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CLOSED SUBALGEBRAS OF GROUP ALGEBRAS

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Introduction. Let G be a compact abelian group with character group Γ. For basic definitions the reader is advised to see Chapters 1 and 9 of Rudin [7]. Each closed subalgebra A of the group algebra $L^1(G)$ induces an equivalence relation $\sim$ on Γ: $\alpha\sim\beta$ if and only if $\hat{f}(\alpha)=\hat{f}(\beta)$ every f in A, where $\hat{f}$ is the Fourier transform of f. We will denote the equivalence classes of $\sim$ by $\{E_\lambda\}$ where λ is in some index set and distinguish one special class $E_0=\{\gamma\in\Gamma : \hat{f}(\gamma)=0$ for every f in $A\}$ called the zero set of A. Since Γ is discrete and $\hat{f}\in C_0(\Gamma)$ for every f in $L^1(G)$, each E_λ, $\lambda\neq 0$, is finite. E_0 may be infinite.

For $\lambda\neq 0$ let P_λ be the trigonometric polynomial whose transform is the characteristic function of E_λ, and A_0 the smallest closed algebra containing every P_λ. Rudin [7, p. 231] has shown that A_0 induces $\sim$ and is contained in every closed subalgebra which induces $\sim$ with zero set E_0. If we define A^0 to be the algebra of all functions whose transforms are constant on every E_λ and zero on E_0 then A^0 will be the largest closed algebra inducing $\sim$.

Rudin [7] asked if there exist distinct closed subalgebras which induce the same equivalence relation. Or, equivalently, does there exist a closed subalgebra A with $A^0\neq A_0$? Kahane [4] gives a negative answer for $G=T$, the circle group.

In §1 we give sufficient conditions for an equivalence relation on the integers Z to be uniquely induced by exactly one closed subalgebra of $L^1(T)$. This result strengthens Theorem 3 of Kahane [4]. We also prove his result on $Z\times Z$.

In §2 we study the algebras A_0 and A^0 in detail and give necessary and sufficient conditions for $A_0=A^0$.

Finally in §3 we will consider factorization in closed subalgebras of $L^1(G)$ and show that although $L^1(G)$ has factorization, there exists a closed subalgebra A without factorization and such that $A\cap L^p(G)$ is L^1-dense in A, where $1<p<2$.

0. **Notation.** If B is a linear subspace of $L^1(G)$ we define its annihilator $B^\perp=\{\phi\in L^\infty(G) : \phi(f)=0$ for every f in $B\}$ where $\phi(f)=\int_G f(-x)\phi(x)\,dx$. ($dx$ denotes normalized Haar measure.)

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Since G is compact we have $L^\infty(G)\subset L^1(G)$, and it makes sense to consider the Fourier transform of $\phi\in L^\infty(G)$, defined $\hat{\phi}(\gamma)=\int_G \phi(x)\gamma(-x)\,dx$ where $\gamma\in\Gamma$. Hence, for a trigonometric polynomial $f=\sum_{i=1}^n a_i\gamma_i$, $\gamma_i\in\Gamma$ we have

$$\begin{aligned}\phi(f) &= \int_G \left(\sum_{i=1}^n a_i\gamma_i(-x)\right)\phi(x)\,dx\\ &= \sum_{i=1}^n a_i\int_G \gamma_i(-x)\phi(x)\,dx = \sum_{i=1}^n a_i\hat{\phi}(\gamma_i).\end{aligned}$$

For f and g in $L^1(G)$ we define convolution $(f*g)(x)=\int_G f(x-y)g(y)\,dy$. Thus, with the above notation, it follows that $\phi(f)=(\phi*f)(0)$.

I. **Synthesis.** In this section we will be concerned specifically with the circle group T and the torus T^2. Z will denote the group of integers. We will exhibit a class of partitions of Z and $Z\times Z$ which are induced by exactly one closed subalgebra of $L^1(T)$ and $L^1(T^2)$, respectively.

First, we will generalize Theorem 2 of Kahane [4] on Z and second, we will prove an analagous result on $Z\times Z$.

For the following definition G will be an arbitrary compact abelian group and Γ its dual group.

