^{-k})2^{-N-k} \geqq -(1+(2p)^{-k})2^{1-k}p^N.\end{aligned} \tag{4.5}$$

A straightforward verification shows that $p<1+2^{k-4}$ implies $\frac{1}{2}(p-1)^{-1}(1-p^{1-k}) >(1+(2p)^{-k})2^{1-k}$. Hence from (4.2) and (4.5) we conclude that

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)}+\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} > 0 \tag{4.6}$$

everywhere on $|z|=r$ for $r\in B_N$. This together with (4.1) and (4.3) implies

$$\frac{d}{d\theta}\arg f(re^{i\theta}) > 0$$

everywhere on $|z|=r$ for $r\in B_N$.

We now estimate $\max_{-\pi\leqq\theta\leqq\pi}|f_1(re^{i\theta})f_3(re^{i\theta})|$ for $r\in B_N$. We have

$$|f_1(re^{i\theta})| = \prod_{n\in J;n<N}\left|1-\frac{re^{i\theta}}{(2p)^n}\right|^{p^n} \leqq \prod_{n\in J;n<N}\left(\frac{2r}{(2p)^n}\right)^{p^n}.$$

Hence

$$\begin{aligned}\log|f_1(re^{i\theta})| &\leqq \sum_{n\in J;n<N} p^n(\log 2+\log r-n\log 2p)\\ &\leqq \sum_{n=1}^{N-1} p^n(2\log 2+(N-n)\log 2p)\\ &\leqq 2(\log 2)(p^N-p)(p-1)^{-1}+(\log 2p)\sum_{n=1}^{N-1}p^n(N-n).\end{aligned}$$

Certainly

$$\sum_{n=1}^{N-1} p^n(N-n) \leqq p^{N-1}+2p^{N-2}+\int_1^{N-2} (N-t)p^t\,dt.$$

Integration by parts yields

$$\int_1^{N-2} (N-t)p^t\,dt = \frac{p^{N-2}}{\log p}\left(2+\frac{1}{\log p}\right)-\frac{p}{\log p}\left(N-1+\frac{1}{\log p}\right).$$

Hence there exists $c_1>0$ and independent of N such that for all $r \in B_N$ and all $\theta \in [-\pi, \pi]$

$$\log|f_1(re^{i\theta})| \leqq c_1 p^N. \tag{4.7}$$

Similarly if $r \in B_N$ and $\theta \in [-\pi, \pi]$, then

$$\begin{aligned} \log|f_3(re^{i\theta})| &\leqq \sum_{n\in J; n>N} p^n \log\left(1+\frac{r}{(2p)^n}\right) \\ &\leqq r \sum_{n\in J; n>N} 2^{-n} \leqq 2^{-N-k}r \leqq 2^{1-k}p^N. \end{aligned} \tag{4.8}$$

Thus if $r \in B_N$ and $\theta \in [-\pi, \pi]$, we see from (4.7) and (4.8) that

$$|f_1(re^{i\theta})f_3(re^{i\theta})| \leqq \exp((c_1+1)p^N). \tag{4.9}$$

We now consider the function $1-z/(2p)^N$ on the circle $|z|=r=(2p)^N(1+\varepsilon)$, $0<\varepsilon<1$. The image of $|z|=r$ is a circle centered at 1 having radius $1+\varepsilon$. We have

$$1-z/(2p)^N = 1-(1+\varepsilon)(2p)^N e^{i\theta}/(2p)^N = 1-(1+\varepsilon)\cos\theta - i(1+\varepsilon)\sin\theta.$$

Let $\alpha=\alpha(\varepsilon)$ in $(0, \pi/3)$ be such that $\cos\alpha=(1+\varepsilon)^{-1}$. As θ increases from $-\alpha(\varepsilon)$ to $\alpha(\varepsilon)$, $\arg(1-re^{i\theta}/(2p)^N)$ increases by π. Furthermore, for $z=(2p)^N(1+\varepsilon)e^{i\theta}$, $-\alpha(\varepsilon)\leqq\theta\leqq\alpha(\varepsilon)$,

$$\begin{aligned} |1-z/(2p)^N|^2 &= (1-(1+\varepsilon)\cos\theta)^2+(1+\varepsilon)^2\sin^2\theta \\ &= 1+(1+\varepsilon)^2-2(1+\varepsilon)\cos\theta \leqq 2\varepsilon+\varepsilon^2<3\varepsilon. \end{aligned}$$

