COLLARING AN $(n-1)$ -MANIFOLD IN AN n -MANIFOLD

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1. Introduction. If M is a locally flat two-sided PL m -manifold in a PL $(m+1)$ -manifold N then clearly [2] M can be approximated pointwise by locally flat embeddings from either side. Using a powerful result of Edwards and Kirby [7] we show conversely that M has a collar on one side if M can be approximated by locally flat embeddings from that side. As an application it follows that M is locally flat (even if M is one-sided in N) if $N \setminus M$ is 1-LC at each point of M , M can be approximated by locally flat embeddings, and $m \geq 4$.

Let I denote the interval $[0, 1]$ and Id the identity mapping. Throughout we assume that M is a closed PL m -manifold, N is a PL n -manifold, $n = m + 1$, and M is topologically embedded in N° with two sides. We choose a metric denoted by d on N and on $M \times [-1, 1]$ we choose the product metric ρ . In case A, B are subsets of N and h is a homeomorphism of N we say that h is an ε -push of (N, A) keeping B fixed if there is an isotopy h_t of N such that $h_0 = \text{Id}$, $h_1 = h$, and for each $t \in I$ h_t is the identity on B and outside the ε -neighborhood of A and $d(h_t, \text{Id}) < \varepsilon$.

2. Preliminary results. Our proof depends heavily upon the following result of Edwards and Kirby [7].

LEMMA 1. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $h: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ is an embedding within δ of $\text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]$ then there is an isotopy $g_t: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ such that $g_0 = h$, $g_1|M \times 0 = h|M \times 0$, and for each $t \in I$ $g_t|M \times \{-\frac{1}{2}, \frac{1}{2}\} = h|M \times \{-\frac{1}{2}, \frac{1}{2}\}$ and $\rho(g_t, \text{Id}) < \varepsilon$.*

Thus we may define an isotopy f_t on $M \times [-1, 1]$ by $f_t = g_t h^{-1}$ on $h(M \times [-\frac{1}{2}, \frac{1}{2}])$ and $f_t = \text{Id}$ elsewhere. Clearly $f_0 = \text{Id}$, $f_1 h = \text{Id}|M \times 0$, and $\rho(f_t, \text{Id}) < 2\varepsilon$. Now let k be a homeomorphism of $M \times [-1, 1]$ commuting with the projection onto M , taking $M \times \frac{1}{4}$ onto $M \times t_0$ for some small $t_0 > 0$, and equal the identity on $M \times [-1, 0]$. Then, given $\varepsilon' > 0$, if ε is small enough and $h(M \times 0) \subset M \times [-1, 0]$ then $H = k f_1 k^{-1}|M \times [0, t_0]$ is an embedding satisfying $H|M \times t_0 = \text{Id}|M \times t_0$, $H|M \times 0 = h|M \times 0$, and $\text{diam } H(x \times [0, t_0]) < \varepsilon'$ for each $x \in M$.

Where two disjoint, close embeddings h_0, h_1 of M into N bound a common complementary domain, let $[h_0, h_1]$ denote the one which is near M . Thus the

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remarks above can be interpreted as saying that $[h|M \times 0, \text{Id}|M \times 0]$ is an ε -product if h is sufficiently close to $\text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]$. Using radial engulfing we shall show that h can be chosen close to $\text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]$ if $h|M \times 0$ is close enough to $\text{Id}|M \times 0$. First we need the codimension three version of Bing's Engulfing Theorem A of [1]. Thus we adopt the following terminology of Bing. Let $\{A_\alpha\}$ be a collection of sets in N^n , O an open subset of N , and U a neighborhood of $\bar{O}$. We say that finite r -complexes of U can be pulled into O along $\{A_\alpha\}$ if for each k -dimensional polyhedron $P \subset U$ and each subpolyhedron $Q \subset O$ such that $R = \overline{P \setminus Q}$ is compact and $R \subset N^\circ$, there is a homotopy $h: R \times I \rightarrow N^\circ$ such that $h_0 = \text{Id}$, $h_1(R) \subset O$, for each $t \in I$ $h_t|Q = \text{Id}|Q$, and for each $x \in R$ $h(x \times I)$ lies in an element of $\{A_\alpha\}$.