1.1. Definition. An equivalence relation $\sim$ on Γ with a distinguished class E_0 is said to have synthesis if it is induced by exactly one closed subalgebra of $L^1(G)$ with zero set E_0.

Kahane [4] has shown that if $\sim$ is an equivalence relation on Z which satisfies: $n\sim m$ implies $|n-m|\leqq M$, where M is a fixed constant, then $\sim$ has synthesis.

The following is a generalization of Kahane's result.

1.2. Theorem. *Let $\sim$ be an equivalence relation on Z with a distinguished class E_0. Suppose that for $n, m\notin E_0$, $n\sim m$ and $|n|\leqq|m|$ we have that $|m-n|\leqq M|n|^{1/2}$ (M a constant). Then $\sim$ has synthesis.*

Proof. It is well known that if $h\in L^1(T)$ and is continuous at $x_0\in T$, then the Cesaro sums $\{U_n\}$ of its Fourier series $\sum_{j=-\infty}^{\infty}\hat{h}(j)e^{ijx}$ converge to h at the point x_0. By a standard computation we may write

$$U_n(x) = \sum_{j=-\infty}^{\infty} \hat{K}_n(j)\hat{h}(j)e^{ijx} \tag{1}$$

where $\hat{K}_n(j)=\max(0;1-|j|/(n+1))$. Now let $\sim$ satisfy the above hypotheses. Also, let A^0 and A_0 be the maximal and minimal subalgebras of $L^1(T)$, respectively, which induce $\sim$. We want to show $A^0=A_0$ or equivalently, $A^{0\perp}=A_0^\perp$.

Obviously, $A^{0\perp}\subset A_0^\perp$ so we need only prove the reverse containment. For this purpose, choose $f\in A^0$ and $\phi\in A_0^\perp$. We will show $\phi(f)=0$.

Since $\phi\in L^\infty(G)$ and $f\in L^1(G)$ we have that $\phi*f$ is continuous. In particular, it is continuous at 0. We shall approximate $(\phi*f)(0)=\phi(f)$ by the Cesaro sums

$U_n(0)$ of (1) where $h=\phi * f$. In fact, we will show $U_n(0) \to 0$ as $n \to \infty$. Since $\phi \in A_0^{\perp}$ it follows that $\phi(P_\lambda)=0$ or

$$\sum_{j \in E_\lambda} \hat{\phi}(j) = 0. \tag{2}$$

Define $j_\lambda = \min \{|j| ; j \in E_\lambda\}$. Now using (2) and the fact that $\hat{f}$ is constant on every E_λ, we have

$$U_n(0) = \sum_{\lambda \neq 0} \sum_{j \in E_\lambda} \{\hat{K}_n(j) - \hat{K}_n(j_\lambda)\} \hat{f}(j) \hat{\phi}(j) \tag{3}$$

where the inner sum extends over those λ for which $j_\lambda \leqq n$. If $j \in E_\lambda$ for such a λ we have $|j - j_\lambda| \leqq M j_\lambda^{1/2}$ or $|j| \leqq j_\lambda + M j_\lambda^{1/2} \leqq (M+1) j_\lambda \leqq (M+1)n$. Hence, it follows that $\hat{K}_n(j_\lambda) - \hat{K}_n(j) \leqq (M+1)/n^{1/2}$. Thus, setting $N=(M+1)n$ and applying the Cauchy-Schwarz inequality and the Plancherel theorem to (3) we obtain

$$|U_n(0)| \leqq \frac{(M+1)}{n^{1/2}} \sum_{j=-N}^{N} |\hat{f}(j)\hat{\phi}(j)| \leqq (M+1)\left[\frac{1}{N} \sum_{j=-N}^{N} |\hat{f}(j)|^2\right]^{1/2} \|\phi\|_2$$

which tends to 0 as $N \to \infty$ since $\phi \in L^2(G)$ and $\hat{f} \in C_0(\Gamma)$.

Daniel Rider [6] has an example of an equivalence relation $\sim$ on Z satisfying: $n \sim m$, $|n| \leqq |m|$ implies $|n-m| \leqq |n|$, and does not have synthesis.

