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CHARACTERIZATION OF TAMING SETS ON 2-SPHERES(1)

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1. Suppose that a 2-sphere S in E^3 is tame modulo a closed subset F, and suppose that F is tame (i.e., F lies on a tame 2-sphere in E^3). Is S tame? And if S is not tame, at which points of F can S be wild? These questions are answered by Theorem 1.1. The author is deeply indebted to C. E. Burgess, who in private conversation pointed out how Lemma 2.3 and linking arguments can be used to establish special cases of Theorem 1.1 (see [6]).

If ε is a positive number, let F^ε denote the closed subset of F consisting of those points of F which lie in components of F of diameter equal to or greater than ε. Let $F\#$ denote the subset of F which consists of those points of F which are degenerate components of F. The set $F\#$ is not necessarily closed.

THEOREM 1.1. *If F is a tame, closed subset of a 2-sphere S in E^3, S is tame modulo F, and $W \subset F\#$, then*

(1) *S is tame modulo* cl $F\#$, *and*

(2) *there is a 2-sphere S' in E^3 which contains F such that* cl W *is the set of points at which S' is wild.*

Theorem 1.1 characterizes those sets which we shall call taming sets. A closed subset F of E^3 is a *taming set* if the following two conditions are satisfied:

(i) F lies on a 2-sphere in E^3, and

(ii) if a 2-sphere S in E^3 contains F and is tame modulo F, then S is tame.

If F is a taming set, then it follows from (i) and [1, Theorem 7] that there is a 2-sphere S in E^3 which contains F and is tame modulo F. By (ii), S is tame. Thus a taming set is tame. Hence we see that Theorem 1.1 characterizes taming sets as those tame, closed subsets of spheres which have no degenerate components.

Special cases of Theorem 1.1 have been established previously. We mention the principal results (hereafter referred to as statements (iii)–(vii) of §1): a closed subset F of a 2-sphere in E^3 is a taming set if

(iii) F is a tame finite graph [9, Theorem 2 and Corollary 1],

(iv) F is a tame Sierpiński curve [4, Theorem 8.2],

(v) F is a tame, nondegenerate treelike continuum [6],

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(vi) F is a tame, nondegenerate, locally connected continuum [8], or

(vii) F is a closed, countable union of taming sets [5, Theorem 3.1].

Although Bing stated (vii) in a slightly different form and for special cases only, his proof is valid in the general case.

Theorem 1.1 will follow from Theorem 1.2. Our first goal will be to give a proof of Theorem 1.2 based upon results from §2 and §3. But first we fix some notation and terminology.

Throughout this paper by a 2-sphere we shall mean a 2-sphere in E^3, and P will invariably denote the plane $\{(x, y, z) \mid z=0\}$ in E^3. If $X \subset E^3$ and $\varepsilon > 0$, then $N_\varepsilon(X)$ will denote the set $\{x \in E^3 \mid \rho(x, X) < \varepsilon\}$, where ρ is the Euclidean metric. An ε-set X is one such that Diam $X \equiv \sup \{\rho(x, y) \mid x, y \in X\} < \varepsilon$, and an ε-map (or homeomorphism) is one which moves no point as far as ε. A sphere S is tame modulo a set X if S is locally tame at each point of $S - X$. A *punctured disk* is the set remaining when the interiors of finitely many disjoint disks, each interior to a disk D, are removed from D.

THEOREM 1.2. *If the closed set F lies on a tame sphere and $\varepsilon > 0$, then F^ε is a taming set.*

Proof. The result is clear if F is a 2-sphere. Hence we may suppose that F is not a 2-sphere and that F lies in P since F is tame. Let S be a 2-sphere containing F^ε ($\varepsilon > 0$).