Combining these two observations we see that if $r=(1+\varepsilon)(2p)^N$, the argument of $f_2(re^{i\theta})$ increases by πp^N as θ increases from $-\alpha(\varepsilon)$ to $\alpha(\varepsilon)$ and that $|f_2(re^{i\theta})|<(3\varepsilon)^{p^N/2}$ for $-\alpha(\varepsilon)\leqq\theta\leqq\alpha(\varepsilon)$.

We now choose $\varepsilon_1>0$ such that $(3\varepsilon_1)^{1/2}e^{c_1+1}<1/2$. We note ε_1 is independent of N. We define $C_N\subset B_N$ by $C_N=\{r : (2p)^N<r<(1+\varepsilon_1)(2p)^N\}$. Combining the above observations about $f_2(re^{i\theta})$ with (4.6) and (4.9) we see that if $r\in C_N$, so that $r=(1+\varepsilon)(2p)^N$ for some ε, $0<\varepsilon<\varepsilon_1$, then there exists $\alpha=\alpha(\varepsilon)$ in $(0, \pi/3)$ such that

$$\begin{aligned} &\text{(i)} \quad |f(re^{i\theta})| < 2^{-p^N} \quad \text{if } -\alpha(\varepsilon)\leqq\theta\leqq\alpha(\varepsilon) \text{ and} \\ &\text{(ii)} \quad \arg f(re^{i\alpha})-\arg f(re^{-i\alpha}) > \pi p^N. \end{aligned} \tag{4.10}$$

We now show that if $a_0\neq 0$, then $d\arg f(re^{i\theta})/d\theta>0$ on $[-\pi, \pi]$ implies $n(r, a_0)$ is equal to the number of values of $\theta\in[-\pi, \pi)$ such that $\operatorname{Im} a_0^{-1}f(re^{i\theta})=0$ and

$\operatorname{Re} a_0^{-1} f(re^{i\theta}) \geqq 1$. Let $a_0 = t_0 e^{i\theta_0}$. We first suppose $a_0 \neq f(re^{i\theta})$ for any $\theta \in [-\pi, \pi]$. Suppose $\theta_1 < \theta_2 < \cdots < \theta_q$ is the set of distinct θ in $[-\pi, \pi)$ such that $\operatorname{Im} a_0^{-1} f(re^{i\theta}) = 0$ and $\operatorname{Re} a_0^{-1} f(re^{i\theta}) > 1$. Set $\theta_{q+1} = \theta_1 + 2\pi$. Trivially, for $1 \leqq j \leqq q$,

$$\arg (f(re^{i\theta_{j+1}}) - a_0) - \arg (f(re^{i\theta_j}) - a_0)$$

is either 2π, 0, or -2π. We certainly have

$$\frac{d}{d\theta} \arg (f(re^{i\theta}) - a_0) = \operatorname{Re} \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta}) - a_0} > 0 \qquad \text{if } \theta = \theta_j,\ 1 \leqq j \leqq q.$$

This fact enables us easily to eliminate two of the above possibilities and to conclude

$$\arg (f(re^{i\theta_{j+1}}) - a_0) - \arg (f(re^{i\theta_j}) - a_0) = 2\pi$$

for $1 \leqq j \leqq q$. Hence by the argument principle $q = n(r, a_0)$. This establishes our contention in the case $a_0 \notin \{f(re^{i\theta}) : -\pi \leqq \theta \leqq \pi\}$.

