TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, January 1970

A 1-LINKED LINK WHOSE LONGITUDES LIE IN THE SECOND COMMUTATOR SUBGROUP(1)

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1. **Introduction.** In this paper we give an example of a link L of two polygonal simple closed curves in S^3 such that the longitudes of L lie in the second commutator subgroup, G'', of its link group $G=\pi_1(S^3-L)$, but L is 1-linked, that is the two simple closed curves of L do not bound disjoint orientable surfaces in S^3. The question of the existence of such a link was raised by Eilenberg in [3] and again by Smythe in [5] and one motivation for this question is the observation that the longitudes of any boundary link lie in the second commutator subgroup of its link group. (A link is a boundary link if and only if it is not 1-linked, see Smythe [5]). In [2], [3], and [5] examples of 1-linking are given. In all these examples the authors proved 1-linking by showing that at least one of the longitudes was not in the second commutator subgroup. It is clear then that in our example L we must invent some other argument to show it is 1-linked.

2. **The example L.** In E^3 let T be the solid torus obtained by rotating the disk $x=0$, $(y-2)^2+z^2 \leqq 1$ about the z-axis. Let l_1 be the simple closed curve in S^3 consisting of the z-axis and the point at infinity. Figure 1 pictures the oriented

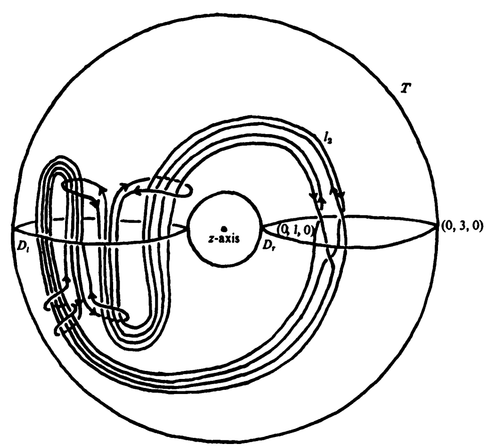

FIGURE 1

Presented to the Society, January 24, 1969; received by the editors January 20, 1969.

(1) Research for this paper was supported in part by the Old Gold Development Fund of the University of Iowa.

simple closed curve l_2 in T as seen by looking down upon it from a point on the positive z-axis. (All the remaining figures in this paper will be drawn from the same viewpoint.) Our example is $L=l_1 \cup l_2$. Let $G=\pi_1(S^3-L)$. To say the longitudes of L belong to G'', the second commutator subgroup of G, we mean that on the boundary of each component of a regular neighborhood of L there is a simple closed curve (scc) lying in G'' (a loop in S^3-L actually determines a class of conjugate elements in G); for short we may say l_1 and l_2 belong to G'' or $L \in G''$. (See [4, p. 123] for a discussion of longitudes.)

Since l_2 is rather complicated some further sets will be needed. Let V be the neighborhood of l_2 in T illustrated by Figure 2; V is a cube with four handles

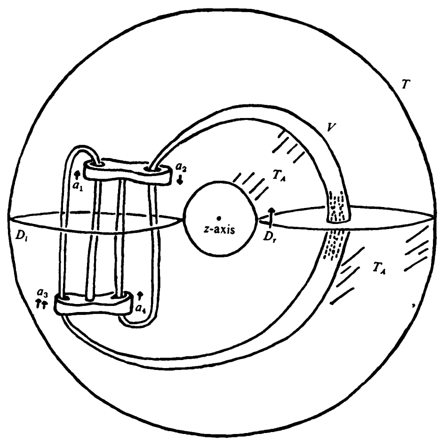

FIGURE 2

a_1, a_2, a_3 and a_4. We may think of the a_i's as disjoint annuli such that Bd $(\bigcup_{i=1}^4 a_i) \subset$ Bd V and Int $(\bigcup_{i=1}^4 a_i) \subset$ Int $T-V$. Let a_1^+ denote the side of a_1 facing the same direction as the small arrow next to a_1 and a_1^- its opposite side. Define similarly a_2^+, a_3^+, and a_4^+, and their opposite sides a_2^-, a_3^-, and a_4^-. (See Figure 2.) Let $D_l=T \cap \{(x, y, z) : x=0, y<0\}$ and $D_r=T \cap \{(x, y, z) : x=0, y>0\}$, D_l and D_r are meridional disks of T. Choose the positive side of D_r, D_r^+ (negative side D_r^-) to be the side of D_r intersecting the component V^+ (V^-) of $V-D_r$ containing the handles a_1 and a_2 (the handles a_3 and a_4).