This yields the following interesting question which we have not been able to answer: Suppose $\sim$ is an equivalence relation on Z satisfying: $n \sim m$, $|n| \leqq |m|$ implies $|n-m| \leqq M|n|^\delta$, where M is some constant and $1/2 < \delta < 1$. For which values of δ does $\sim$ have synthesis?

The more ambitious task of classifying all synthetic partitions is beyond our reach. However, suppose S is a "Sidon set" (see [7, pp. 121–130] for the definition and facts about Sidon sets) and A is a closed subalgebra of $L^1(T)$ with $Z - E_0 \subset S$. Then $A^0 = A_0$. In fact, it will follow that if B is a closed subalgebra of A then $B^0 = B_0$.

The problem of discovering synthetic equivalence relations on $Z \times Z$ is open. Although we will prove an extension of Kahane's result to $Z \times Z$, the correspondingly stronger result of (1.2) has eluded us.

We begin with some facts and definitions which may all be found in Zygmund [8, Vol. II].

The torus will be denoted T^2. Let $R(n) = \{(i,j) \in Z \times Z : |i| \leqq n, |j| \leqq n\}$. For $h \in L^1(T^2)$ let

$$S_n(\bar{x}) = \sum_{j \in R(n)} \hat{h}(j) e^{i(j \cdot \bar{x})} \tag{1}$$

where $\bar{x} \in T^2$, and $\cdot$ represents the usual dot product.

By a standard manipulation we may write the Cesaro sums $\{U_n(\bar{x})\}$ of (1) by

$$U_n(\bar{x}) = \frac{1}{(n+1)^2} \sum_{j \in R(n)} (n - |j_1| + 1)(n - |j_2| + 1) \hat{h}(\bar{j}) e^{i(j \cdot \bar{x})} \tag{2}$$

where $\bar{j} = (j_1, j_2)$.

1.3. THEOREM [8, VOL. II, P. 304]. *Suppose* $h \in L^{\infty}(T^2)$ *and* h *is continuous at* $\bar{x}_0 \in T^2$. *Then* $|h(\bar{x}_0) - U_n(\bar{x}_0)| \to 0$ *as* $n \to \infty$.

1.4. THEOREM. *Let* $\sim$ *be an equivalence relation on* $Z \times Z$ *which satisfies*; $(n_1, m_1) \sim (n_2, m_2)$ *implies* $|n_1 - n_2| \leqq M$ *and* $|m_1 - m_2| \leqq M$ *for* $(n_i, m_i) \notin E_0$, $i = 1, 2$, *where* M *is some constant. Then* $\sim$ *has synthesis.*

Proof. The proof is similar to that of (1.2). We define

$$\hat{K}_n(i, j) = \max(0; 1 - |i|/(n+1)) \max(0; 1 - |j|/(n+1))$$

$i_\lambda = \min(|i|; (i, j) \in E_\lambda)$, and $j_\lambda = \min(|j|; (i, j) \in E_\lambda)$. Now if $\phi \in A_0^\perp$, $f \in A^0$ we let $h = \phi * f$, $\bar{x}_0 = 0$ and rewrite (2) as

$$U_n(0) = \sum_{(i,j) \in Z \times Z} \hat{K}_n(i, j) \hat{\phi}(i, j) \hat{f}(i, j),$$

$$= \sum_{\lambda \neq 0} \sum_{(i,j) \in E_\lambda} \{\hat{K}_n(i, j) - \hat{K}_n(i_\lambda, j_\lambda)\} \hat{f}(i, j) \hat{\phi}(i, j)$$

where the inner sum extends over those λ for which $i_\lambda \leqq n$ and $j_\lambda \leqq n$. If $(i, j) \in E_\lambda$ for such a λ then $|i| \leqq (M+1)n$ and $|j| \leqq (M+1)n$. Hence $\hat{K}_n(i_\lambda, j_\lambda) - \hat{K}_n(i, j) \leqq C/n$ where C is a constant independent of n. Thus, as in the proof of (1.2) we have

$$|U_n(0)| \leqq \frac{C}{n} \left[\sum_{(i,j) \in R(n)} |\hat{f}(i, j)|^2 \right]^{1/2} \|\phi\|_2$$

which tends to 0 as $n \to \infty$.