At this point we insert some informal comments on the proof. Lemma 2.3 asserts that it is sufficient for our purposes to show that any sufficiently small unknotted simple closed curve J in $E^3 - S$ is homotopically trivial in a small subset of $E^3 - F^\varepsilon$. Now if F^ε were a wild simple closed curve on S, then J might very well be homotopically nontrivial in $E^3 - F^\varepsilon$ and yet not be entangled in the "hole" of F^ε at all. However, since F^ε is flat, the homotopic nontriviality of J in $E^3 - F^\varepsilon$ is essentially equivalent to the entangling of J in the holes of F^ε. Hence any such entanglement should be reflected in the entanglement of J in the holes of other sets on S which closely approximate F^ε in some sense. In particular, since the holes in F^ε can be approximated by the holes in a finite graph on S and since every arc on S can be homeomorphically approximated by tame arcs on S, the entanglement of J with F^ε should be reflected in the entanglement of J with a tame finite graph on S which has holes much like those in F^ε. This intuitive idea is made precise in Lemmas 3.2 and 3.3. But Lemma 2.4 shows that J cannot be badly entangled in any tame finite graph on S. Although our actual proof uses a finite union of tame arcs on S rather than a tame finite graph, the preceding discussion gives an intuitive outline of the proof.

Let $p \in F^\varepsilon$, and let N be a spherical neighborhood of diameter less than ε centered at p. Let N_1 be a concentric spherical neighborhood whose closure lies in N. By Lemma 2.4 there is another concentric neighborhood N_2 whose closure lies in N_1 so that if J is any unknotted simple closed curve in $(E^3 - S) \cap N_2$ and T is any taming set on S, J bounds a disk in $N_1-(N_1 \cap T)$. Let J be any unknotted simple closed curve in $(E^3-S) \cap N_2$. We may assume that J is in general position with respect to P. Let D be the circular disk $\mathrm{cl}\,(N \cap P)$. Let $M=\mathrm{Bd}\; D \cup (D \cap F^\varepsilon)$. Then M is a continuum because Diam $D<\varepsilon$. Let C_0 denote a polyhedral cube with handles in E^3-J which is formed by thickening a punctured disk D_0 in P as in Lemma 3.3. By Lemma 3.3 there is a set generator G for $\pi(C_0)$ so that $G \cap N_1$ is a subset of a taming set T on S. The simple closed curve J bounds a disk E in $N_1-(N_1 \cap T)$ by our choice of N_2. Therefore $E \cap G=\varnothing$. Hence by Lemma 3.2, J bounds a disk E' in E^3-C_0, hence in E^3-M. Since Bd N is a round sphere and Bd $N-M$ is simply connected, E' can be cut off geometrically just inside Bd N so as to miss M. We conclude that J can be shrunk to a point in $N-M$, hence in $N-F^\varepsilon$. Since S, p, and N were arbitrarily chosen, it follows from Lemmas 2.2 and 2.3 that F^ε is a taming set.

The idea of cutting E' off geometrically near Bd N was suggested in a seminar on topology at the University of Wisconsin. We originally used a more complicated argument to show that the existence of E' implies that J can be shrunk in $N-M$.

Proof of Theorem 1.1. In order to prove conclusion (1) it suffices to show that S is locally tame at each point of $F-\mathrm{cl}\; F\#$. Let $p \in F-\mathrm{cl}\; F\#$. Let D be a disk on S with tame boundary such that $p \in \mathrm{Int}\; D$ and $D \subset (S-\mathrm{cl}\; F\#)$. Such a disk exists by [2, Theorem 1]. Let $F'=\mathrm{Bd}\; D \cup (F \cap D)$. Then F' is the closed countable union of the taming sets $\{(F \cap D)^{1/i}\}_{i=1}^{\infty}$ and Bd D (Theorem 1.2). Hence F' is a taming set by Bing's theorem given in the introduction as statement (vii). By Lemma 2.1, Int D is locally tame. Hence S is locally tame at p.

We now indicate without proof some procedures and lemmas sufficient to prove conclusion (2) of Theorem 1.1.

LEMMA 1.3. *Let F be a compact subset of P, and let p be a degenerate component of F. Then there is a sequence $\{A_i\}$ of annuli in P such that*

(1) Diam $A_i \to 0$ *as* $i \to \infty$,

(2) $\{p\} \cup A_{i+1} \subset$ *bounded component of* $P-A_i$ *for each* i, *and*

(3) $F-\{p\} \subset \bigcup_{i=1}^{\infty} \mathrm{Int}\; A_i$.

LEMMA 1.4. *Let F, p, and $\{A_i\}$ be as in the conclusion of Lemma 1.3. Fix m and let D denote the disk in P bounded by the inner boundary of A_m. Let U be an arbitrary open set in E^3 which contains* Int D. *Then* Bd D *bounds a disk D' in* Bd $D \cup U$ *such that p is the single wild point of D' and* $\bigcup_{i=m+1}^{\infty} A_i \subset D'$.