The next observation is important. Suppose l is a loop in $T-V$ and as we go around l we count 1 or -1 (± 2 in the case of a_3) each time we pass through a handle of V, the sign being determined by whether we went through the handle with or against the direction of the small arrow next to the handle, for instance if we pass from a_3^+ to a_3^- on l add -2. Using the right-hand rule we see that the sum total of the number of times l goes through the handles of V gives the algebraic linking number of l with respect to l_2, denoted by Lk (l, l_2). (See [1, p. 81] for a definition of algebraic linking.) Obviously the number of times l passes through D_r

(± 1 each time, depending on which direction one went through D_r) gives the algebraic linking number of l with respect to l_1.

We first show $L \in G''$ and then show L is 1-linked.

3. $L \in G''$. Let S_0 be the surface (compact, orientable) having one boundary component and of genus 2. To show $L \in G''$ we show that there is a map f of S_0 into $T-V$ such that $f|\text{Bd } S_0$ is a homeomorphism onto Bd D_l and the image of the homology generators of S_0 belong in G'. We do not build f but rather we construct the desired image S of S_0 in $T-V$. We start with a disk with four holes S' as illustrated in Figure 3 (in Figure 3, V appears as a dotted 1-dimensional object

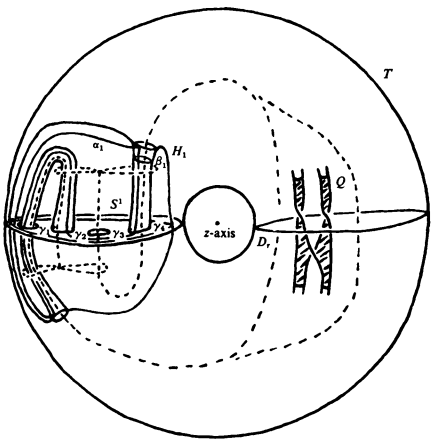

FIGURE 3

except at the right-hand side of the illustration). We add a handle H_1 (H_1 is an annulus) to S' by starting at γ_2, following V up through its handle a_1, down through the hole bounded by γ_1, through handle a_3, then, like a fountain, the annulus widens out, reverses direction in T and goes back through S' in a simple closed curve parallel to Bd $S' \cap$ Bd T, then narrows down again around V, reverses direction in T, goes through a_2 and ends at γ_4. (See Figure 3.) Notice the figure eight $\alpha_1\beta_1$ in $S' \cup H_1$ is such that $(\alpha_1\beta_1) \cap D_r = \varnothing$, hence $\text{Lk}(\alpha_1, l_1) = \text{Lk}(\beta_1, l_1) = 0$ and, again using the comment of §2, $\text{Lk}(\alpha_1, l_2) = 1-2+1 = 0$ and $\text{Lk}(\beta_1, l_2) = 0$. Hence $\alpha_1, \beta_1 \in G'$.

Since it would complicate Figure 3 too much to add the second handle H_2, we just imagine it in Figure 3 as an annulus attaching γ_1 to γ_3 by going down through a_3, up through S' in a simple closed curve parallel to Bd $S' \cap$ Bd T, down through a_2, γ_4 and up through a_4. Again $S' \cup H_2$ contains a figure eight $\alpha_2\beta_2$ disjoint from $\alpha_1\beta_1$ such that $\text{Lk}(\alpha_2, l_1) = \text{Lk}(\beta_2, l_1) = \text{Lk}(\alpha_2, l_2) = \text{Lk}(\beta_2, l_2) = 0$ (where $\text{Lk}(\alpha_2, l_2) = -2+1+1 = 0$). Hence $\alpha_2, \beta_2 \in G'$. It can be checked that $S = S' \cup H_1 \cup H_2$ is the image of S_0. It then follows that Bd S is homotopic in $T-V$ to $(\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}) \cdot (\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1})$. Since $\alpha_1, \alpha_2, \beta_1, \beta_2 \in G'$, Bd $S \in G''$ and it follows that $l_1 \in G''$. Let

Q be the disk indicated in the right side of Figure 3, Q bounds a portion of l_2. The four scc's making up $\mathrm{Cl}\,[(l_2-Q)\cup(\mathrm{Bd}\,Q-l_2)]$ are homotopic to $\mathrm{Bd}\,S$ in $T-l_2$ and it follows that $l_2\in G''$.