II. **The algebras** A_0 **and** A^0. Throughout this section, G will be an arbitrary compact abelian group and A a closed subalgebra of $L^1(G)$ with induced equivalence relation $\sim$ and zero set E_0. We will be concerned with finding conditions which will guarantee $A_0 = A^0$. It will be shown in (2.1) that although A_0 and A^0 have the same maximal ideal space, their annihilators $A_0^\perp$ and $A^{0\perp}$ respectively, can be characterized in (2.3) quite simply. We will construct a linear subspace B of $L^\infty(G)$ consisting of trigonometric polynomials such that $A^{0\perp} = w^*$-closure of B and $A_0^\perp = (L^2\text{-closure of } B) \cap L^\infty(G)$.

Denote the maximal ideal space of A by $\Delta(A)$ and let Γ' be the quotient space obtained from $\Gamma \backslash E_0$ by identifying equivalent elements of Γ. If we let γ' be the image of γ under the quotient map, it is clear that γ' induces a nonzero homomorphism h on A defined by $h(f) = \hat{f}(\gamma)$ for $f \in A$. However, the converse is also true.

2.1. THEOREM. *Every nonzero homomorphism on A is induced by an element of* Γ'.

Proof. Define $T: A \to C_0(\Gamma')$ by $T(f)(\gamma') = \hat{f}(\gamma)$. It is easy to show that $T(A_0)$ is a separating, conjugate-closed subalgebra of $C_0(\Gamma')$. Since $A_0 \subset A$ (Rudin [7, p. 232]) we have by the Stone-Weierstrass theorem that $T(A)$ is dense in $C_0(\Gamma')$ under the uniform norm. It follows that the adjoint map $T^*: \Gamma' \to \Delta(A)$ carries

Γ' homeomorphically onto a closed subset of $\Delta(A)$ (see for example [5, p. 76]). It remains to show $T^*(\Gamma')=\Delta(A)$. Choose $h \in \Delta(A)$. Then for $f \in A$

$$|h(f)|^n = |h(f^n)| \leqq \|f^n\|_A = \|f^n\|_{L^1(G)}.$$

By the spectral radius formula we obtain

$$|h(f)| \leqq \|\hat{f}\|_{C_0(\Gamma)} = \|\hat{f}\|_{C_0(\Gamma')}.$$

Hence, the function $h'(T(f))=h(f)$ is a continuous homomorphism on $T(A)$ and may be extended to all of $C_0(\Gamma')$ since $T(A)$ is dense in $C_0(\Gamma')$. Thus, there exists $\gamma' \in \Gamma'$ with $h(f)=T^*(\gamma')f$ for every $f \in A$. Consequently, T^* is onto.

The above theorem suggests that Banach algebra methods of distinguishing A_0 and A^0 will be of little help. Also, the fact that A_0 is generated as a vector space as well as an algebra by the P_λ leads us to study the annihilators $A_0^\perp$ and $A^{0\perp}$. For this purpose we introduce the following notation.

Let $\sim$ have equivalence classes $\{E_\lambda\}$. For $\lambda \neq 0$ and $E_\lambda=\{\gamma_1, \ldots, \gamma_n\}$ define

$$B_\lambda = \left\{\phi \in L^\infty(G) : \phi = \sum_{i=1}^n a_i\gamma_i \text{ where } a_1, \ldots, a_n \text{ are complex numbers with } \sum_{i=1}^n a_i = 0\right\}$$

and let B_0 be the linear space of $L^\infty(G)$ generated by the elements of E_0. (If $E_0=\varnothing$, let $B_0=\varnothing$.) Finally, let B be the linear space of $L^\infty(G)$ generated by the elements of $\bigcup B_\lambda$.