LEMMA 2. *Suppose $r \leq n-3$ and $\{A_\alpha\}$ is a collection of sets in N^n such that finite r -complexes in U can be pulled into O along $\{A_\alpha\}$. Then for each compact k -dimensional polyhedron $P \subset U$, each q -dimensional polyhedron $Q \subset O$ such that $R = \overline{P \setminus Q} \subset N^\circ$, $k, q \leq r$, and each $\varepsilon > 0$ there is an isotopy h_t of N^n such that $h_0 = \text{Id}$, $h_1(O) \supset P$, for each $t \in I$ $h_t|Q = \text{Id}|Q$, and for each $x \in N^n$ either $h(x \times I)$ is a point or else lies in the ε -neighborhood of the sum of some $k+1$ elements of $\{A_\alpha\}$ if $k \leq n-4$ and some $k+3$ elements of $\{A_\alpha\}$ if $k = n-3$.*

A proof can be constructed using piping (see Lemma 48 of [11]) and following the proof of Lemma 2.7 of [5].

PROPOSITION 3. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for each pair of disjoint embeddings $h_0, h_1: M \rightarrow N$ within δ of $\text{Id}|M$, there are strong ε -deformation retractions of $[h_0, h_1]$ onto $h_0(M)$ and $h_1(M)$.*

Proof. It is easy to show using the local contractibility of M that there is a neighborhood U of M and a $\delta > 0$ such that for any δ -embedding $h: M \rightarrow N$, U $\varepsilon/2$ -retracts onto $h(M)$ (see Proposition 2 of [3]). Since M is two-sided δ can be chosen so small that $[h_0, h_1]$ is defined for disjoint δ -embeddings of M and there is an $\varepsilon/2$ retraction r of U onto $[h_0, h_1]$. Now if h_0 and h_1 are sufficiently close to $\text{Id}|M$ then there is a homotopy r_t of $[h_0, h_1]$ in U with $r_0 = \text{Id}|[h_0, h_1]$, for each $t \in I$ $r_t|_{h_1(M)} = \text{Id}|_{h_1(M)}$, $d(r_t, \text{Id}|M) < \varepsilon/2$, and $r_1[h_0, h_1] = h_1(M)$. Thus rr_t is a strong ε -deformation retraction of $[h_0, h_1]$ onto $h_1(M)$.

PROPOSITION 4. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for each pair $h_0, h_1: M \rightarrow N$, $n \geq 5$, of disjoint embeddings within δ of $\text{Id}|M$ each neighborhood U of $[h_0, h_1]$, and each open set $O \supset h_0(M)$ there is an ε -push H of (N, M) fixed on $h_0(M)$ and outside U such that $H[h_0, h_1] \subset O$.*

Proof. The proof is standard using Proposition 3 to construct the sets $\{A_\alpha\}$ of Lemma 2. Then using the standard dual skeleton argument H is constructed by pushing O out over the $(n-3)$ -skeleton across to the dual 2-skeleton and then out over the rest of $[h_0, h_1]$. The first and last pushes are made using Lemma 2 and the

middle push (see Lemma 8.1 of [9]) preserves simplexes of some triangulation of N ; thus H can be made an ε -push of (N, M) fixed outside U .

LEMMA 5. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $h: M \times 0 \rightarrow M \times [-1, 1]$ is a locally flat embedding within δ - of $\text{Id}|M \times 0$ such that $h(M \times 0) \cap M \times 0 = \emptyset$ then there is a homeomorphism $H: M \times I \rightarrow [M \times 0, h]$ such that $H_0 = \text{Id}|M$, $H_1 = h$, and $\text{diam } H(x \times I) < \varepsilon$ for each $x \in M$.*

