APPROXIMATION BY POLYNOMIALS SUBORDINATE TO A UNIVALENT FUNCTION⁽¹⁾

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This paper is concerned with approximating a function $f(z)$ analytic and univalent in the unit disk $E = \{z: |z| < 1\}$ by polynomials which are also univalent in E . We are interested in such approximations where exactly one polynomial of each degree n , $n \geq 1$, is used and such that the polynomials are monotonically subordinate to each other.

Recall that $f(z)$ is called subordinate to $g(z)$ in E if both functions are analytic in E and if there exists a function $\varphi(z)$ analytic in E and satisfying $|\varphi(z)| < 1$, $\varphi(0) = 0$ such that $f(z) = g(\varphi(z))$. The existence of such a function $\varphi(z)$ is implied by the conditions that $g(z)$ is univalent in E , $f(0) = g(0)$ and $f(E) \subset g(E)$. If $f(z)$ is subordinate to $g(z)$ in E we will write $f(z) \subset g(z)$ in E .

Theorem 1 asserts that if $f(z)$ is analytic and univalent in E then there exists a sequence $\{p_n(z)\}$, $n = 1, 2, \dots$, such that $p_n(z)$ is a polynomial of degree n which is univalent in E ,

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \dots$$

and $p_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$. This convergence is uniform in each compact subset of E . The idea of proving this is to appropriately relate $f(z)$ to the partial sums $s_n(z)$ of the power series for $f(z)$. Although $s_n(z)$ are not, in general, univalent in E (they are in $|z| < \frac{1}{4}$ [12]) they are univalent in $|z| < r$, $0 < r < 1$, for all large n once r is given. This eventually leads to a chain of univalent polynomials

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \dots$$

such that $p_{n_j}(z)$ has degree n_j , $n_1 < n_2 < n_3 < \dots$, and $p_{n_j}(z) \rightarrow f(z)$ as $j \rightarrow \infty$. These polynomials are so related that it is still possible to find polynomials of the remaining degrees which fill in the chain and such that $p_n(z) \rightarrow f(z)$.

This method is equally adaptable when $f(z)$ maps E one-to-one onto a convex domain in order to produce a similar chain of convex, univalent polynomials. In fact, our argument may be applied to any one of a number of classes of univalent functions.

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The possibility of such an approximation to convex mappings was implicitly raised by G. Pólya and I. J. Schoenberg in [10]. They considered analytic functions

$$f(z) = \sum_{k=1}^{\infty} a_k z^k$$

which map E one-to-one onto a convex domain. For each integer n , $n \geq 1$, the de la Vallée Poussin means of $f(z)$ are defined by

$$V_n(z) = \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1) \cdots 1}{(n+1)(n+2) \cdots (2n)} a_n z^n.$$

They showed that $V_n(z)$ is a convex, univalent function, $V_n(z) \subset f(z)$, and $V_n(z) \rightarrow f(z)$ in E , and conjectured that

$$V_1(z) \subset V_2(z) \subset V_3(z) \subset \cdots.$$

This conjecture remains open and its validity would yield a somewhat more explicit proof of the existence of such a chain except that we also require $p_n(z)$ to have the exact degree n .

Disregarding the demand that $p_n(z)$ be a polynomial, it is easy to find a chain

$$f_1(z) \subset f_2(z) \subset f_3(z) \subset \cdots$$

such that $f_n(z) \rightarrow f(z)$ and we do not have $f_n(z) = f(z)$ for all n . One simply sets $f_n(z) = f(r_n z)$ where $\{r_n\}$ is an increasing sequence of real numbers in the interval $0 < r < 1$ such that $r_n \rightarrow 1$. A less trivial situation in the case $f(z)$ is univalent and convex in E is obtained by considering the functions

$$f(z, \alpha) = \frac{1}{\alpha} \int_0^\alpha f(z e^{i\theta}) d\theta.$$

If $\{\alpha_n\}$ is any sequence of real numbers in the interval $0 < \alpha < 2\pi$ which monotonically converges to zero then $f(z, \alpha_n)$ is univalent and convex, $f(z, \alpha_1) \subset f(z, \alpha_2) \subset f(z, \alpha_3) \subset \cdots$ and $f(z, \alpha_n) \rightarrow f(z)$ in E [6].