We now suppose there exist $\theta_1 < \theta_2 < \cdots < \theta_q$ as above and in addition $\{\tilde{\theta}_1, \ldots, \tilde{\theta}_{q'}\}$ is the set of all distinct $\theta \in [-\pi, \pi)$ such that $f(re^{i\theta}) = a_0$. We remark that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$$

on $|z| = r$ implies $f'(re^{i\tilde{\theta}_j}) \neq 0$ for $1 \leqq j \leqq q'$. Let D_1 and D_2 be the components of $\Sigma - \{f(re^{i\theta}) : -\pi \leqq \theta \leqq \pi\}$ such that for some $\delta > 0$, $\{te^{i\theta_0} : t_0 - \delta < t < t_0\} \subset D_1$ and $\{te^{i\theta_0} : t_0 < t < t_0 + \delta\} \subset D_2$. By the above argument $a \in D_1$ implies $n(r, a) = q + q'$ and $a \in D_2$ implies $n(r, a) = q$. In $|z| < r$, f assumes the value a_0 at most q times counting multiplicities; this follows from the fact f is an open mapping and $a_0 \in \bar{D}_2$. The fact that $f'(re^{i\tilde{\theta}_j}) \neq 0$ for $1 \leqq j \leqq q'$ now implies $n(r, a_0) \leqq q + q'$. Because f is continuous and $a_0 \in \bar{D}_1$ we conclude $n(r, a_0) \geqq q + q'$. Thus, in this case as well, $n(r, a_0)$ is the number of values of $\theta \in [-\pi, \pi)$ for which $\operatorname{Im} a_0^{-1} f(re^{i\theta}) = 0$ and $\operatorname{Re} a_0^{-1} f(re^{i\theta}) \geqq 1$.

We draw two conclusions from this fact. First, for a fixed $r \in C_N$, $n(r, te^{i\theta})$ is a nonincreasing function of t on $[0, \infty)$ for each $\theta \in [-\pi, \pi]$. Hence $n(r, 0) \geqq n(r, a)$ for all $a \in \Sigma$.

Secondly, suppose $r \in C_N$ and $|a| > 2^{-pN}$. We have $r = (2p)^N(1+\varepsilon)$ for some ε, $0 < \varepsilon < \varepsilon_1$. By (4.10i) if θ is such that $\operatorname{Im} a^{-1} f(re^{i\theta}) = 0$ and $\operatorname{Re} a^{-1} f(re^{i\theta}) \geqq 1$, then $\theta \notin [-\alpha(\varepsilon), \alpha(\varepsilon)]$. Let $\alpha < \theta_1 < \cdots < \theta_q < 2\pi - \alpha$ be the q values of θ in $[0, 2\pi)$ for which $\operatorname{Im} a^{-1} f(re^{i\theta}) = 0$ and $\operatorname{Re} a^{-1} f(re^{i\theta}) \geqq 1$. From the above remarks $n(r, a) = q$. $d \arg f(re^{i\theta})/d\theta > 0$ for all $\theta \in [0, 2\pi]$ implies $\arg f(re^{i\theta_{j+1}}) - \arg f(re^{i\theta_j}) \geqq 2\pi$ for $1 \leqq j \leqq q-1$. Hence

$$\arg f(re^{i(2\pi-\alpha)}) - \arg f(re^{i\alpha}) \geqq 2\pi(q-1).$$

Let $A_N = \{a : |a| > 2^{-pN}\}$. We thus conclude

$$\arg f(re^{i(2\pi-\alpha)}) - \arg f(re^{i\alpha}) \geqq 2\pi\Big(-1 + \sup_{a \in A_N} n(r, a)\Big). \tag{4.11}$$

By the argument principle

$$\arg f(re^{i(2\pi-\alpha)})-\arg f(re^{-i\alpha}) = 2\pi n(r,0). \tag{4.12}$$

We combine (4.10ii), (4.11), and (4.12) to conclude that for any $r \in C_N$

$$-1+\sup_{a\in A_N} n(r,a)+(1/2)p^N \leqq n(r,0). \tag{4.13}$$

For $r \in C_N$ we have

$$n(r,0) = \sum_{n\in J;n\leqq N} p^n \leqq \frac{p^{N+1}-p}{p-1} < \frac{p^{N+1}}{p-1}.$$

Hence $p^N > p^{-1}(p-1)n(r,0)$. Thus from (4.13)

$$-1+\sup_{a\in A_N} n(r,a) \leqq n(r,0)-\tfrac{1}{2}p^N \leqq n(r,0)\frac{p+1}{2p}.$$