Outline of Proof. The proof is implicit in Wright's proof in [10]. From the remarks following Lemma 1 it is sufficient to show that h can be extended to $M \times [-\frac{1}{2}, \frac{1}{2}]$ so that $\rho(h, \text{Id}|M \times [-\frac{1}{2}, \frac{1}{2}]) < \delta$ where δ is given by Lemma 1 with ε replaced by some positive number depending on ε . Thus we need only to produce a $\delta > 0$ such that for each locally flat δ -approximation h of $\text{Id}|M \times 0$, each extension of h to $M \times [-\frac{1}{2}, \frac{1}{2}]$ so that for each $t \in [-\frac{1}{2}, \frac{1}{2}] \rho(h(x, t), (x, 0)) < \delta$, and each number $t_0 \in (0, 1)$ there is an ε -push H of $(M \times [-1, 1], M \times 0)$ fixed on $h(M \times [-1, t_0])$ such that $h(M \times t_0)$ is separated from $Hh(M \times 1)$ by $M \times t'$ for some $t' \in (-\varepsilon, \varepsilon)$. However for δ chosen by Proposition 4, an embedding $h: M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-1, 1]$ such that $\rho(h(x, t), (x, 0)) < \delta$ for all $t \in [-\frac{1}{2}, \frac{1}{2}]$, and $0 < t_0 < t_1 < \frac{1}{2}$ we apply Proposition 4 with $h_0(x) = (x, \lambda)$ for each $x \in M$ ($\lambda = \delta$ if $h(M \times 0)$ is separated from $M \times \delta$ by $h(M \times \frac{1}{2})$ and $\lambda = -\delta$ in the other case), $h_1(x) = h(x, t_1)$ for all $x \in M$, U = the component of $M \times [-1, 1] \setminus h(M \times t_0)$ containing $[h_0, h_1]$, and O = the component of $M \times [-1, 1] \setminus M \times \eta$ which does not contain $M \times 0$ where $0 < |\eta| < |\lambda|$ and $[M \times \eta, h_0] \subset [h_0, h_1]$. Thus there is an ε -push G of $(M \times [-1, 1], M \times 0)$ fixed outside U (and hence on $h(M \times t_0)$) such that $G[h_0, h_1] \subset O$ and therefore $Gh(M \times (t_0, t_1))$ contains $M \times \eta$. The embedding H is obtained now by choosing a fine partition $0 = t_0 < t_1 < \dots < t_k = \frac{1}{2}$ and an embedding $H': M \times [-\frac{1}{2}, \frac{1}{2}] \rightarrow M \times [-\delta, \delta]$ extending $h|M \times 0$ such that for each i , $H'(M \times (t_i, t_{i+1})) \supset M \times \eta_i$ for some $\eta_i \in (-\delta, \delta)$ and $\text{diam } H'(M \times [t_i, t_{i+1}])$ is small. Finally move η_i to $\pm i/2k$ by a homeomorphism F of $M \times [-1, 1]$ leaving $h(M \times 0)$ fixed. We define $H = FH'$. This completes the outline of the proof.

3. The main results.

THEOREM 6. *Suppose M is a closed PL m -manifold, $m \geq 4$, N is a PL n -manifold, and $n = m + 1$. If M is topologically embedded in the interior of N as a two-sided subset then M has a collar on one side if M can be pointwise approximated by locally flat embeddings on that side.*

Proof. Clearly it is sufficient to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any pair of disjoint locally flat embeddings $h_0, h_1: M \rightarrow N$ within δ of $\text{Id}|M$, there is a homeomorphism $H: M \times I \rightarrow [h_0, h_1]$ such that $H_0 = h_0$, $H_1 = h_1$, and $\text{diam } H(x \times I) < \varepsilon$ for each $x \in M$. H is constructed roughly as follows: Using a sequence of small engulfings move collars on $h_0(M)$ and $h_1(M)$ so that the whole collars are very close. Then apply local contractibility [7] to make the collars

agree on a little stretch in the middle. Thus $[h_0, h_1]$ is homeomorphic to $M \times I$ by the standard push-pull technique. We now give a rigorous argument.

First select $\delta_1 > 0$ and $\eta_1 > 0$ so that $d(x, y) < \eta_1$, $x, y \in M \Rightarrow d(h(x), h(y)) < \varepsilon/3$ for any δ_1 -embedding of M into N . Next pick η_2 using Lemma 5 with ε replaced by η_1 . Now there are $\delta_2 > 0$ and $\delta_3 > 0$ such that for any δ_2 -approximation h of Id/M and any pair (x, y) of points in $h(M)$ with $d(x, y) < 3\delta_3$, $d(h^{-1}(x), h^{-1}(y)) < \eta_2$. Finally we choose $\delta_4 > 0$ using Proposition 4 with ε replaced by $\min \{\varepsilon/3, \delta_3\}$. Let $\delta = \min \{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Now let $h_0, h_1: M \rightarrow N$ be disjoint locally flat embeddings within δ of $\text{Id}|M$. We can now assume that δ is so small that $h_0(M)$ is two-sided; then identifying M with $M \times 0$ extend h_0 to an embedding (still denoted h_0) of $M \times [-1, 1]$ into N such that

$$x, y \in M \times [-1, 1], \rho(x, y) < \eta_1 \Rightarrow d(h_0(x), h_0(y)) < \varepsilon/3.$$