Interest in (continuous as well as discrete) chains of subordinate functions has occurred elsewhere and we would like to point out the recent papers by C. Pommerenke [7], [8], [9] as well as [1] by A. Bielecki and Z. Lewandowski.

The earliest consideration of this kind presumably is due to C. Carathéodory in his idea of kernels.

LEMMA 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in E and set $s_n(z) = \sum_{k=0}^n a_k z^k$.*

Let $\{n_j\}$ be a strictly increasing sequence of positive integers and let $\{r_j\}$ be a sequence of real numbers such that $0 < r_j < 1$ and $r_j \rightarrow 1$ as $j \rightarrow \infty$. Then $s_{n_j}(r_j z) \rightarrow f(z)$ uniformly in each disk $|z| \leq r$, $0 < r < 1$.

Proof. Suppose that $|z| \leq r$ and $0 < r < 1$.

$$\begin{aligned} |f(z) - s_{n_j}(r_j z)| &= \left| \sum_{k=0}^{n_j} a_k (1 - r_j^k) z^k + \sum_{k=n_j+1}^{\infty} a_k z^k \right| \\ &\leq (1 - r_j) \sum_{k=0}^{n_j} k |a_k| r^k + \sum_{k=n_j+1}^{\infty} |a_k| r^k \\ &\leq (1 - r_j) \sum_{k=0}^{\infty} k |a_k| r^k + \sum_{k=n_j+1}^{\infty} |a_k| r^k. \end{aligned}$$

Given a positive number ε there is an integer N such that $\sum_{k=n+1}^{\infty} |a_k| r^k < \varepsilon/2$ if $n \geq N$. Since $n_j \geq j$ this implies that for $j \geq N$, $\sum_{k=n_j+1}^{\infty} |a_k| r^k < \varepsilon/2$. As $r_j \rightarrow 1$ there is an integer N' such that if $j \geq N'$ then

$$(1 - r_j) \sum_{k=0}^{\infty} k |a_k| r^k < \frac{\varepsilon}{2}.$$

Choosing $N'' = \max(N, N')$ we see that $|f(z) - s_{n_j}(r_j z)| < \varepsilon$ for $j \geq N''$ and this proves the lemma.

THEOREM 1. Let $f(z)$ be analytic and univalent in E . There exists a sequence of polynomials $\{p_n(z)\}$, $n = 1, 2, 3, \dots$, where $p_n(z)$ has degree n and is univalent in E , such that

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \dots$$

in E and $p_n(z) \rightarrow f(z)$ in E . The convergence is uniform in compact subsets of E .

Proof. Let the power series for $f(z)$ be $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and let $s_n(z) = \sum_{k=0}^n a_k z^k$.

Then $s_n(z) \rightarrow f(z)$ and $s'_n(z) \rightarrow f'(z)$ as $n \rightarrow \infty$ uniformly in each disk $|z| \leq r$, where $0 < r < 1$.

Let us introduce the notation:

$$\begin{aligned} \Delta_g(u, v) &= (g(u) - g(v))/(u - v), \quad \text{if } u \neq v, \\ &= g'(u), \quad \text{if } u = v, \end{aligned}$$

for each function $g(z)$ analytic in E . If $u \neq v$, $|u| \leq r$, $|v| \leq r$, $0 < r < 1$, we may write

$$g(u) - g(v) = \int_v^u g'(z) dz,$$

where the path of integration is the line segment from v to u . Since the points on this line segment satisfy $|z| \leq r$, it is clear that

$$\max_{|u| \leq r, |v| \leq r} |\Delta_g(u, v)| = \max_{|z| \leq r} |g'(z)|.$$

Applying this to $g_n(z) = f(z) - s_n(z)$ we conclude that $\Delta_{g_n}(u, v)$ is uniformly small for large n if $|u| \leq r$, $|v| \leq r$, since $g'_n(z)$ is uniformly small for large n in $|z| \leq r$.