Indication of proof. The disk D' can be constructed as a feeler which winds in and out of P in U and is knotted like the arc described by Fox and Artin in [11, Example 1.2].

Proof of conclusion (2) of Theorem 1.1. We start with a tame sphere S which contains a large disk D in P with $F \subset \mathrm{Int}\; D$. (Since F is tame, we lose no generality in assuming $F \subset P$.) Let $\{w_i\}$ denote a countable dense subset of W and proceed inductively to insert for each i a Fox-Artin feeler in S with wild point at w_i as indicated by Lemmas 1.3 and 1.4. The two lemmas show that each adjusted sphere can be made to contain F. That the adjusted spheres converge to a sphere S' can be insured by methods described in [2, Theorem 7]. We can insure the wildness of S' at each point of $\{w_i\}$ and hence at each point of $\mathrm{cl}\,(W)$ by picking at stage i an arc A_i on the adjusted sphere which contains w_i and misses $\bigcup_{j\neq i}\{w_j\}$ and then by keeping A_i in all succeeding adjusted spheres. The arc A_i is necessarily wild at w_i; hence S', which contains A_i, is wild at w_i. We can make sure that S' is tame modulo $\mathrm{cl}\,(W)$ by requiring that each point of $S-\mathrm{cl}\,(W)$ lie in a neighborhood which is not adjusted after some given stage.

2. **Taming sets.** We now recall three well-known results and prove an elementary lemma regarding taming sets.

LEMMA 2.1 (BING [4, THEOREM 8.3] AND LOVELAND [18, THEOREM 18]). *If U is an open subset of a 2-sphere S in E^3, T is a taming set on S, and U is locally tame at each point of $U-(U\cap T)$, then U is locally tame.*

Again we note that Bing states his Theorem 8.3 for special kinds of taming sets only, but that his proof is sufficient to establish Lemma 2.1.

Let S denote a 2-sphere in E^3 and F a closed subset of S. We say that F satisfies (C, F, S) if for each $\varepsilon>0$ there is a $\delta>0$ so that each unknotted δ-simple closed curve in E^3-S can be shrunk to a point in an ε-subset of E^3-F. The following is Lemma 1 of [7].

LEMMA 2.2. *Suppose F is a closed subset of a 2-sphere S in E^3. Then F satisfies (C, F, S) if for each $p\in F$ and for each open set N containing p there is an open subset V containing p so that each unknotted simple closed curve in $(E^3-S)\cap V$ can be shrunk to a point in $N-(N\cap F)$.*

The following is a consequence of [18, Theorem 16] and the proof of [18, Theorem 13]. Loveland's Theorem 13 [18] states that in the case considered $(*, F, S)$ is satisfied if and only if (A', F, S) is satisfied. Loveland's proof that $(*, F, S)$ is satisfied if (A', F, S) is satisfied, however, uses only the weaker hypothesis that (C, F, S) is satisfied.

LEMMA 2.3. *Suppose that F is a closed subset of a 2-sphere in E^3 and that $F=F^\varepsilon$ for some $\varepsilon>0$. Then if F satisfies (C, F, S) for each 2-sphere S containing F, F is a taming set.*

LEMMA 2.4. *Let S be a 2-sphere and ε a positive number. Then there is a $\delta>0$ so that if F is any taming set on S and J is an unknotted δ-simple closed curve in E^3-S, then J bounds an ε-disk in E^3-F.*