It is interesting to note that the singularities of f on S_0 can be made to consist of two disjoint scc's parallel to $\mathrm{Bd}\,S_0$ and two disjoint scc's in each of the two handles. Under f one meridian curve is sewed to the other and the remaining two meridian curves are sewed to the two curves parallel to $\mathrm{Bd}\,S_0$.

4. L **is 1-linked.** The following nine lemmas combined with the assumption that l_1, l_2 bound disjoint orientable surfaces S_1, S_2', respectively, (i.e. L is not 1-linked) will be shown to lead to a contradiction. It should be noted that the linking properties developed in the following lemmas echo those of §3.

LEMMA 1. *If l_1, l_2 bound disjoint orientable surfaces S_1, S_2', respectively, then l_2 bounds an orientable surface S_2 such that $S_2\subset\mathrm{Int}\,T$, S_2 is in general position relative to D_r and at most one component of l_2-D_r is contained in a component of S_2-D_r intersecting both sides of D_r.*

Proof. By the existence of S_1 we may suppose $S_2'\subset\mathrm{Int}\,T$ and S_2' is in general position relative to D_r. Let $l(i)$, $i=1$, 2, 3 and 4, be the component (open arc) of l_2-D_r going through the handle a_i of V and let $C(i)$ be the component of $S_2'-D_r$ containing $l(i)$. Suppose $C(1)$ and $C(2)$ intersect both sides of D_r. If every component of $S_2'-D_r$ which intersects D_r^+ also intersects D_r^-, then we could, by going around the components of $S_2'-D_r$, find a loop l in S_2' such that $\mathrm{Lk}\,(l, \mathrm{Bd}\,D_r)>0$ (counting $+1$ each time we pass through D_r going from D_r^- to D_r^+). But from this it follows that $S_1\cap S_2'\neq\varnothing$, contradiction. Hence there is some component X of $S_2'-D_r$ intersecting only D_r^+. Then X separates $E^3=\mathrm{Int}\,T-D_r$ into exactly two components, one of which contains $C(1)\cup C(2)$; call this component Ext X, the other Int X. Replace X by the punctured disk (or disks) $\mathrm{Cl}\,(\mathrm{Int}\,X)\cap D_r$. (In this process we will also have to cut off all other parts of $S_1\cup S_2'$ in Int X.) In any case by repeating this process a finite number of times, it follows that at least one of the resulting $C(1)$, $C(2)$ intersects only D_r^+.

By a similar reasoning relative to the components $C(3)$, $C(4)$ we may suppose at least one of them intersects only D_r^-. By general position of S_2' and D_r, either $C(1)=C(2)$ or $C(3)=C(4)$. Hence let $S_2=S_2'$, and S_2 satisfies the conclusion of this lemma.

Let S be an orientable surface in $T-(D_r\cup V)$ such that $\mathrm{Bd}\,S=\mathrm{Bd}\,D_l$, $\mathrm{Int}\,S\subset\mathrm{Int}\,(T-(D_r\cup V))$ and S is in general position relative to $\bigcup_{i=1}^4 a_i$. If l is a loop in S, let $\mathrm{Lk}\,(l, a_i)$ be the linking number of l with respect to the handle a_i only and $\mathrm{Lk}\,(l, a_i\cup a_j)$ the linking number of l with respect to the handle a_i and a_j only.