Let $O \subset h_0(M \times [-1, 1])$ be a neighborhood of $h_0(M)$ so small that $x, y \in O$, $d(x, y) < 3\delta_3 \Rightarrow \rho(h_0^{-1}(x), h_0^{-1}(y)) < \eta_2$. Now apply Proposition 4 to obtain a $\min \{\varepsilon/3, \delta_3\}$ -push F of (N, M) , fixed on $h_0(M)$, such that $F[h_0, h_1] \subset O$. Then $d(h_0(x), Fh_1(x)) \leq d(h_0(x), h_1(x)) + d(h_1(x), Fh_1(x)) < 3\delta_3$. Thus $\rho(x, h_0^{-1}Fh_1(x)) < \eta_2$. Therefore we can apply Lemma 5 and obtain an embedding $G: M \times I \rightarrow M \times [-1, 1]$ such that $G_0 = \text{Id}|M$, $G_1 = h_0^{-1}Fh_1$, and $\text{diam } G(x \times I) < \eta_1$. Thus $\text{diam } h_0G(x \times I) < \varepsilon/3$. Now define H to be $F^{-1}h_0G$. Then $H_0 = F^{-1}h_0G_0 = F^{-1}h_0 = h_0$, $H_1 = F^{-1}h_0G_1 = F^{-1}h_0h_0^{-1}Fh_1 = h_1$, and for each $x \in M$, $H(x \times I) = F^{-1}(h_0G(x \times I))$ has diameter $< \varepsilon$. This completes the proof of Theorem 6.

For the proof of the next theorem we need one more preliminary result in order to apply Theorem 2.

PROPOSITION 7. *Suppose that the closed m -manifold M embedded in the interior of N^n , $n = m + 1$, separates N into two components U and V and U is 1-ULC. For each $\varepsilon > 0$ there is a neighborhood O of $U \cup M$ such that for each closed set $C \subset U$ there is a closed set B , $C \subset B \subset U$, and a homotopy h_t of O in N such that:*

1. $h_0 = \text{Id}$,
2. $h_1(O) \subset U$,
3. $h_t|B = \text{Id}|B$ for each $t \in I$,
4. $h_t(O \setminus B) \cap C = \emptyset$ for each $t \in I$, and
5. $d(h_t, \text{Id}) < \varepsilon$ for each $t \in I$.

Proof. The proof is in two steps.

Step 1. There is a neighborhood P of $N \setminus U$ such that for any closed set $C \subset U$ there is a closed neighborhood D of $N \setminus U$, $C \cap D = \emptyset$, and an $\varepsilon/2$ -map r of P into $N \setminus C$ such that $r|D = \text{Id}|D$ and $r(P \cap U) \subset U \setminus C$.

Step 2. There is a neighborhood O' of $U \cup M$ such that for any closed set $B \subset U$ there is a closed set E , $B \subset E \subset U$, and a strong $\varepsilon/2$ -retraction r_t of O' onto E such that $r_t(N \setminus E) \subset P$ for each $t \in I$.