Proof. Choose 2δ in the range $0<2\delta<\varepsilon/6$ so that each 2δ-set on S lies in the interior of an $\varepsilon/6$-disk on S. Let F be a taming set on S and J an unknotted δ-simple closed curve in E^3-S. Then J bounds a 2δ-disk D in E^3. The set $D\cap S$ is a 2δ-set on S, hence lies in the interior of an $\varepsilon/6$-disk E on S. By Bing's polyhedral approximation theorem for open subsets of spheres [1, Theorem 7], there is a homeomorphism f from S into E^3 that is fixed on F so that $f(S)$ is locally polyhedral modulo F and thus is tame, so that $[f(S) \cap D] \subset [f(\text{Int } E) \cap \text{Int } D]$, and so that $\text{Diam} f(E) < \varepsilon/6$. By the Tietze Extension Theorem there is a map g from D into $D \cup E$ so that g is the identity on the component K of $D - f(S)$ which contains J and g takes $D - K$ into $f(E)$. Then $g(D)$ is a singular $\varepsilon/3$-disk. Because $f(S)$ is tame, there is an $\varepsilon/3$-homeomorphism $h: E^3 \to E^3$ that is the identity on J and so that $h \circ g(D) \cap f(S) = \varnothing$. Then $h \circ g(D)$ is a singular ε-disk in $E^3 - f(S)$ bounded by J with no singularities near J. Thus by Dehn's Lemma [19], which one can adjust for nonpiecewise-linear maps by using [1, Theorem 7], J bounds a nonsingular ε-disk in $E^3 - f(S)$, hence in $E^3 - F$.

3. **Set generators.** Lemmas 3.2 and 3.3 are the goals of this section. By a 3-*manifold* we shall always mean a compact, connected, polyhedral 3-manifold-with-boundary in E^3.

A *set generator G for $\pi(K)$*, where K is a 3-manifold, is a closed, connected subset of $\text{Int } K$ such that for each loop $f: S^1 \to K$ (S^1 denotes the 1-sphere or circle) there is a homotopy $H: S^1 \times I \to K$ such that $H(s, 0) = f(s)$ for each $s \in S^1$ while $H(S^1 \times 1) \subset G$. The basic properties of set generators are summarized in Lemma 3.1.

Remark. The symbol $\pi(K)$ denotes the fundamental group of K, but the presence of $\pi(K)$ in our terminology is merely meant to suggest the properties enjoyed by set generators. As a group $\pi(K)$ will play no role in our discussion.

Lemma 3.1. *Let K be a* 3-*manifold and G a set generator for $\pi(K)$.*

(1) *If $H: G \times I \to \text{Int } K$ is continuous and $H(g, 0) = g$ for each $g \in G$, then $H(G \times 1)$ is a set generator for $\pi(K)$.*

(2) *If C is a polyhedral cube in E^3 such that $C \cap K$ is a polyhedral annulus which is common to* Bd C *and* Bd K, *then G is a set generator for $\pi(K \cup C)$.*

(3) *If D is a polyhedral disk in K such that* Bd $D \subset$ Bd K, Bd D *is not nullhomotopic in* Bd K (Bd $D \nsim 0$ in Bd K), *and* Int $D \subset$ Int K, *then $G \cap D \neq \varnothing$.*

Proof. (1) and (2) are obvious. To prove (3) we proceed as follows. Suppose that $G \cap D = \varnothing$. Under this false assumption we show the existence of a disk E such that Bd $E \subset$ Bd K, Int $E \subset$ Int K, and E separates K into two 3-manifolds K_1 and K_2 such that $G \subset \text{Int } K_1$ while K_2 is not simply connected. Indeed, if D separates K, let $D = E$ and let K_1 and K_2 be the closures of the two components of $K - D$. Since G is connected and does not intersect D we may assume that $G \subset \text{Int } K_1$. Because Bd $D \nsim 0$ in Bd K, Bd K_2 is not simply connected. Hence by [20, p. 224], K_2 is not simply connected. This proves the existence of K_1 and K_2 when D separates K. If D does not separate K, then thicken D slightly in $K - G$ to form a handle C for the 3-manifold $K_0 = \text{cl}(K - C)$. Let A be an arc on Bd K_0 which joins the two components of $K_0 \cap C$. Then a regular neighborhood K_2 of $C \cup A$ in $K - G$ is a solid torus which intersects the 3-manifold $K_1 = \text{cl}(K - K_2)$ in a single disk E. Further, $G \subset \text{Int } K_1$. We have thus established the existence of the desired disk E and 3-manifolds K_1 and K_2. Let $f: S^1 \to \text{Int } K_2$ be a nontrivial loop in K_2 and $H: S^1 \times I \to K$ a homotopy such that $h(s, 0) = f(s)$ for each $s \in S^1$ while $H(S^1 \times 1) \subset G$. Since E is an absolute retract, there is a map $H_0: S^1 \times I \to K_2$ such that H_0 agrees with H on the component R of $(S^1 \times I) - H^{-1}(E)$ which contains $S^1 \times \{0\}$, while $H_0[(S^1 \times I) - R] \subset E$. This shows that f is homotopic in K_2 to a loop in E, a contradiction. This completes the proof of (3).