Certainly

$$\begin{aligned} S(r) &= \int_{A_N} n(r,a)\,dm(a)+\int_{\Sigma-A_N} n(r,a)\,dm(a) \\ &\leqq m(A_N)\left(1+\frac{p+1}{2p}\,n(r,0)\right)+m(\Sigma-A_N)n(r,0). \end{aligned}$$

Thus $S(r)-1\leqq n(r,0)(m(A_N)(p+1)/2p+m(\Sigma-A_N))$. We let J' be the set of positive integers congruent to 0 mod $2k$ and define $\tilde{E}=\bigcup_{N\in J'} C_N$. As r tends to infinity through values in $\tilde{E}$, N also tends to infinity and thus $m(A_N)\to 1$ and $m(\Sigma-A_N)\to 0$. Consequently $\liminf_{r\to\infty;r\in\tilde{E}} n(r,0)/(S(r)-1)\geqq 2p/(p+1)$; this certainly implies $\liminf_{r\to\infty;r\in\tilde{E}} n(r,0)/S(r)\geqq 2p/(p+1)$.

Finally we observe that the lower logarithmic density of $\tilde{E}$ is not less than

$$\liminf_{n\to\infty} \frac{m_l(\tilde{E}\cap[1,(2p)^{2kn}))}{m_l[1,(2p)^{2kn})}.$$

We have $m_l(\tilde{E}\cap[1,(2p)^{2kn}))=(n-1)\log(1+\varepsilon_1)$ and $m_l[1,(2p)^{2kn})=2kn\log 2p$. Consequently the lower logarithmic density of $\tilde{E}$ is at least $\log(1+\varepsilon_1)/2k\log 2p>0$. It is in fact easy to verify that the logarithmic density of $\tilde{E}$ is $\log(1+\varepsilon_1)/2k\log 2p$.

Before proceeding to Example 2 we prove the assertion in §2 that for some functions $f(z)$ the exceptional set A_2 of Theorem 2 is uncountable. Let $f(z)$ be a function having uncountably many Valiron deficient values such that $0<c_1<T(r)/r<c_2<\infty$ for some c_1 and c_2 and all $r>r_0$. Thus $c_1 r<T(r)<T(1)+S(r)\log r$ for $r>r_0$. Hence for this function the exceptional set E_2 of Theorem 2 has finite logarithmic measure. Let $CE_2=[1,\infty)-E_2$, $E_2^r=E_2\cap[1,r)$, and $CE_2^r=CE_2\cap[1,r)$. Given $\varepsilon>0$, there exists r_0 such that $m_l(E_2\cap[r_0,\infty))<\varepsilon$. Thus for $r>r_0$,

$$\int_{E_2^r}\frac{S(t)}{t}\,dt \leqq T(r_0)+\varepsilon S(r) \leqq T(r_0)+\varepsilon T(er) < 2c_2\varepsilon er.$$

Thus

$$\int_{E_2^r} \frac{S(t)}{t}\,dt = o(T(r)) \qquad (r \to \infty). \tag{4.14}$$

It is trivial that if $\liminf_{r\to\infty; r\notin E_2} n(r, a)/S(r) \geqq 1$, then

$$\liminf_{r\to\infty} \frac{\int_{CE_2^r} (n(t, a)/t)\,dt}{\int_{CE_2^r} (S(t)/t)\,dt} \geqq 1.$$

This fact, combined with (4.14) and the inequality

$$\frac{N(r, a)}{T(r)} \geqq \frac{\int_{CE_2^r} (n(t, a)/t)\,dt}{T(1)+\int_{CE_2^r} (S(t)/t)\,dt+\int_{E_2^r} (S(t)/t)\,dt},$$

implies that $\liminf_{r\to\infty; r\notin E_2} n(r, a)/S(r) < 1$ for all Valiron deficient values of f. Hence for this function A_2 is uncountable.

EXAMPLE 2. *Let*

$$f(z) = \prod_{n=1}^{\infty} \left(1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right).$$

There exists a disk D in the plane and a sequence $r_N \to \infty$ such that $n(r_N, a) > 16S(r_N)/15$ for all $a \in D$ and all r_N. In addition there exists an arc L in the plane and a number $\beta < 1$ such that $n(r_N, a) < \beta S(r_N)$ for all $a \in L$ and all r_N.