We introduce now another set which will be useful in investigating the linking properties of S with respect to $\bigcup_{i=1}^4 a_i$. Let $T_A=T\cap xy$-plane, T_A is an annulus in T which intersects each a_i in two arcs (see Figure 2). Adjust S slightly in $T-V$ so that it is in general position relative to both T_A and $\bigcup_{i=1}^4 a_i$. Note that the components of $(S \cap \operatorname{Int} T_A) - \bigcup_{i=1}^4 a_i$ are scc's or open arcs whose closures are either arcs or scc's. Let $\Gamma = \{\gamma : \gamma \text{ is a component of } (S \cap T_A) - (\bigcup_{i=1}^4 a_i)\}$. We now alter S so that $S \cap (T_A \cup (\bigcup_{i=1}^4 a_i))$ is in a certain sense simpler. By adjusting S close to $\bigcup_{i=1}^4 a_i$ we may suppose that if l is a scc of $S \cap (\bigcup_{i=1}^4 a_i)$ that bounds a disk in $\bigcup_{i=1}^4 a_i$, then $l \cap T_A = \varnothing$ and if l does not bound a disk in $\bigcup_{i=1}^4 a_i$, then $l \cap T_A$ consists of exactly two points. Suppose $\gamma \in \Gamma$ and $\mathrm{Cl}\,\gamma$ is an arc starting and ending on the same side of some a_i. We may then adjust S near T_A by pushing γ down through a_i, changing two (or more) scc's of $S \cap a_i$ to a scc l which bounds a disk on a_i and adjust S close to a_i so that $l \cap T_A = \varnothing$. Since this process reduces the number of scc's of $S \cap (\bigcup_{i=1}^4 a_i)$ which separate an a_i, it is finite. Now remove those scc's of $S \cap (\bigcup_{i=1}^4 a_i)$ which bound a disk in $\bigcup_{i=1}^4 a_i$ by cutting S off on $\bigcup_{i=1}^4 a_i$, let $S(r)$ be the component of the resulting surface which contains $\mathrm{Bd}\, S$. Let $\Gamma(r) = \{\gamma : \gamma \text{ is a component of } (S(r) \cap T_A) - (\bigcup_{i=1}^4 a_i)\}$ and $\Delta(r) = \{\delta : \delta \text{ is a component of } S(r) - (\bigcup_{i=1}^4 a_i)\}$. Note that if $\mathrm{Cl}\,\gamma$ is an arc intersecting both sides of an a_i, then we obtain a spiral in $S(r) \cap T_A$ going around a_i; eventually it reverses itself and here we obtain an arc intersecting only one side of a_i, contradicting the form of $S(r)$. Therefore if $\gamma \in \Gamma(r)$ and $\mathrm{Cl}\,\gamma$ is an arc, then its endpoints lie in different components of $\mathrm{Bd}\, T \cup (\bigcup_{i=1}^4 a_i)$.

To say $\gamma \in \Gamma(r)$ ends on a_i^+ means γ approaches a_i from its positive side (denoted by $\gamma \cap a_i^+ \neq \varnothing$). Similarly we may define $\gamma \cap a_i^- \neq \varnothing$ and, for $\delta \in \Delta(r)$, $\delta \cap a_i^+ \neq \varnothing$ and $\delta \cap a_i^- \neq \varnothing$. We say two distinct elements γ_1, γ_2 of $\Gamma(r)$ abut on a_i if they approach a point of $T_A \cap (\bigcup_{i=1}^4 a_i)$ from opposite sides of a_i and $\delta_1, \delta_2 \in \Delta(r)$ abut on a_i if they approach a scc of $S \cap a_i$ from opposite sides of a_i. If γ ends on a_i, then by the form of $S(r)$, it follows that $S \cap a_i$ contains a scc l such that $l \cap T_A$ is two points b_1, b_2 where γ approaches b_1. Let $\mathrm{Op}\,\gamma(a_i^+)$ (or $\mathrm{Op}\,\gamma(a_i^-)$) be the arc of $\Gamma(r)$ which approaches b_2 from the positive side of a_i, a_i^+ (or the negative side of a_i, a_i^-). It helps to draw in Figures 2, 4 and 5 the various arcs γ which arise in the next lemmas.