Since $f(z)$ is univalent in E , $f'(z)$ does not vanish, and, therefore,

$$m_f(r) = \min_{|u| \leq r, |v| \leq r} |\Delta_f(u, v)| > 0,$$

where $0 < r < 1$. Because $\Delta_{s_n}(u, v) = \Delta_f(u, v) - \Delta_{g_n}(u, v)$ there exists an integer N such that if $n \geq N$ then $m_{s_n}(r) \geq m_f(r)/2$. In particular, this implies that $s_n(z)$ is univalent in $|z| \leq r$ for all $n \geq N$.

Let $D(r)$ denote the image of $|z| < r$ under $f(z)$ and let $D_n(r)$ denote the image of $|z| < r$ under $s_n(z)$, where $0 < r \leq 1$. Also, let $d(A, B)$ denote the distance between the sets A and B and let δA denote the boundary of the set A . If we set $d(r) = d(D(r), \delta D(1))$, where $0 < r < 1$, then $d(r) < \infty$ since $D(r)$ is bounded and $D(1)$ is not the whole plane. Also, $d(r) > 0$ as $f(z)$ is an open mapping. There is an integer N' such that

$$|f(z) - s_n(z)| < d(r)/2$$

for $|z| \leq r$, $0 < r < 1$, if $n \geq N'$. Consequently, if $n \geq N'$ then $d_n(r) = d(D_n(r), \delta D(1)) > d(r)/2$.

Let r_1 be any fixed number such that $0 < r_1 < 1$ and choose an integer $n_1 \geq \max(N, N')$, where $N = N(r_1)$, $N' = N'(r_1)$ was obtained by the two previous arguments. This implies that $m_{s_{n_1}}(r_1) \geq m_f(r_1)/2$ and $d_{n_1}(r_1) > d(r_1)/2$.

We continue our argument with the assumption that $f(z)$ is not a polynomial. Then the selection of n_1 can be made so that $s_{n_1}(z)$ has exactly the degree n_1 . If $E_{n_1}(r) = f^{-1}(D_{n_1}(r))$ then there is a number ρ_1 , $0 < \rho_1 < 1$, such that $E_{n_1}(r_1)$ is contained in $|z| \leq \rho_1$, as, otherwise, $d_{n_1}(r_1) = 0$. Choose the number r_2 such that $\rho_1 < r_2 < 1$ and $r_2 > (1 + r_1)/2$. Then $d(D_{n_1}(r_1), \delta D(r_2)) > 0$ and, as always, $d(r_2) > 0$.

Arguing as before we conclude that there is an integer $N = N(r_2)$ such that if $n \geq N$ then $m_{s_n}(r_2) \geq m_f(r_2)/2$. Also there is an integer $N' = N'(r_2)$ such that if $n \geq N'$ then $|f(z) - s_n(z)| < d^*/2$ for $|z| \leq r_2$, where

$$d^* = \min(d(r_2), d(D_{n_1}(r_1), \delta D(r_2))).$$

This implies that $d_n(r_2) > d(r_2)/2 > 0$ and

$$d(D_{n_1}(r_1), \delta D_n(r_2)) > \frac{1}{2}d(D_{n_1}(r_1), \delta D(r_2)) > 0$$

for $n \geq N'$. The last condition and the facts that $D_{n_1}(r_1)$ is connected and $D_{n_1}(r_1)$ and $D_n(r_2)$ have a common point, namely a_0 , implies that $D_{n_1}(r_1) \subset D_n(r_2)$.