LEMMA 3.2. *Let K be a 3-manifold, J a simple closed curve in $E^3 - K$, and G a set generator for $\pi(K)$. Then J bounds a disk in $E^3 - K$ if and only if J bounds a disk in $E^3 - G$.*

Proof. If J bounds a disk in $E^3 - K$, then J certainly bounds a disk in $E^3 - G$. Suppose that J bounds a disk D in $E^3 - G$. Then we may assume that D is locally polyhedral modulo J and that D is in general position with respect to Bd K. We proceed by induction on the number of components of $D \cap \text{Bd } K$. If $D \cap \text{Bd } K = \varnothing$, we are done. Otherwise there is a subdisk E of D such that Bd $E \subset \text{Bd } K$ and Int $E \cap \text{Bd } K = \varnothing$. If Bd E is nullhomotopic in Bd K, then, since $G \subset \text{Int } K$, the trivial intersections of D with Bd K can be removed by cutting D off near Bd K. This reduces the number of components of $D \cap \text{Bd } K$, and the result follows by induction. If Bd E is not nullhomotopic in Bd K, then Int $E \not\subset \text{Int } K$ by Lemma 3.1(3); and so Int $E \subset E^3 - K$. In this case thicken E into a polyhedral cube C so that $C \cap K$ is an annulus and $D \cap C = E \subset \text{Int}(C \cup K)$. By Lemma 3.1(2), G is a set generator for $\pi(C \cup K)$. Also $J \subset E^3 - (C \cup K)$, $D \cap G = \varnothing$, and $D \cap \text{Bd}(C \cup K)$ has fewer components than does $D \cap \text{Bd } K$. Thus by induction J bounds a disk in the complement of $C \cup K$, thus in the complement of K. This completes the inductive proof.

LEMMA 3.3. *Let M denote a subcontinuum of a circular disk D in P such that M contains* Bd D *and* $M - \text{Bd } D \subset S$, *where S is a 2-sphere in E^3. Let C_0 denote a polyhedral cube with handles formed by thickening a punctured disk D_0 in P which has the following properties*:

(1) $M \subset \text{Int } D_0$.

(2) *No two components of $P - D_0$ lie in the same component of $P - M$.*

Then given $\varepsilon > 0$, there is a set generator G for $\pi(C_0)$ such that $G - N_\varepsilon(\text{Bd } D)$ is a subset of a taming set on S.

Proof. We may assume that $2\varepsilon < \text{Diam } D$. Let D' be the circular subdisk of D that is concentric with D so that $\rho(\text{Bd } D, \text{Bd } D') = \varepsilon/2$. Choose $\alpha > 0$ so that $\alpha < \min(\varepsilon/4, \rho(\text{Bd } C_0, M))$. Choose $\beta > 0$ so that if $x, y \in S$ and $0 < \rho(x, y) < \beta$, then x and y are endpoints of an $\alpha/2$-arc on S. Choose $\gamma > 0$ so that $\gamma < \min(\beta/3, \alpha/4)$. Let E_0 be a polyhedral punctured disk in P which has the following properties:

(3) $M \subset \text{Int } E_0 \subset E_0 \subset N_\gamma(M)$.

(4) $E_0 \cap D' \subset N_\gamma(\text{Int } D \cap M)$.