Before proving $f(z)$ has the required properties, we observe that

$$1-\left(\frac{z}{e^{2^n}}\right)^{2^n} = \prod_{i=1}^{2^n} \left(1-\frac{z}{e^{2^n}\omega_i}\right)$$

where $\omega_1, \ldots, \omega_{2^n}$ are the distinct roots of $\omega^{2^n}=1$. Thus $f(z)$ has 2^n zeros evenly distributed on the circle of radius $\exp(2^n)$. The order of $f(z)$ is certainly zero.

For any integer $N \geqq 2$ we let

$$f_1(z) = \prod_{n=1}^{N-1} \left(1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right)$$

$$f_2(z) = 1-\left(\frac{z}{e^{2^N}}\right)^{2^N}$$

and

$$f_3(z) = \prod_{n=N+1}^{\infty} \left(1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right).$$

As before we do not indicate the dependence of f_1, f_2, and f_3 on N.

We consider the behavior of f on $|z| = r_N = (1+\varepsilon_N)\exp(2^N)$ for a value of ε_N in $(0, 1)$ as yet undetermined. On $|z| = r_N$ we have

$$\begin{aligned}\log|f_1(z)| &= \sum_{n=1}^{N-1} \log\left|1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right| \\ &\leqq \sum_{n=1}^{N-1} \log\left(1+\left(\frac{e^{2^N}(1+\varepsilon_N)}{e^{2^n}}\right)^{2^n}\right).\end{aligned}$$

Since $\log(1+x)<\log x+1/x$ for every $x>0$, we have

$$\begin{aligned}\log|f_1(z)| &\leqq \sum_{n=1}^{N-1}\log\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n}+\sum_{n=1}^{N-1}\left(\frac{e^{2^n}}{e^{2N}(1+\varepsilon_N)}\right)^{2^n}\\ &\leqq \sum_{n=1}^{N-1}2^n\log\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)+\frac{(N-1)e^{2N-1}}{e^{2N}(1+\varepsilon_N)}\\ &\leqq \sum_{n=1}^{N-1}2^n\log\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)+(N-1)e^{-2^{N-1}}.\end{aligned}\tag{4.15}$$

Similarly we have everywhere on $|z|=r_N$

$$\log|f_1(z)| = \sum_{n=1}^{N-1}\log\left|1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right| \geqq \sum_{n=1}^{N-1}\log\left(\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n}-1\right).$$

Since $x\geqq 2$ implies $\log(x-1)>\log x-2/x$, we have

$$\begin{aligned}\log|f_1(z)| &\geqq \sum_{n=1}^{N-1}2^n\log\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)-2\sum_{n=1}^{N-1}\left(\frac{e^{2^n}}{e^{2N}(1+\varepsilon_N)}\right)^{2^n}\\ &\geqq \sum_{n=1}^{N-1}2^n\log\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)-2(N-1)e^{-2^{N-1}}.\end{aligned}\tag{4.16}$$

Hence from (4.15) and (4.16) we conclude

$$\log\operatorname*{Max}_{|z|=r_N}|f_1(z)|-\log\operatorname*{Min}_{|z|=r_N}|f_1(z)| \leqq 3(N-1)e^{-2^{N-1}}.\tag{4.17}$$

Everywhere on $|z|=r_N$ we have

$$\begin{aligned}\log|f_3(z)| &= \sum_{n=N+1}^{\infty}\log\left|1-\left(\frac{z}{e^{2^n}}\right)^{2^n}\right|\\ &\leqq \sum_{n=N+1}^{\infty}\log\left(1+\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n}\right) \leqq \sum_{n=N+1}^{\infty}\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n}\\ &\leqq \sum_{n=N+1}^{\infty}\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}} \leqq 2(1+\varepsilon_N)e^{2N}e^{-2^{N+1}} = 2(1+\varepsilon_N)e^{-2^N}.\end{aligned}\tag{4.18}$$