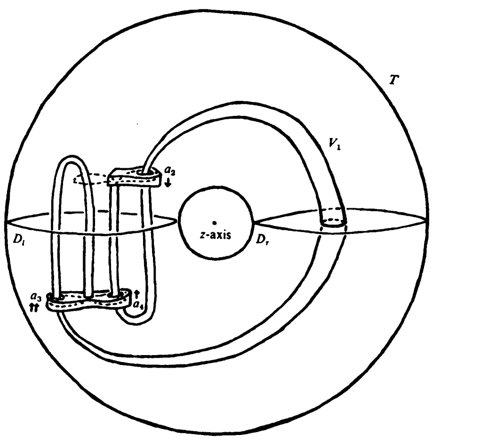

FIGURE 4

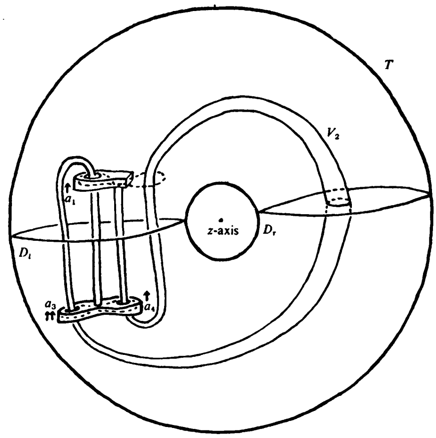

FIGURE 5

LEMMA 2. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_2^- \neq \varnothing$, Bd $S(r) \not\subset \delta$ *and* δ *contains no* $\gamma \in \Gamma(r)$ *so that* Cl γ *is a scc. Then* $\delta \cap a_3^- \neq \varnothing$.

Proof. Suppose $\delta \cap a_3^- = \varnothing$. Since Bd $S(r) \not\subset \delta$ and δ contains no γ so that Cl γ is a scc, there is an arc γ_1 of $\Gamma(r)$ in δ going from a_2^- to a_4^-. Let $\gamma_2 = \text{Op}\, \gamma_1(a_4^-)$, γ_2 must end on a_2^+. Let $\gamma_3 = \text{Op}\, \gamma_2(a_2^+)$, $\gamma_3 \neq \gamma_1$ and γ_3 must end on a_4^-. Let $\gamma_5 = \text{Op}\, \gamma_4(a_4^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence γ_1 must end on a_3^- and $\delta \cap a_3^- \neq \varnothing$.

Also we have the following

LEMMA 2′. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_3^- \neq \varnothing$, Bd $S(r) \not\subset \delta$ *and* δ *contains no* $\gamma \in \Gamma(r)$ *so that* Cl γ *is a scc. Then* $\delta \cap a_2^- \neq \varnothing$.

LEMMA 3. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_3^+ \neq \varnothing$, Bd $S(r) \not\subset \delta$ *and* δ *contains no* γ *so that* Cl γ *is a scc. Then* $\delta \cap a_1^+ \neq \varnothing$.

Proof. Suppose $\delta \cap a_1^+ = \varnothing$. Since Bd $S(r) \not\subset \delta$ and δ contains no γ so that Cl γ is a scc, there is an arc γ_1 of $\Gamma(r)$ in δ going from a_3^+ to a_2^-. Let $\gamma_2 = \text{Op}\, \gamma_1(a_2^-)$, by the proof of Lemma 2, γ_2 must end on a_3^-. Let $\gamma_3 = \text{Op}\, \gamma_2(a_3^-)$, $\gamma_3 \neq \gamma_1$, and γ_3 must end on a_2^-. Let $\gamma_4 = \text{Op}\, \gamma_3(a_2^-)$. Continuing in this manner, it follows that $\Gamma(r)$ must be an infinite set, contradiction. Hence γ_1 must end on a_1^+ and $\delta \cap a_1^+ \neq \varnothing$.

Also we have the following

LEMMA 3′. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_2^+ \neq \varnothing$, Bd $S(r) \not\subset \delta$ *and* δ *contains no* $\gamma \in \Gamma(r)$ *so that* Cl γ *is a scc. Then* $\delta \cap a_4^- \neq \varnothing$.

The next lemma follows easily by examining the various arcs of $\Gamma(r)$ in δ.

LEMMA 4. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_1^- \neq \varnothing$, Bd $S(r) \not\subset \delta$ *and* δ *contains no* $\gamma \in \Gamma(r)$ *so that* Cl γ *is a scc. Then* $\delta \cap a_2^+ \neq \varnothing$.

Also we have the following.

LEMMA 4′. *Suppose* $\delta \in \Delta(r)$, $\delta \cap a_4^+ \neq \varnothing$, $\mathrm{Bd}\, S(r) \not\subset \delta$ *and* δ *contains no* $\gamma \in \Gamma(r)$ *so that* $\mathrm{Cl}\,\gamma$ *is a scc. Then* $\delta \cap a_3^+ \neq \varnothing$.