Note that $E_0 \subset \text{Int } D_0$ and that, roughly speaking, E_0 is a much better approximation to M than is D_0. Let G_0 be a connected finite graph in $D \cap E_0$ that contains those boundary components of E_0 in D. Then G_0 is clearly a set generator for $\pi(C_0)$. The graph G_0 is the union of finitely many γ-arcs $A_1, A_2, \ldots, A_n$ so that if two intersect then their intersection is a common endpoint. Let $v_1, v_2, \ldots, v_j$ denote the endpoints of $A_1, A_2, \ldots, A_n$. To each v_i assign a point w_i as follows: If $v_i \in D-D'$, let $w_i = v_i$. If $v_i \in D'$, let w_i be a piercing point of S in $S \cap M$ so that $\rho(v_i, w_i) < \gamma$. This is possible because of (4) and the fact that the set of nonpiercing points on S is 0-dimensional [3]. We may require that $w_1, w_2, \ldots, w_j$ be distinct. We say that A_i is of type 1 if the endpoints of A_i are in D', that A_i is of type 2 if the endpoints of A_i are in $D-D'$, and that A_i is of type 3 otherwise. To each A_i assign an arc B_i and a homeomorphism $h_i: A_i \to B_i$ as follows.

Case 1. If A_i is of type 1 and v_r and v_s are the endpoints of A_i, then w_r and w_s are piercing points of S in $S \cap M$ so that $\rho(w_r, w_s) < 3\gamma < \beta$. Hence there is an $\alpha/2$-arc B_i' on S of which w_r and w_s are endpoints. It follows from [2, Theorem 1] and [4, Theorem 8.5] that there is an $\alpha/2$-arc B_i from w_r to w_s such that B_i is locally tame modulo w_r and w_s. Hence B_i is tame by [12, Lemma 6.1]. Let $h_i: A_i \to B_i$ be any homeomorphism that sends v_r to w_r and v_s to w_s.

Case 2. If A_i is of type 2, let $B_i = A_i$ and let $h_i: A_i \to B_i$ be the identity map.

Case 3. If A_i is of type 3 with endpoints v_r and v_s, let B_i be the straight line segment joining w_r and w_s and let $h_i: A_i \to B_i$ be any homeomorphism that sends v_r to w_r and v_s to w_s.

We now show that $G = \bigcup_{i=1}^n B_i$ satisfies the requirements of Lemma 3.3. Let $h: G_0 \to G$ be the continuous map defined piecewise by $h|A_i = h_i$. Define $H: G_0 \times I \to E^3$ by $H(g, t) = (1-t)g + th(g)$ for each $g \in G_0$ and $t \in I$. Thus by Lemma 3.1(1) we can show that G is a set generator for $\pi(C_0)$ by showing that $H(G \times I) \subset \text{Int } C_0$. If $g \in A_i$, $0 \leqq t \leqq 1$, and A_i is of type 2, then $H(g, t) = g \in \text{Int } C_0$. If $g \in A_i$ and A_i is of type 1 or 3, then $B_i \cap M \neq \varnothing$ while $\text{Diam}\,(A_i \cup B_i) < \alpha$ because (1) $\text{Diam } A_i < \gamma$, (2) $\text{Diam } B_i < \alpha/2$, (3) $\rho(A_i, B_i) < \gamma$, and (4) $\gamma < \alpha/4$. Hence $H(\{g\} \times I)$, which is in the convex hull of $A_i \cup B_i$, is in the convex hull of an α-set which intersects M. Since $\rho(M, \text{Bd } C_0) > \alpha$, $H(\{g\} \times I) \subset \text{Int } C_0$. Thus G is a set generator for $\pi(C_0)$. Now $\bigcup \{B_i | A_i \text{ is of type } 1\}$ is by statements (iii) and (vii) of §1 a taming set on S. Hence in order to complete the proof it suffices to show that if $g \in A_i$ where A_i is of type 2 or 3, then $H(g, 1) \in N_\varepsilon(\text{Bd } D)$. But if $g \in A_i$, where A_i is of type 2 or 3, then $\rho(g, \text{Bd } D) < \varepsilon/2 + \gamma$ because (1) $\text{Diam } A_i < \gamma$, (2) $A_i \cap (D-D') \neq \varnothing$, and (3) $\rho(x, \text{Bd } D') < \varepsilon/2$ for each $x \in D-D'$. Thus $\rho(H(g, 1), \text{Bd } D) < \varepsilon/2 + \gamma + \alpha < \varepsilon$ because $\rho(H(g, 1), H(g, 0)) < \alpha$ for each $g \in G$. This completes the proof.