We also have on $|z|=r_N$ using calculations in (4.18)

$$\begin{aligned}\log|f_3(z)| &\geqq \sum_{n=N+1}^{\infty}\log\left(1-\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n}\right)\\ &\geqq -2\sum_{n=N+1}^{\infty}\left(\frac{e^{2N}(1+\varepsilon_N)}{e^{2n}}\right)^{2^n} \geqq -4(1+\varepsilon_N)e^{-2^N}.\end{aligned}\tag{4.19}$$

(4.18) and (4.19) imply

$$\log\operatorname*{Max}_{|z|=r_N}|f_3(z)|-\log\operatorname*{Min}_{|z|=r_N}|f_3(z)| < 6(1+\varepsilon_N)e^{-2^N}.\tag{4.20}$$

Combining (4.17) and (4.20) we conclude that if $\delta > 0$, there exists N_0 such that $N > N_0$ implies

$$\max_{|z|=r_N} |f_1(z)f_3(z)| < (1+\delta) \min_{|z|=r_N} |f_1(z)f_3(z)|. \tag{4.21}$$

For $N > N_0$ we now specify the value of ε_N. We first let ε_N' be such that

$$1+\varepsilon_N' = (1+\varepsilon_N)^{2^N}. \tag{4.22}$$

We determine ε_N' (and hence ε_N) so that

$$2^{1/2}\varepsilon_N' \max_{|z|=r_N} |f_1(z)f_3(z)| = 1, \tag{4.23}$$

where as before $r_N = (1+\varepsilon_N)\exp(2^N)$. This is certainly possible. If $\varepsilon_N' = 0$, the left side of (4.23) is 0; if $\varepsilon_N' = 1$, the left side of (4.23) is greater than 1 by (4.16) and (4.19). The left side of (4.23) is a strictly increasing, continuous function of ε_N' for ε_N' in [0, 1]. Thus the choice of ε_N' in (0, 1) is in fact unique. We note also $0 < \varepsilon_N < \varepsilon_N'$.

For $N \geqq N_0$ we now consider the behavior of $f_2(z)$ on the circle

$$|z| = r_N = (1+\varepsilon_N)\exp(2^N).$$

The image under $f_2(z)$ of $|z| = r_N$ is a circle of radius $1+\varepsilon_N'$ centered at 1. Thus $|f_2(r_N e^{i\theta})| \geqq \varepsilon_N'$ for all $\theta \in [-\pi, \pi]$. This fact together with (4.21) and (4.23) implies that for all $N > N_0$ and all $\theta \in [-\pi, \pi]$

$$\begin{aligned} 2^{-1/2}(1+\delta)^{-1} &= \varepsilon_N'(1+\delta)^{-1} \max_{|z|=r_N} |f_1(z)f_3(z)| \\ &< \varepsilon_N' \min_{|z|=r_N} |f_1(z)f_3(z)| \leqq |f(r_N e^{i\theta})|. \end{aligned} \tag{4.24}$$

We let $D = \{a : |a| \leqq 2^{-1/2}(1+\delta)^{-1}\}$. From the argument principle and (4.24) we conclude for $N \geqq N_0$ that if $a \in D$, then $n(r_N, a) = n(r_N, 0) = 2^{N+1}-2$.

We again consider the image under $w = f_2(z)$ of $|z| = r_N$ for $N \geqq N_0$. The circle $\{w : |w-1| = 1+\varepsilon_N'\}$ is traversed 2^N times as θ increases from $-\pi$ to π. Let $Q = \{x+iy : x \leqq -|y|\}$. Clearly there exist 2^N-1 disjoint intervals $[a_n, b_n] \subset (-\pi, \pi)$ such that for $1 \leqq n \leqq 2^N-1$

$$\begin{aligned} &\text{(i)} && f_2(r_N e^{i\theta}) \in Q \quad \text{if } \theta \in [a_n, b_n] \\ &\text{(ii)} && \arg f_2(r_N e^{ib_n}) - \arg f_2(r_N e^{ia_n}) = \pi/2 \qquad \text{and} \\ &\text{(iii)} && |f_2(r_N e^{i\theta})| \leqq 2^{1/2}\varepsilon_N' \quad \text{if } \theta \in [a_n, b_n]. \end{aligned} \tag{4.25}$$

For notational convenience we do not indicate the dependence of the intervals $[a_n, b_n]$ on N.