2.2. Lemma. *For any closed subalgebra A of $L^1(G)$, we have $B\subset A^\perp$.*

Proof. Pick $f \in A$ and an index λ. If $\lambda\neq 0$ and $\phi \in B_\lambda$, then $\phi=\sum_{i=1}^n a_i\gamma_i$ with $\sum_{i=1}^n a_i=0$ where $E_\lambda=\{\gamma_1, \ldots, \gamma_n\}$. Then

$$\phi(f) = \int f(x)\phi(-x)\,dx = \int f(x)\sum_{i=1}^n a_i\gamma_i(-x)\,dx = \sum_{i=1}^n a_i\hat{f}(\gamma_i) = \hat{f}(E_\lambda)\sum_{i=1}^n a_i = 0.$$

If $\lambda=0$ and $\phi \in B_0$, then $\phi=\sum_{i=1}^k b_i\beta_i$ with $\beta_i \in E_0$ for $i=1, \ldots, k$. Then, similarly, $\phi(f)=\sum_{i=1}^n b_i\hat{f}(\beta_i)=0$ since $\hat{f}(\beta_i)=0$ for $i=1, \ldots, k$. Since the elements of the B_λ generate B and are contained in the linear space $A^\perp$, we have $B\subset A^\perp$.

2.3. Theorem. *Let A be a closed subalgebra of $L^1(G)$, then*
(1) $A_0^\perp=(L^2$-closure of $B) \cap L^\infty(G)$
(2) $A^{0\perp}=w^*$-closure of B.

Proof. (i) By (2.2) we have $A^\perp\supset B$. Now choose $\phi \in (L^2$-closure of $B) \cap L^\infty(G)$. We will show $\phi \in A_0^\perp$.

Now, there exists a sequence $\phi_1, \phi_2, \ldots$ in B with $\|\phi-\phi_n\|_2 \to 0$ as $n \to \infty$. Suppose $P_\lambda \in A_0$. Applying the Cauchy-Schwarz inequality and using the fact that $\phi_n(P_\lambda)=0$ for every n, we have

$$\begin{aligned} |\phi(P_\lambda)| &= \left| \int \phi(x) P_\lambda(-x)\, dx \right| \leqq \left| \int (\phi(x)-\phi_n(x)) P_\lambda(-x)\, dx \right| + |\phi_n(P_\lambda)| \\ &\leqq \|\phi-\phi_n\|_2 \cdot \|P_\lambda\|_2 \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Thus, $\phi(P_\lambda)=0$. Consequently, since linear combinations of the P_λ are dense in A_0, we have $\phi \in A_0^\perp$. This yields

$$(L^2\text{-closure of } B) \cap L^\infty(G) \subset A_0^\perp.$$

(ii) Choose $\phi \in A_0^\perp$. Clearly $\phi \in L^\infty(G)$. We will show $\phi \in$ (L^2-closure of B). Let $\lambda_1, \lambda_2, \ldots$ be the indices for which $\hat{\phi}(E_\lambda) \neq \{0\}$ and $\{\beta_1, \beta_2, \ldots\} = E_0 \cap$ support of $\hat{\phi}$. If $E_\lambda = \{\gamma_1, \ldots, \gamma_n\}$ then $0 = \phi(P_\lambda) = \sum_{i=1}^n \hat{\phi}(\gamma_i)$. So $\phi_\lambda = \sum_{i=1}^n \hat{\phi}(\gamma_i)\gamma_i \in B_\lambda \subset B$. By the Plancherel theorem ϕ is the L^2-limit of the sums $\sum_{i=1}^k (\phi_{\lambda_i} + \beta_i) \in B$. Hence, $\phi \in$ (L^2-closure of B) $\cap L^\infty(G)$. We have proven

$$A_0^\perp \subset (L^2\text{-closure of } B) \cap L^\infty(G).$$

Consequently, (i) and (ii) yield (1).

(iii) Using (2.2) we have $A^{0\perp} \supset B$. But every annihilator in $L^\infty(G)$ is w^*-closed. Thus,

$$A^{0\perp} \supset w^*\text{-closure of } B.$$

(iv) We will show $A^{0\perp} \subset w^*$-closure of B. Let $D = \{f \in L^1(G) : \phi(f) = 0$ for every $\phi \in B\}$. Then $D^\perp = w^*$-closure of B. Hence, it will suffice to show $A^0 \supset D$, for then $A^{0\perp} \subset D^\perp$.

Pick $g \in L^1(G)$ with $g \notin A^0$. Then either there exists $\gamma_1 \sim \gamma_2$ with $\gamma_1, \gamma_2 \notin E_0$ and $\hat{g}(\gamma_1) \neq \hat{g}(\gamma_2)$ or there exists $\beta \in E_0$ with $\hat{g}(\beta) \neq 0$. If the former is true, let $\Phi = \gamma_1 - \gamma_2 \in B$ then $\Phi(g) \neq 0$ and hence $g \notin D$. Otherwise, let $\Psi = \beta \in B$ and conclude $\Psi(g) \neq 0$. Consequently $A^0 \supset D$, or

$$A^{0\perp} \subset w^*\text{-closure of } B.$$

Combining (iii) and (iv), we have proven (2).