4. **Applications of Theorem 1.1.** The fact that certain sets are taming sets has played an important role in several papers published previously. Notable examples are Bing's papers [4] and [5], Eaton's results in [10], and Gillman's work on piercing points [12]. In another paper we use the fact that a tame, nondegenerate,

locally connected continuum is a taming set in giving the following extension of a result announced by White [21]:

THEOREM 4.1. *A 2-sphere S in E^3 is tame if for each $p \in S$, each $\varepsilon > 0$, and each component V of $E^3 - S$, there is a map $f: \Delta \to V \cup S$ such that $f|\mathrm{Bd}\,\Delta$ is a tame loop in S that links p on S and $f(\mathrm{Int}\,\Delta)$ is an ε-subset of V (Δ denotes a standard disk).*

However, in this paper we prove only some of the most immediate consequences of our characterization of taming sets.

THEOREM 4.2. *Let F be a closed subset of a 2-sphere S in E^3 such that*

$$F = \bigcup_{i=1}^{\infty} (F_i - (F_i)^{\#}),$$

where each F_i is closed and tame. Then F is a taming set.

Proof. F is a closed countable union of taming sets by Theorem 1.2. Hence F is a taming set by statement (vii) of §1.

COROLLARY 4.3. *If $\{F_i\}_{i=1}^n$ is a finite collection of tame continua on a 2-sphere S in E^3, then $\bigcup_{i=1}^n F_i$ is tame.*

Proof. The set $\bigcup \{F_i | F_i$ is nondegenerate$\}$ is a tame set by Theorem 4.2. But a tame, closed subset of a sphere plus a finite number of points on the sphere is clearly tame.

EXAMPLE 4.4. Theorem 4.2 and its corollary are in a sense best possible. Indeed any arc in E^3 which is wild only at an end point lies on a 2-sphere and is a closed countable union of tame continua. Each such arc is also the union of two closed, tame sets which, by Theorem 4.2, must have a degenerate component in common.

Theorem 1.1 shows that the following definition has reasonable consequences. If F is a closed subset of a 2-sphere, then we say that F is *locally tame* at p if $N_\varepsilon(p) \cap F$ lies on a tame sphere for some $\varepsilon > 0$.

THEOREM 4.5. *A closed set F on a 2-sphere S is tame if it is locally tame modulo a taming set T on S such that $F^{\#} \subset T$.*

Proof. For each $p \in F - (F \cap T)$, there is a small disk $D_p \subset S - T$ so that $p \in \mathrm{Int}\, D_p$ and $D_p \cap F$ is tame. There is a countable subcollection $\{D_{p_i}\}_{i=1}^{\infty}$ so that $F - (F \cap T) \subset \bigcup_{i=1}^{\infty} \mathrm{Int}\, D_i$. Then $F \cup T$ is a closed countable union of the taming sets T and $\{(F \cap D_{p_i})^{1/j}\}_{i,j}$. Thus $F \cup T$ is tame.

Theorem 4.5 has a corollary which extends a result proved by Doyle and Hocking [9, Corollary 6] to the effect that if p is an isolated wild point of a 2-sphere S, then any arc on S which contains p is wild at p.

COROLLARY 4.6. *If p is an isolated wild point of any continuum M on a 2-sphere S, then any nondegenerate subcontinuum of S which contains p is wild at p.*

COROLLARY 4.7. *If p is a point of a 2-sphere S in E^3, then p is a piercing point of S if and only if p lies on a nondegenerate tame subcontinuum of S.*

Proof. This follows from Theorem 1.1, Theorem 4.5, the fact that p lies on an arc A on S that is tame modulo p, and from Gillman's characterization of piercing points [12, see Theorem 11] as being those which lie on tame arcs on S.

5. **Final remarks.** One of the advantageous properties of taming sets is the following

THEOREM 5.1 ([18, THEOREM 16] AND [12]). *A closed subset F of a 2-sphere in E^3 is a taming set if and only if $(*, F, S)$ is satisfied for each 2-sphere S containing F; i.e., F is a taming set if and only if each 2-sphere containing F can be side approximated missing F. (See Loveland [18] for a precise definition of $(*, F, S)$.)*