If $\theta \in [a_n, b_n]$ for some n, $1 \leqq n \leqq 2^N-1$, it follows from (4.23) and (4.25iii) that

$$\begin{aligned} |f(r_N e^{i\theta})| &\leqq |f_2(r_N e^{i\theta})| \max_{|z|=r_N} |f_1(z)f_3(z)| \\ &\leqq 2^{1/2}\varepsilon_N' \max_{|z|=r_N} |f_1(z)f_3(z)| = 1. \end{aligned} \tag{4.26}$$

We now estimate $\operatorname{Re}(zf_i'(z)/f_i(z))$ for $i=1, 2$, and 3 by the same methods as those employed in obtaining estimates (4.2), (4.3), and (4.5) of Example 1. We recall that these estimates depended only on the moduli and not on the arguments of the zeros of $f(z)$. We see easily that everywhere on $|z|=r_N$

$$\operatorname{Re}\frac{zf_1'(z)}{f_1(z)} > \frac{1}{2}\sum_{n=1}^{N-1} 2^n = 2^{N-1}-1$$

and

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} > \left(\frac{1}{2}\right)2^N = 2^{N-1}.$$

For $N \geqq N_0$ we have on $|z|=r_N$

$$\begin{aligned}\operatorname{Re}\frac{zf_3'(z)}{f_3(z)} &\geqq -r_N\sum_{n=N+1}^{\infty}\frac{2^n}{e^{2^n}-r_N} \geqq -2r_N\sum_{n=N+1}^{\infty}\frac{2^n}{e^{2^n}} \\ &\geqq -4r_N 2^{N+1}e^{-2^{N+1}} \geqq -2^{N+4}e^{-2^N} > -1.\end{aligned}$$

Thus, for $N\geqq N_0$, $d\arg(f_1f_3)(r_Ne^{i\theta})/d\theta>0$ and $d\arg f(r_Ne^{i\theta})/d\theta>0$ for all

$$\theta\in[-\pi,\pi].$$

Because of (4.25ii) and the fact that $d\arg(f_1f_3)(r_Ne^{i\theta})/d\theta>0$ for all $\theta\in[-\pi,\pi]$, we see that for each n, $1\leqq n\leqq 2^N-1$, there exists a subinterval $[c_n, d_n]\subset(a_n, b_n)$ such that

$$\arg f(r_Ne^{id_n})-\arg f(r_Ne^{ic_n}) = \pi/2.$$

By adjoining additional points to the set $\{c_n : 1\leqq n\leqq 2^N-1\}\cup\{d_n : 1\leqq n\leqq 2^N-1\}$, we obtain a sequence $-\pi=\theta_0<\theta_1<\cdots<\theta_{p(N)}=\pi$ such that

$$\text{(4.27)}\quad \begin{array}{ll}\text{(i)} & \arg f(r_Ne^{i\theta_{j+1}})-\arg f(r_Ne^{i\theta_j}) < 2\pi, \qquad 0\leqq j\leqq p(N)-1,\\ \text{(ii)} & \text{for at least } 2^N-1 \text{ values of } j,\\ & \arg f(r_Ne^{i\theta_{j+1}})-\arg f(r_Ne^{i\theta_j}) = \pi/2 \quad\text{and}\\ & |f(r_Ne^{i\theta})| \leqq 1 \quad \text{for } \theta\in[\theta_j, \theta_{j+1}].\end{array}$$

The second condition in (4.27ii) follows from (4.26).

For $0\leqq j\leqq p(N)-1$, let

$$A_j = \{tf(re^{i\theta}) : 0 < t \leqq 1 \text{ and } \theta_j < \theta \leqq \theta_{j+1}\}.$$

Since $d\arg f(r_Ne^{i\theta})/d\theta>0$ for all $\theta\in[-\pi,\pi]$, we conclude from the same discussion as in Example 1 that if $a\neq 0$, then $n(r, a)$ is equal to the number of values of θ in $[-\pi,\pi)$ for which $\operatorname{Im} a^{-1}f(re^{i\theta})=0$ and $\operatorname{Re} a^{-1}f(re^{i\theta})\geqq 1$. Since this is also the number of distinct values of j such that $a\in A_j$, we have $S(r_N)=\sum_{j=0}^{p(N)-1} m(A_j)$.