If we set $N'' = \max(N, N')$ then for $n \geq N''$ $m_{s_n}(r_2) \geq (m_f(r_2)/2)$, $d(D_{n_1}(r_1), \delta D_n(r_2)) > 0$ and $d_n(r_2) > 0$. The first two conditions and $s_n(0) = s_{n_1}(0) = a_0$ show that $s_n(r_2 z)$ is univalent in $|z| \leq 1$ and $s_{n_1}(r_1 z) \subset s_n(r_2 z)$ in E . Since $f(z)$ is not a polynomial there is such an integer $n = n_2$ so that $n_2 \geq N''$, $n_2 > n$, and $s_{n_2}(z)$ has degree n_2 . Moreover, since $d_{n_2}(r_2) > 0$ this whole argument may be repeated.

Namely, we first consider the set $E_{n_2}(r_2)$ to show that there is a number r_3 such that $(1 + r_2)/2 < r_3 < 1$ and $d(D_{n_2}(r_2), \delta D(r_3)) > 0$.

We then find an integer N associated with the condition $m_{s_n}(r_3) \geq m_f(r_3)/2$ and an integer N' associated with the inequality $|f(z) - s_n(z)| < d^*/2$ for $|z| \leq r_3$, where

$$d^* = \min(d(r_3), d(D_{n_2}(r_2), \delta D(r_3))).$$

This leads to an integer n_3 so that $n_3 > n_2$, $s_{n_3}(z)$ has degree n_3 , $m_{s_{n_3}}(r_3) \geq m_f(r_3)/2$, $d(D_{n_2}(r_2), \delta D_{n_3}(r_3)) > 0$, and $d_{n_3}(r_3) > 0$. In particular, $s_{n_3}(r_3 z)$ is univalent in $|z| \leq 1$ and $s_{n_2}(r_2 z) \subset s_{n_3}(r_3 z)$ in E .

This argument, therefore, can be continued indefinitely and it yields the following conclusion. There is a sequence of real numbers $\{r_j\}$, a strictly increasing sequence of integers $\{n_j\}$ and a sequence of polynomials $\{p_{n_j}(z)\}$ such that $0 < r_j < 1$, $r_j \rightarrow 1$ and $p_{n_j}(z) = s_{n_j}(r_j z)$ are univalent in $|z| \leq 1$. Moreover, $p_{n_j}(z)$ has degree n_j and

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \dots$$

in E . In addition,

$$m_{p_{n_j}}(1) \geq m_f(r_j)/2$$

and

$$d(D_{n_j}(r_j), \delta D_{n_{j+1}}(r_{j+1})) > 0.$$

The fact that $r_j \rightarrow 1$ follows from the demand that

$$(1 + r_j)/2 < r_{j+1} < 1.$$

We also note that because of Lemma 1 $p_{n_j}(z) \rightarrow f(z)$ uniformly in every disk $|z| \leq r$, $0 < r < 1$.

The remaining part of our argument consists of interjecting appropriate polynomials between successive pairs in the chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \dots$$

Suppose that $p(z)$ and $q(z)$ are such a pair and have degrees n and k , respectively, where $n < k$. Then $m_p(1) > 0$ and therefore $p(z)$ is univalent in $|z| < R$ for some $R > 1$. If $1 < \rho < R$ and $P(z) = p(\rho z)$ then $m_P(1) > 0$. Let

$$Q(z) = P(z) + bz^{n+1}$$

so that

$$|\Delta_Q(u, v)| \geq m_P(1) - (n+1)|b|$$

in $|z| \leq 1$. This implies that for all sufficiently small values of b

$$m_Q(1) \geq \frac{1}{2}m_P(1)$$

so that, in particular, $Q(z)$ is univalent in E . If $A = p(E)$ and $B = q(E)$ then we have the condition $d(A, \delta B) > 0$. Because of the kind of arguments made earlier in this proof it follows that for $\rho > 1$ and sufficiently near 1 we can conclude that $p(E) \subset P(E) \subset q(E)$ from the relations $d(p(E), \delta P(E)) > 0$, $d(P(E), \delta q(E)) > 0$, and $p(0) = P(0) = q(0)$.