LEMMA 5. *There is a loop* l *in* S *so that either* (1) $\mathrm{Lk}\,(l, a_1 \cup a_2) \neq 0$ *or* (2) $\mathrm{Lk}\,(l, a_3 \cup a_4) \neq 0$.

Proof. We first note that if there is a loop l' in $S(r)$ satisfying (1) or (2) of this lemma, then there is a loop l in S satisfying (1) or (2). For, while retaining property (1) or (2), we may push l' off the disks of $S(r)$ obtained by cutting S off on $\bigcup_{i=1}^4 a_i$. Pushing a $\mathrm{Cl}\,\gamma$ down through some a_i does not alter the linking properties of l' with the various a_i. Hence reversing the steps needed to arrive at $S(r)$ from S, it follows that l' gives rise to an l in S satisfying condition (1) or (2).

If there is an element γ of $\Gamma(r)$ so that $\mathrm{Cl}\,\gamma$ is a scc intersecting either a_1, a_2, a_3 or a_4, then $l' = \mathrm{Cl}\,\gamma$ satisfies condition (1) or (2). Suppose then no such scc exists.

Let δ_0 be the element of $\Delta(r)$ containing $\mathrm{Bd}\, S(r)$ and suppose $\delta_0 \cap a_2^+ \neq \varnothing$. By Lemma 2, δ_0 abuts a δ_1 on a_2 so that $\delta_1 \cap a_3^- \neq \varnothing$ (assuming $\delta_0 \neq \delta_1$). By Lemma 3, δ_1 abuts a δ_2 on a_3 so that $\delta_2 \cap a_1^+ \neq \varnothing$ (again assuming $\delta_0 \neq \delta_2$). By Lemma 4, δ_2 abuts a δ_3 on a_1 so that $\delta_3 \cap a_2^+ \neq \varnothing$. Continuing this reasoning we obtain a sequence $\delta_0, \delta_1, \ldots, \delta_i$ which eventually repeats an element of $\Delta(r)$. Let m be the first integer so that $n < m$, $\delta_n = \delta_m$ and $\delta_i \neq \delta_j$ if $i \neq j$ and $n < i$, $j < m$. If $m - n = 1$, then $\mathrm{Cl}\,(\bigcup_{i=n}^m \delta_i)$ contains a loop l' so that $\mathrm{Lk}\,(l', a_i) = \pm 1$ for some $i = 1$, 2 or 3 and $\mathrm{Lk}\,(l', a_j) = 0$ for $j \neq i$. Hence l' satisfies either (1) or (2). If $m - n \geqq 2$, then it follows that $\mathrm{Cl}\,(\bigcup_{i=n}^m \delta_i)$ contains a loop l' so that $\mathrm{Lk}\,(l', a_1 \cup a_2) \neq 0$.

If $\delta_0 \cap a_3^- \neq \varnothing$ or $\delta_0 \cap a_1^+ \neq \varnothing$, then we may start at either of these places and, repeating the same pattern as before, obtain a sequence $\delta_0, \ldots, \delta_i$ so that $\mathrm{Cl}\,(\bigcup_{i=n}^m \delta_i)$ contains a loop l' satisfying (1) or (2).

If $\delta_0 \cap a_2^- \neq \varnothing$, then making use of Lemmas 2′, 3′ and 4′ it follows that $S(r)$ contains a loop l' satisfying (1) or (2). Since δ_0 must intersect one of a_2^+, a_3^-, a_1^+, or a_2^-, Lemma 5 follows.

Let $V_i = V$ minus its ith handle, $i = 1$, 2, 3 and 4, see Figures 4 and 5 for V_1 and V_2, respectively. Let S_i be a surface in $T - (D_r \cup V_i)$ such that $\mathrm{Bd}\, S_i = \mathrm{Bd}\, D_l$, $\mathrm{Int}\, S_i \subset \mathrm{Int}\,(T - (D_r \cup V_i))$ and S_i is in general position relative to $(\bigcup_{j=1}^4 a_j) - a_i$. As before, from S_i we may obtain a surface $S_i(r)$ so that for each arc γ of $\Gamma_i(r) = \{\gamma : \gamma$ is a component of $(S_i \cap T_A) - ((\bigcup_{j=1}^4 a_j) - a_i)\}$ such that $\mathrm{Cl}\,\gamma$ is not a scc we have that its endpoints lie in different components of $\mathrm{Bd}\, T \cup ((\bigcup_{j=1}^4 a_j) - a_i)$.