III. **Factorization.** If a commutative Banach algebra B has an approximate identity there exists a bounded set U of B such that x belongs to the closure of xU for every x in B. It is this property that motivates the following definition.

3.1. Definition. Let B be a commutative Banach algebra. We say that B has a *pseudo-identity* (p.i.) if and only if x belongs to the closure of xB for every x in B.

It is well known that $L^1(G)$ has an approximate identity which may be chosen to be a (p.i.) for $L^p(G)$, $1 \leqq p < \infty$. But if $1 < p \leqq \infty$, it will follow from the Hausdorff-Young theorem that $L^p(G)$ has no approximate identity.

Cohen [1] has shown that any Banach algebra B with approximate identity has factorization, i.e., $B=B\cdot B$. In particular, $L^1(G)=L^1(G)*L^1(G)$. One may ask if this property is transferred to all the closed subalgebras of $L^1(G)$. We will use a result of Daniel Rider [6] to prove the existence of a closed subalgebra A without factorization and $A\cap L^p(G)$ is L^1-dense in A for $1<p<2$.

We begin with a definition.

3.2. Definition. Let $1<p\leqq\infty$. A closed subalgebra A of $L^1(G)$ is a p-algebra if and only if $A\cap L^p(G)$ is L^1-dense in A.

Clearly, A_0 is a p-algebra for every p and every algebra A.

For what follows, we will adopt the notation: $A^2=A*A$ and $A^j=A^{j-1}*A$ for $j=3, 4, \ldots$, where A is a closed subalgebra of $L^1(G)$.

3.3. Lemma. *Let A be a p-algebra. Then there exists an integer k such that $A^m\subset A_0$ for $m\geqq k$.*

Proof. *Case* (i). Assume $p\geqq 2$. Then $A_0\subset A=L^1$-closure of $(A\cap L^p(G))\subset L^1$-closure of $(A\cap L^2(G))\subset A_0$. Hence, $A=A_0$ and so $A^m\subset A_0$ for every integer m.

Case (ii). Assume $1<p<2$. Choose k such that $p^{(k-1)}>q$ where $p^{-1}+q^{-1}=1$. Now suppose $m\geqq k$ and $f=f_1*\cdots*f_m$ where $f_j\in A$ for $j=1, 2, \ldots, m$. We will show $f\in A_0$. Let $\varepsilon>0$ be given.

Since A is a p-algebra, we may choose $h=h_1*\cdots*h_m$ with $h_j\in A\cap L^p(G)$ for $j=1, 2, \ldots, m$ and $\|h-f\|_1<\varepsilon$. If we define $\phi_j=\hat{h}_j$ for $j=1, 2, \ldots, m$ it will follow from the Hausdorff-Young theorem that $\phi_j\in L^q(\Gamma)$ for each j.

Let K be any finite subset of Γ. Then by the Hölder inequality we have

$$\begin{aligned}\sum_{\gamma\in K}|\hat{h}(\gamma)| &= \sum_{\gamma\in K}|\phi_1\cdots\phi_m|(\gamma) \leqq \Big[\sum_{\gamma\in K}|\phi_1|^q(\gamma)\Big]^{q^{-1}}\Big[\sum_{\gamma\in K}|\phi_2\cdots\phi_m|^p(\gamma)\Big]^{p^{-1}} \\ &\leqq \|\phi_1\|_q\Big[\sum_{\gamma\in K}|\phi_2\cdots\phi_m|^p(\gamma)\Big]^{p^{-1}}.\end{aligned}\tag{1}$$