As $Q(z) \rightarrow P(z)$ as $b \rightarrow 0$ uniformly in $|z| \leq 1$ it is also possible to deduce that

$$d(p(E), \delta Q(E)) > 0 \quad \text{and} \quad d(Q(E), \delta q(E)) > 0$$

for sufficiently small b . Choosing $b \neq 0$ and sufficiently small so that these several

conditions hold we see that $Q(z)$ is univalent in E , has degree $n+1$ and $p(z) \subset Q(z)$ in E . Moreover, $m_Q(1) > 0$ and $d(Q(E), \delta q(E)) > 0$ so that the argument that just was applied to $p(z)$ and $q(z)$ can again be applied to $Q(z)$ and $q(z)$. Continuing in this manner we obtain the chain of univalent polynomials

$$p(z) \subset p_{n+1}(z) \subset p_{n+2}(z) \subset \cdots \subset p_{k-1}(z) \subset q(z),$$

where $p_j(z)$ has degree j . In fact, there is an infinite such chain between $p(z)$ and $q(z)$, but for our purposes we end with $p_{k-1}(z)$.

Now we return to the earlier chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots.$$

If $n_1 \neq 1$ set $p_1(z) = a_0 + b_1 z$ where $b_1 \neq 0$ and is sufficiently small so the closed disk $|w - a_0| \leq |b_1|$ is contained in $p_{n_1}(E)$. Between successive pairs of polynomials in this new chain interject appropriate polynomials, as just described. This produces the chain

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots,$$

where $p_n(z)$ is univalent in E and is a polynomial of degree n . Moreover, a subsequence of $\{p_n(z)\}$, namely $\{p_{n_j}(z)\}$, converges to $f(z)$ uniformly in compact subsets of E . The proof will be complete if we can show that if $\{\varepsilon_j\}$ is any sequence of positive numbers such that $\varepsilon_j \rightarrow 0$ then the interjected polynomials can be so chosen to fill out the chain

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \cdots$$

so that

$$|p_n(z) - p_{n_j}(z)| < \varepsilon_j$$

for z in E and for $n_j < n < n_{j+1}$, $j = 1, 2, \dots$

In order to do this consider the previous argument where $p(z)$ and $q(z)$ are successive pairs in the sequence $\{p_{n_j}(z)\}$ and have degrees n and k , respectively, with $n < k$. That argument yielded real numbers R_μ and δ_μ , $\mu = n+1, \dots, k-1$, such that $R_\mu > 1$, $\delta_\mu > 0$ and successively defining $p_\mu(z)$ by

$$p_{\mu+1}(z) = p_\mu(\rho_\mu z) + b_{\mu+1} z^{\mu+1}$$

the chain between $p(z)$ and $q(z)$ results, where ρ_μ and b_μ are restricted only by the conditions $1 < \rho_\mu < R_\mu$ and $0 < |b_\mu| < \delta_\mu$. The recursive definitions lead to the formula

$$(1) \quad p_\mu(z) = p_n(\rho_n \rho_{n+1} \cdots \rho_{\mu-1} z) + b_{n+1}(\rho_{n+1} \rho_{n+2} \cdots \rho_{\mu-1})^{n+1} z^{n+1} \\ + b_{n+2}(\rho_{n+2} \rho_{n+3} \cdots \rho_{\mu-1})^{n+2} z^{n+2} + \cdots + b_{\mu-1} \rho_{\mu-1}^{\mu-1} z^{\mu-1} + b_\mu z^\mu.$$