LEMMA 6. *There is a loop* l *in* S_1 *so that either* (1) $\mathrm{Lk}\,(l, a_2) \neq 0$ *or* (2)

$$\mathrm{Lk}\,(l, a_3 \cup a_4) \neq 0.$$

Proof. As in Lemma 5, we note that if $S_1(r)$ contains a loop satisfying (1) or (2) then so does S_1. If there is an element γ of $\Gamma_1(r)$ so that $\mathrm{Cl}\,\gamma$ is a scc, then $l' = \mathrm{Cl}\,\gamma$ satisfies (1) or (2) of this lemma. Suppose then no such scc exists. Note that $S_1(r) \cap a_3 = \varnothing$ and $S_1(r) \cap a_4 \neq \varnothing$. There is an element γ_1 of $\Gamma_1(r)$ going from a_4^+ to a_2^+. Let $\gamma_2 = \mathrm{Op}\,\gamma_1(a_2^+)$. If γ_2 ends on $\mathrm{Bd}\, T$, then the arc γ_2 abuts on a_2 would start and end on a_2^-, contradiction. Hence γ_2 must end on a_4^-. Let $\gamma_3 = \mathrm{Op}\,\gamma_2(a_4^-)$. Continuing in this manner, it follows that $\Gamma_1(r)$ must be infinite, contradiction.

The proof of the next lemma is (geometrically) analogous to the proof of Lemma 6.

LEMMA 7. *There is a loop l in S_4 so that either* (1) $\mathrm{Lk}\,(l, a_3) \neq 0$ *or* (2)

$$\mathrm{Lk}\,(l, a_1 \cup a_2) \neq 0.$$

LEMMA 8. *There is a loop l in S_2 so that either* (1) $\mathrm{Lk}\,(l, a_1) \neq 0$ *or* (2)

$$\mathrm{Lk}\,(l, a_3 \cup a_4) \neq 0.$$

Proof. Again note that if $S_2(r)$ contains a loop satisfying (1) or (2), then so does S_2, hence assume no $\mathrm{Cl}\,\gamma$ is a scc. Since $S_2 \cap a_1 \neq \varnothing$, there are arcs γ', γ'' in $\Gamma_2(r)$ so that both γ', γ'' start on a_1^-, γ' ends on a_3^+, γ'' ends on a_4^+ and γ', γ'' end on the innermost scc of $S_2(r) \cap a_1$ (see Figure 5). Let γ_0', γ_0'' be the arcs γ', γ'' abut on a_3, a_4, respectively. Since not both γ_0', γ_0'' can end on Bd T, it follows that $\gamma_0' = \gamma_0''$ and the scc $l = \mathrm{Cl}\,(\gamma' \cup \gamma'' \cup \gamma_0')$ plus an arc in $S_2(r) \cap a_1$ satisfies (2) of this lemma.

We also have the following

LEMMA 9. *There is a loop l in S_3 so that either* (1) $\mathrm{Lk}\,(l, a_4) \neq 0$ *or* (2)

$$\mathrm{Lk}\,(l, a_1 \cup a_2) \neq 0.$$

THEOREM. *L is* 1-*linked.*

Proof. Suppose l_1, l_2 bound disjoint orientable surfaces S_1, S_2', respectively. Let $l(i)$, $C(i)$, $i = 1, 2, 3$, and 4 be as given in the proof of Lemma 1. By Lemma 1, l_2 bounds an orientable surface S_2 such that $S_2 \subset \mathrm{Int}\, T$, S_2 is in general position relative to D_r and at most one component $C(i)$ of $S_2 - D_r$ intersects both sides of D_r. Suppose this $i = 1$.

Let U be a closed regular neighborhood of $C(3) \cup C(4) \cup D_r$ chosen so Bd U contains a surface S such that $\mathrm{Int}\, S \subset \mathrm{Int}\,(T - (D_r \cup l_2))$ and $\mathrm{Bd}\, S = \mathrm{Bd}\, D_l$. Further, U may be chosen so that $S \cap C(i) = \varnothing$ for $i = 2$, 3 and 4, since these $C(i)$'s intersect just one side of D_r. Since $S \cap l_2 = \varnothing$, we may push S off V (we may need to use a cut and paste argument on $S \cup (\bigcup_{i=2}^4 C(i))$ to get S off the handle of V intersecting a_3 since l_2 goes through this handle twice).