However, the fact that $(*, F, S)$ is satisfied implies that $(*, F', S)$ is satisfied for each closed subset F' of F. In view of this fact and since any nondegenerate closed set F has closed subsets which are not taming sets (Theorem 1.1), it follows from Theorem 5.1 and conclusion (2) of Theorem 1.1 that there are many examples of closed subsets F of spheres S and S' so that $(*, F, S)$ is satisfied while $(*, F, S')$ is not. Furthermore, even though in a particular instance $(*, F, S)$ is not satisfied, it may very well be that $(*, F, \text{Int } S)$ or $(*, F, \text{Ext } S)$ is satisfied. In view of these remarks it is valuable to develop for $(*, F, S)$, $(*, F, \text{Int } S)$, and $(*, F, \text{Ext } S)$ analogues to the theorems on taming sets. Many such results appear in [18]. As our final result, Theorem 5.3, we prove the analogue to statement (vii) of §1. Theorem 5.3 answers in the affirmative a question raised by Loveland [18, p. 515]. Notation and concepts which now appear but did not appear in §§1–4 of this paper are explained in [18].

LEMMA 5.2. *Let F be a closed subset of a 2-sphere S in E^3 so that $(*, F, S)$ is satisfied. Then given $\varepsilon > 0$, there is a taming set F' on S such that $F \subset F' \subset N_\varepsilon(F)$.*

Proof. From plane topology and the results of [2] it follows that there is a finite family $D_1, \ldots, D_n$ of disjoint punctured disks on S such that $F \subset \bigcup_{i=1}^n \text{Int } D_i \subset \bigcup_{i=1}^n D_i \subset N_\varepsilon(F)$ and such that $\bigcup_{i=1}^n \text{Bd } D_i$ consists of a finite number of tame simple closed curves. Then $F \cup (\bigcup_{i=1}^n \text{Bd } D_i)$ satisfies $(*, F \cup (\bigcup_{i=1}^n \text{Bd } D_i), S)$ by [18, Theorem 21]. Thus by [18, Theorems 6 and 16] there is a null sequence $\{E_i\}_{i=1}^\infty$ of disjoint disks on S so that if $M = S - \bigcup_{i=1}^\infty \text{Int } E_i$, then M is a taming set and $F \cup (\bigcup_{i=1}^n \text{Bd } D_i) \subset M - \bigcup_{i=1}^\infty \text{Bd } E_i$. From [18, Theorems 16, 19, and 21] it follows that $M \cap (\bigcup_{i=1}^n D_i)$ is a taming set and hence satisfies the requirements of the lemma.

THEOREM 5.3. *Let $\{F_n\}$ be a countable collection of closed sets on a 2-sphere S in E^3, and let V be a component of $E^3 - S$. Suppose further that $F = \bigcup_{i=1}^\infty F_n$ is closed and that $(*, F_n, V)$ is satisfied for each n. Then $(*, F, V)$ is satisfied.*

Proof. By the Hosay-Lininger Theorem ([14] and [16]), there is a homeomorphism f from $S \cup V$ into E^3 that moves no point as far as 1 and so that $f(S)$ is tame from $E^3-f(S \cup V)$. Lister [17] has shown that $(*, F_n, V)$ is satisfied if and only if $(*, f(F_n), f(V))$ is satisfied. Since $f(S)$ is tame from $E^3-f(S \cup V)$, it follows from Lister's theorem just mentioned, that $(*, f(F_n), f(S))$ is satisfied for each n. By Lemma 5.2, for each n there is a taming set T_n on S so that $f(F_n) \subset T_n \subset N_{1/n}(f(F_n))$. But since $f(F)$ is closed, $T = \bigcup_{i=1}^{\infty} T_n$ is closed. By statement (vii) of §1, T is a taming set. Hence by Theorem 5.1 and the remark following it, $(*, T, f(S))$ and $(*, f(F), f(S))$ are satisfied. Thus Lister's theorem implies the desired result.

COROLLARY 5.4. *If F is a closed subset of a 2-sphere S and $F\# = \varnothing$, then properties (A, F, S), (B, F, S), (C, F, S), and $(*, F, S)$ are equivalent, and each of these four properties implies that F is a taming set. If U is a component of $E^3 - S$, and $F\# = \varnothing$, then (A, F, U), (B, F, U), (C, F, U), and $(*, F, U)$ are also equivalent.*

Proof. Theorems 1.1 and 5.3 allow one to extend Theorems 8, 9, 10, and 13 of [18] in a straightforward manner to yield Corollary 5.4.

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