Let ε be any given positive number. Since $p_n(z)$ is uniformly continuous in E there exist numbers ρ_μ such that $1 < \rho_\mu < R_\mu$ and the product $\rho_{n+1} \rho_{n+2} \cdots \rho_{k-1}$ is so close to 1 that

$$|p_n(z) - p_n(\rho_n \rho_{n+1} \cdots \rho_{\mu-1} z)| < \varepsilon/2$$

for z in E , $\mu = n+1, \dots, k-1$. Let such a choice of ρ_μ be made. From (1) it follows that if $z \in E$ then

$$\begin{aligned} |p_\mu(z) - p_n(\rho_n \rho_{n+1} \cdots \rho_{\mu-1} z)| &\leq |b_{n+1}|(\rho_{n+1} \cdots \rho_{\mu-1})^{n+1} + |b_{n+2}|(\rho_{n+2} \cdots \rho_{\mu-1})^{n+2} \\ &\quad + \cdots + |b_{\mu-1}| \rho_\mu^{n-1} + |b_\mu| \\ &\leq (\rho_{n+1} \rho_{n+2} \cdots \rho_{k-1})^{n+1} \{|b_{n+1}| + |b_{n+2}| + \cdots + |b_\mu|\} \\ &\leq (\rho_{n+1} \rho_{n+2} \cdots \rho_{k-1})^{n+1} \{|b_{n+1}| + |b_{n+2}| + \cdots + |b_{k-1}|\}. \end{aligned}$$

Now choose the numbers b_μ , $\mu = n+1, \dots, k-1$, so that $0 < |b_\mu| < \delta_\mu$ and the last expression in the previous line does not exceed $\varepsilon/2$. This produces an appropriate chain

$$p(z) \subset p_{n+1}(z) \subset \cdots \subset p_{k-1}(z) \subset q(z)$$

satisfying the additional condition

$$|p_\mu(z) - p_n(z)| < \varepsilon$$

for z in E , $\mu = n+1, \dots, k-1$. This argument applied to $p_{n_j}(z)$ and $p_{n_{j+1}}(z)$, with $\varepsilon = \varepsilon_j$, yields the desired conclusion that $p_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$ uniformly in compact subsets of E . This completes the proof in the case $f(z)$ is not a polynomial.

The case when $f(z)$ is a univalent polynomial can be treated in an even simpler way. Suppose that $f(z) = \sum_{k=0}^N a_k z^k$, $a_N \neq 0$, is univalent in E . Let $\{r_j\}$ be a sequence of real numbers such that $0 < r_1 < r_2 < r_3 < \cdots$ and $r_j \rightarrow 1$ and set $f_j(z) = f(r_j z)$. Choose $b_1 \neq 0$ so small that $p_1(z) = a_0 + b_1 z$ is subordinate to $f_1(z)$ say by demanding the disk $|w - a_0| \leq |b_1|$ is covered by $f_1(E)$. The relation between $p_1(z)$ and $p_N(z) = f_1(z)$ implies the existence of polynomials $p_n(z)$, $n = 2, \dots, N-1$ univalent in E such that $p_n(z)$ has degree n and

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots \subset p_{N-1}(z) \subset p_N(z),$$

as was already shown in an earlier argument in this proof.

Next choose $b_{N+1} \neq 0$ so small that

$$p_{N+1}(z) = f_2(z) + b_{N+1} z^{N+1}$$

is univalent in E and such that $d(p_N(E), \delta p_{N+1}(E)) > 0$ and $d(p_{N+1}(E), \delta f_3(E)) > 0$.

This is possible since $f_2(z)$ is univalent in a disk $|z| < R$ with $R > 1$, $d(f_1(E), \delta f_2(E)) > 0$ and $d(f_2(E), \delta f(E)) > 0$.

At the next step we select $b_{N+2} \neq 0$ so small that $p_{N+2}(z) = f_3(z) + b_{N+2} z^{N+1}$ is univalent in E and such that $d(p_{N+1}(E), \delta p_{N+2}(E)) > 0$ and $d(p_{N+2}(E), \delta f_4(E)) > 0$.

This process can be continued to yield the chain

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots,$$

where each $p_n(z)$ is a polynomial of degree n univalent in E . Moreover, given any sequence of positive real numbers $\{\varepsilon_j\}$ such that $\varepsilon_j \rightarrow 0$ we can choose the numbers b_{N+j} such that $|b_{N+j}| < \varepsilon_j$. This ensures that $p_n(z) \rightarrow f(z)$ uniformly in E .