Let J be some set of 2^N-1 values of j satisfying (4.27ii). Certainly

$$\sum_{j\in J}(\arg f(r_Ne^{i\theta_{j+1}})-\arg f(r_Ne^{i\theta_j})) = \frac{\pi}{2}(2^N-1). \tag{4.28}$$

Since the total increase of $\arg f(r_N e^{i\theta})$ on $[-\pi, \pi]$ is $2\pi n(r_N, 0) = 2\pi(2^{N+1}-2)$, we conclude

$$\sum_{j\notin J} (\arg f(r_N e^{i\theta_{j+1}}) - \arg f(r_N e^{i\theta_j})) = 2\pi(2^{N+1} - 2^{N-2} - 7/4). \tag{4.29}$$

Because for any j

$$m(A_j) < (1/2\pi)(\arg f(r_N e^{i\theta_{j+1}}) - \arg f(r_N e^{i\theta_j})), \tag{4.30}$$

we conclude from (4.29) and (4.30) that

$$\sum_{j\notin J} m(A_j) < 2^{N+1} - 2^{N-2} - 7/4. \tag{4.31}$$

If $j \in J$, it follows from (4.27ii) that $m(A_j) < 1/8$. Hence

$$\begin{aligned} S(r_N) &= \sum_{j\notin J} m(A_j) + \sum_{j\in J} m(A_j) \\ &< 2^{N+1} - 2^{N-2} - \frac{7}{4} + \frac{1}{8}(2^N - 1) = \left(\frac{15}{16}\right) 2^{N+1} - \frac{15}{8}. \end{aligned}$$

Since $n(r_N, a) = 2^{N+1} - 2$ for all $a \in D$ and all $N > N_0$, we have for such values of a and N

$$\frac{n(r_N, a)}{S(r_N)} > \frac{2^{N+1}-2}{(15/16)2^{N+1} - 15/8} = \frac{16}{15}.$$

This establishes the first of the required properties for $f(z)$.

We now prove $f(z)$ has the second property as well. We have shown there exists a disk D, a sequence r_N, and a number $\gamma > 1$ such that $n(r_N, a) > \gamma S(r_N)$ for all $a \in D$ and all r_N. This implies there exists $\alpha > 0$, $\beta < 1$, and sets D_N such that $m(D_N) > \alpha$ and such that $a \in D_N$ implies $n(r_N, a) < \beta S(r_N)$. To show this, for $\beta < 1$ we let $D_N(\beta) = \{a : n(r_N, a) < \beta S(r_N)\}$. We then have

$$\begin{aligned} S(r_N) &= m(D)S(r_N, D) + m(D_N(\beta))(S(r_N, D_N(\beta)) \\ &\quad + m(\Sigma - (D \cup D_N(\beta)))S(r_N, \Sigma - (D \cup D_N(\beta))) \\ &\geqq m(D)\gamma S(r_N) + (1 - m(D) - m(D_N(\beta))\beta S(r_N) \\ &= (\gamma m(D) + \beta(1 - m(D)) - \beta m(D_N(\beta)))S(r_N). \end{aligned} \tag{4.32}$$

For β sufficiently close to 1, (4.32) implies that $m(D_N(\beta))$ is bounded away from 0. This establishes our contention. We take $D_N = D_N(\beta)$ for such a β.

We conclude that there exists $a_0 \neq 0$ belonging to D_N for infinitely many values of N. Let $a_0 = t_0 e^{i\theta_0}$ and let $L = \{te^{i\theta_0} : t \geqq t_0\}$. Thus for a subsequence of r_N, which we again denote by r_N, we have $n(r_N, a_0) < \beta S(r_N)$. Since $d \arg f(r_N e^{i\theta})/d\theta > 0$ for $\theta \in [-\pi, \pi]$ implies $n(r_N, te^{i\theta})$ is a nonincreasing function of t on $[0, \infty)$ for each $\theta \in [-\pi, \pi]$, we have for $a \in L$ that $n(r_N, a) \leqq n(r_N, a_0) < \beta S(r_N)$. On this subsequence r_N we conclude f has both the required properties.

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