There are disjoint arcs x_1, x_2 in $\mathrm{Cl}\,(C(3) \cup C(4)) \cap D_r$ so that either (1) $l' = l(3) \cup l(4) \cup x_1 \cup x_2$ is a scc or (2) $l(3) \cup x_1$ and $l(4) \cup x_2$ are scc's. In Case (2) let D be a disk in Int D_r so that $x_1 \cup x_2 \subset \mathrm{Bd}\, D$ and the arcs x_1', x_2' forming $\mathrm{Bd}\, D - \mathrm{Int}\,(x_1 \cup x_2)$ form a scc $l'' = l(3) \cup l(4) \cup x_1' \cup x_2'$ which admits an orientation compatible with the orientation on $l(3)$ and $l(4)$ induced by the orientation of l_2. Let d_n, $n = 1, 2, \ldots, m$ be the disks in D_r bounded by the scc's of $\mathrm{Cl}\,(C(3) \cup C(4)) \cap D_r$. Let $C' = \mathrm{Cl}\,(C(3) \cup C(4)) \cup (\bigcup_{n=1}^m d_n)$ and $C'' = C' \cup D$. It then follows that $l' \sim 0$ in C' and $l'' \sim 0$ in C'' (using integer coefficients). Since $S \cap C' = S \cap C'' = \varnothing$, by Alexander Duality for each loop l in S, $\mathrm{Lk}\,(l, a_3 \cup a_4) = 0$.

If both $C(1)$, $C(2)$ intersect just one side of D_r, then by a similar argument as used above, it follows that for each loop l in S, $\mathrm{Lk}(l, a_1 \cup a_2)=0$. But this is impossible by Lemma 5. If $C(1)$ intersects both sides of D_r, then, since $C(2)$ intersects only one side of D_r, $C(1) \cap C(2)=\varnothing$ and it follows that for each loop l in S, $\mathrm{Lk}(l, a_2)=0$, but this contradicts Lemma 6. The cases $i=2$, 3 and 4 are similar. Hence the surfaces S_1, S_2' could not have existed and L is 1-linked.

5. **Concluding remarks.** Since our example $L=l_1 \cup l_2$ is 1-linked it follows that if l_1 bounds the orientable surface S_1 in S^3-l_2, then S_1 contains a loop l such that $l \nsim 0$ in $H_1(S^3-l_2)$. In particular we have the following

THEOREM. *If $K=k_1 \cup k_2$ is a link of two components, then K is a boundary link if and only if k_1 bounds an orientable surface S_1 in S^3-k_2 such that the inclusion of $H_1(S_1)$ in $H_1(S^3-k_2)$ is trivial.*

Proof. The only if part follows immediately from the definition of boundary link. If the surface S_1 exists then k_2 bounds an orientable surface S_2 in S^3-k_1. Put S_2 in general position with respect to the homology generators H (figure eights) of S_1. Since the inclusion of $H_1(S_1)$ in $H_1(S^3-k_2)$ is trivial, we may add handles to S_2 minus a small regular neighborhood of H to form an orientable surface S_2' such that $\mathrm{Bd}\, S_2'=k_2$ and $S_2' \subset S^3-(k_1 \cup H)$. Since S_1-H is a disk minus a finite number of points we may put S_2' in general position relative to S_1-H and cut S_2' off on S_1. We then obtain an orientable surface S_2'' such that $\mathrm{Bd}\, S_2''=k_2$ and $S_1 \cap S_2''=\varnothing$. Hence K is a boundary link.

This theorem and our example L motivate the following

QUESTION. Does there exist a link $K=k_1 \cup \cdots \cup k_n$, $2<n$, such that each k_i bounds an orientable surface S_i in $S^3-(K-k_i)$, and the inclusion of $H_1(S_i)$ in $H_1(S^3-(K-k_i))$ is trivial but K is 1-linked?

Such a link K of the question would also be an example of a 1-linked link whose longitudes all lie in the second commutator subgroup.

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