THEOREM 2. Suppose that $f(z)$ is analytic and univalent in E and maps E onto a convex domain. There exists a sequence $\{p_n(z)\}$, $n=1, 2, \dots$, such that $p_n(z)$ is a polynomial of degree n that maps E one-to-one onto a convex domain,

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \dots$$

in E , and $p_n(z) \rightarrow f(z)$ uniformly in each compact subset of E .

Proof. A function $g(z)$ analytic in E and satisfying $g'(0) \neq 0$ maps E one-to-one onto a convex domain if and only if

$$\operatorname{Re} \left\{ \frac{zg''(z)}{g'(z)} + 1 \right\} > 0$$

for z in E . Thus, if $0 < r < 1$ then such a function satisfies

$$m_g^*(r) = \min_{|z| \leq r} \operatorname{Re} \left\{ \frac{zg''(z)}{g'(z)} + 1 \right\} > 0.$$

The proof of this theorem is essentially the same as that given for Theorem 1 where the role of $m_g(r)$ in that theorem is now replaced by $m_g^*(r)$. Again letting $s_n(z)$ be the n th partial sum of the power series for $f(z)$, we note that $s'_n(z) \rightarrow f'(z)$ and $s''_n(z) \rightarrow f''(z)$ uniformly in $|z| \leq r$, $0 < r < 1$. Consequently, there exists an integer $N = N(r)$ such that if $n \geq N$ then

$$m_{s_n}^*(r) \geq m_f^*(r)/2.$$

In particular, such functions $s_n(z)$ are univalent and convex in $|z| \leq r$. Once this can be done it is clear that the other parts of the argument in Theorem 1 can be combined with this to yield (in the case $f(z)$ is not a polynomial) the polynomials $p_{n_j}(z) = s_{n_j}(r_j z)$ such that $0 < r_j < 1$, $r_j \rightarrow 1$, $p_{n_j}(z)$ is univalent and convex in E and has degree n_j , and $n_1 < n_2 < n_3 < \dots$. Also,

$$p_{n_1}(z) \subset p_{n_2}(z) \subset p_{n_3}(z) \subset \dots$$

in E ,

$$m_{p_{n_j}}^*(1) \geq m_f^*(r_j)/2$$

and

$$d(D_{n_j}(r_j), \delta D(r_{j+1})) > 0.$$

In order to interject convex univalent polynomials between two polynomials $p(z)$ and $q(z)$ which are consecutive in the sequence $\{p_{n_j}(z)\}$ first notice that $m_p^*(1) > 0$ and therefore $p(z)$ is univalent and convex in $|z| < R$ for some $R > 1$. Setting $P(z) = p(\rho z)$ for $1 < \rho < R$ we see that $m_\rho^*(1) > 0$, and, consequently, if

$$Q(z) = P(z) + bz^{n+1}$$

then

$$m_Q^*(1) \geq \frac{1}{2}m_P^*(1) > 0$$

for all sufficiently small values of b . This implies that $Q(z)$ is univalent and convex

in E . All the other arguments of Theorem 1 are applicable. In particular, at this point the relations between $p(z)$ and $q(z)$ now hold between $Q(z)$ and $q(z)$ so that the argument may be repeated. This produces the appropriate chain

$$p_1(z) \subset p_2(z) \subset p_3(z) \subset \cdots,$$

and, as before, a more careful construction of the interjected polynomials shows that such polynomials can be chosen which also satisfy $p_n(z) \rightarrow f(z)$ uniformly in compact subsets of E .

This outlines the proof when $f(z)$ is not a polynomial, and an argument like that given at the end of Theorem 1 takes care of the case when $f(z)$ is a polynomial.