By repeated applications of the Hölder inequality to (1) and the fact that $p^{(m-1)}\geqq p^{(k-1)}>q$ we obtain

$$\begin{aligned}\sum_{\gamma\in K}|\hat{h}(\gamma)| &\leqq C\Big[\sum_{\gamma\in K}|\phi_m(\gamma)|^{p^{(m-1)}}\Big]^{p^{-(m-1)}} \\ &\leqq C\Big[\sum_{\gamma\in K}|\phi_m(\gamma)|^{q}\Big]^{p^{-(m-1)}} \leqq C\|\phi_m\|_q\end{aligned}\tag{2}$$

where C is a constant independent of K. Since K was arbitrary, it follows that $\hat{h}\in L^1(\Gamma)\subset L^2(\Gamma)$. Hence by the Plancherel theorem we have $h\in(L^2(G)\cap A)\subset A_0$. Since ε was arbitrary, we conclude $f\in A_0$.

3.4. Theorem. *If A is a p-algebra such that $A\subset L^1$-closure of A^2 then $A=A_0$.*

Proof. From (3.3) we may pick k such that $A^m \subset A_0$ for $m \geqq k$. It follows that $A_0 \subset A \subset L^1$-closure of $A^2 \subset \cdots \subset L^1$-closure of A^{2^t} where t is chosen to satisfy $2^t > k$. Hence $A = A_0$.

The following corollaries are immediate.

3.5. Corollary. *If A is a p-algebra with (p.i.) then $A = A_0$.*

3.6. Corollary. *If A is a p-algebra with factorization then $A = A_0$.*

Rider [6] has constructed a closed subalgebra A of $L^1(T)$ which is generated by a function $f \in L^p(T)$ $(1 < p < 2)$ such that $A^0 = A \neq A_0$. It follows from (3.5) that A does not have a (p.i.) and from (3.6) that A does not have factorization.

Now let G be a locally compact abelian group and I a closed ideal of $L^1(G)$. Define $Z(I) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \text{ for every } f \in I\}$. The problem of "spectral synthesis" is concerned with the following question: Do there exist two distinct closed ideals I_1 and I_2 of $L^1(G)$ such that $Z(I_1) = Z(I_2)$? Malliavin [7, p. 172] has shown that this will always be the case if G is not compact. For G compact, $Z(I) = E_0$, and it is clear that $Z(I_1) = Z(I_2)$ implies $I_1 = I_2$.

Helson [3] has shown that if I_1 and I_2 are two distinct closed ideals of $L^1(G)$, G not compact, with $I_1 \subset I_2$ and $Z(I_1) = Z(I_2)$ then there exists a closed ideal I with $Z(I) = Z(I_1) = Z(I_2)$ and $I_1 \subsetneqq I \subsetneqq I_2$. One is struck with the following question for G compact: Suppose A is a closed subalgebra of $L^1(G)$ with $A \neq A_0$. Does there exist a closed subalgebra A_1 with $A \supsetneqq A_1 \supsetneqq A_0$? We will use Rider's example and (3.3) to show that this is not generally the case.

For this purpose, let A be a closed subalgebra of $L^1(T)$ with $A \neq A_0$ and generator $f \in L^p(G)$, $1 < p < 2$. Now, functions of the form $h(f)$ where h is a polynomial with $h(0) = 0$ are dense in A. If $\tilde{f}$ represents the image of f under the quotient map $A \to A/A_0$, we have that the functions $h(\tilde{f})$ are dense in the quotient algebra A/A_0. Now by (3.3) there exists a positive integer k such that $f^n \in A_0$ for $n \geqq k$. Since every finite dimensional linear subspace of A/A_0 is closed, it follows that A/A_0 is a finite dimensional vector space with generators $\tilde{f}, \tilde{f}^2, \ldots, \tilde{f}^{(k-1)}$ and thus has dimension less than k.

Suppose we could always find a closed subalgebra B_1 to satisfy $B \supsetneqq B_1 \supsetneqq B_0$ whenever B was a closed subalgebra with $B \neq B_0$. Then by induction we may choose k distinct closed subalgebras of $L^1(G)$ say $A_1, A_2, \ldots, A_k$ with $A \supsetneqq A_1 \supsetneqq \cdots \supsetneqq A_k \supsetneqq A_0$. But then $A/A_0 \supsetneqq A_1/A_0 \supsetneqq \cdots \supsetneqq A_k/A_0 \supsetneqq \{0\}$ implies A/A_0 has dimension greater than k. Contradiction.

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