REMARKS. 1. The method used to prove Theorems 1 and 2 also applies if $f(z)$ maps E one-to-one onto a domain starlike with respect to the point $f(0)$. This yields univalent polynomials $\{p_n(z)\}$ which are starlike with respect to $f(0)$. One simply needs to note that a function $g(z)$ analytic in E and satisfying $g'(0) \neq 0$ is univalent and starlike (with respect to $g(0)$) in E providing

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z) - g(0)} \right\} > 0.$$

The role of $m_\theta(r)$ in Theorem 1 is now replaced by

$$\min_{|z| \leq r} \operatorname{Re} \left\{ \frac{zg'(z)}{g(z) - g(0)} \right\}.$$

One can readily state suitable general conditions on a given function $f(z)$ analytic and univalent in E in order that the ideas of Theorem 1 yield a proof of the existence of such polynomials which additionally have the given property of $f(z)$. One such general statement is the following.

Let $w = F(u_1, u_2, \dots, u_n)$ be defined for certain complex n -tuples $(u_1, u_2, \dots, u_n)$. Then $f(z)$ has a certain "given property" if

$$w = G(z) = F(z, f(z), f'(z), \dots, f^{(n-1)}(z))$$

is a nonconstant analytic function in E and if the values of $G(z)$ are assumed to lie in a given open set. This includes as special cases starlike and convex (univalent) functions; namely, the value region for $G(z)$ in each case is $\operatorname{Re} w > 0$ and the functions $w = F(u_1, \dots, u_n)$ are

$$w = u_1 u_3 / (u_2 - f(0)) \quad \text{and} \quad w = u_1 u_4 / u_3 + 1,$$

respectively. This idea also can be expressed if $G(z)$ is replaced by a function of several variables. In particular, this would include the case of all univalent functions (Theorem 1). There the value region is $w \neq 0$ for the function $\Delta_f(u, v)$.

Finally, we mention one more example of this type of approximation, namely for the case $f(z)$ is close-to-convex. Such functions are defined by the conditions that they be analytic in E and that there exist an analytic, univalent and convex

function $g(z)$ such that $\operatorname{Re} \{f'(z)/g'(z)\} > 0$ in E . Such functions $f(z)$ are univalent in E [4]. If $0 < r < 1$ then $|g'(z)| \geq m_g(r)$ and, consequently,

$$\left| \frac{f'(z)}{g'(z)} - \frac{s'_n(z)}{g'(z)} \right| \leq \frac{1}{m_g(r)} |f'(z) - s'_n(z)|$$

for $|z| \leq r$. This shows that

$$s'_n(z)/g'(z) \rightarrow f'(z)/g'(z)$$

uniformly in $|z| \leq r$. Setting

$$\tilde{m}_f(r) = \min_{|z| \leq r} \operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\}$$

we see that $\tilde{m}_f(r)$ can play the role of $m_f(r)$ in the argument of Theorem 1. This produces an appropriate chain of polynomials each of which is close-to-convex (relative to $g(z)$) in E .

2. The proof of Theorem 1 is very constructive and when combined with precise inequalities for univalent functions gives specific information on how well $p_n(z)$ approximates $f(z)$. What is sought is an estimate on

$$\max_{|z| \leq r} |f(z) - p_n(z)|$$

in terms of n and r . We may assume that $f'(0) = 1$, as otherwise the factor $|f'(0)|$ will generally appear. It is known that $s_n(z)$ are univalent in

$$|z| < 1 - 6(\log n)/n$$

for $n \geq 17$ [5], and, in fact, this can be slightly improved using the more recent result, $m_f(r) \geq (1 - r^2)/r^2$ (see [2, p. 120] or [3]). The quantity $d(r)$ can also be estimated from below in a fairly simple fashion to get

$$d(r) \geq \frac{1}{4}((1 - r)/(1 + r))^2.$$

One only needs consider the interplay of these numbers $1 - 6(\log n)/n$ and $\frac{1}{4}((1 - r)/(1 + r))^2$, noticing that the interjected polynomials bear no significance as the sequence $\{\varepsilon_j\}$ in the proof is arbitrary. One finally applies these relations between r_j and n_j into the proof of Lemma 1 along with the estimate $|a_k| < ek$. A similar procedure is possible for other classes of univalent functions. In particular, use could be made of the results in [11, pp. 404–408].

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