Now to complete the proof let $O=O'$ from Step 2 where $B=\text{closure of } U \setminus D$. Then define $h_t(x)=x$ for $x \in B$ and $t \in I$ and $h_t(x)=r|_B r_t(x)$ for $x \in O \cap D$. Since $r_t(x) \in P$ for $x \in O \cap D$, $h_t(x)$ is defined for all $t \in I$ and $x \in O \cap D$. Since $r|_B \cap D = r_t|_B \cap D = \text{Id}|_B \cap D$, h_t is continuous for all $t \in I$. It is clear that h_t satisfies the conclusion of the proposition. Next we prove Step 1. Since M is an ANR and separates N there is a polyhedral neighborhood P of $N \setminus U$ and an $\varepsilon/8$ -retraction f of P onto $N \setminus U$. Now let $\eta = \min \{\varepsilon/8, d(M, C)/2\}$. Since U is 1-ULC, U is ULC^{n-1} . Thus there is a sequence $0 < \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_n = \eta$ such that each map $f: s^i \rightarrow U$ with $\text{diam } f(s^i) < 2\varepsilon_i$ can be extended to a map of the $(i+1)$ -ball B^i into U such that $\text{diam } f(B^{i+1}) < \varepsilon_{i+1}$. Now take a triangulation T' of P with mesh less than $2\varepsilon_1/3$ and so small that for each $\sigma \in T'$ $\text{diam } f(\sigma) < 2\varepsilon_1/3$. Since $f|_M = \text{Id}|_M$ there is a neighborhood D' of $N \setminus U$ such that $d(f|_{D'}, \text{Id}|_{D'}) < 2\varepsilon_1/3$. Choose a refinement T of T' so fine that the simplicial neighborhood D of $N \setminus U$ in T' is contained in D' . Define $r|_D = \text{Id}|_D$ and extend r skeletonwise to P as follows. For each vertex $\sigma^\circ \in P \setminus D$ pick $r(\sigma^\circ) \in U$ satisfying $d(r(\sigma^\circ), f(\sigma^\circ)) < 2\varepsilon_1/3$. Then for each 1-simplex $\sigma' \in P \setminus D$, $\text{diam } r(\sigma') < 2\varepsilon_1$. Suppose r has been extended to $r: D \cup |T^k| \rightarrow N$ so that for each $\sigma^{k+1} \in T$, $\text{diam } r(\sigma^{k+1}) < 2\varepsilon_{k+1}$. Thus for each $\sigma^{k+1} \in T$ we can extend $r|_{\sigma^{k+1}}$ to σ^{k+1} so that $\text{diam } r(\sigma^{k+1}) < 2\varepsilon_{k+2}$. By induction we have r defined on all of P so that $\text{diam } r(\sigma) < \eta$ for all simplexes $\sigma \in T$. However, $d(r(\sigma), N \setminus U) < 2\varepsilon_1/3 < \eta$ and $\text{diam } r(\sigma) < \eta$ imply that $r(\sigma) \subset N \setminus C$ for each $\sigma \in T$. Moreover for each point $p \in P$ there is a vertex v of T such that

$$d(p, r(P)) \leq d(p, v) + d(v, r(v)) + d(r(v), r(p)) < \eta + (\varepsilon/8 + \eta) + \eta < \varepsilon/2.$$

This completes the proof of Step 1. Step 2 can be proved similarly.

LEMMA 8. *Suppose that M^m is a closed two-sided submanifold of the interior of N^n , $n=m+1 \geq 5$. If $N \setminus M$ is 1-ULC then for each $\varepsilon > 0$ there is a PL ε -push H of (N, M) such that $H(M) \cap M = \emptyset$.*

Proof. Suppose that W is a connected open neighborhood of M which is separated into U and V by M . Then apply Proposition 7 with N replaced by W and ε by $\varepsilon/3n$ to obtain polyhedral neighborhoods O_1 of $W \setminus U$ and O_2 of $W \setminus V$. Apply Proposition 7 for each closed subset of $W \setminus M$ containing $W \setminus O_1 \cap O_2$ and let $\{A_\alpha\}$ be the tracks of all points under all such homotopies of $O_1 \cap O_2$ into $O_1 \cap U$ and $O_2 \cap V$. Now take a triangulation T of $O_1 \cap O_2$ with mesh less than $\varepsilon/3$ and apply Lemma 2 with P replaced by $|T^{n-3}|$, N by O_1 , O by $O_1 \cap U$ and U by $O_1 \cap O_2$. Thus there is an $\varepsilon/3$ -push H_1 of (W, M) fixed on $U \setminus O_1$ such that $H_1(U) \supset T^{n-3}$. Similarly there is an $\varepsilon/3$ -push H_2 of (W, M) fixed on $V \setminus O_2$ such that $H_2(V) \supset \tilde{T}^2$ the dual 2-skeleton of T^{n-3} . Using Lemma 8.1 of [9] there is an $\varepsilon/3$ -push H_3 of (W, M) such that $H_3 H_1(U) \cup H_2(V) = W$. Thus $H_2^{-1} H_3 H_1(U) \cup V = W \supset M$. Let $H = H_2^{-1} H_3 H_1$. Then H is an ε -push of (N, M) such that $H(M) \subset V$.

Clearly $H^{-1}(M) \subset U$ therefore if M can be approximated sufficiently close by locally flat embeddings then it can be approximated from both sides and so M is

bicollared. In fact, we can construct a double cover $\tilde{N}$ of N in the case that M is one-sided in N and $\tilde{M}$ (the part of $\tilde{N}$ covering M) is two-sided in $\tilde{N}$. Moreover, if M can be approximated then so can $\tilde{M}$. Therefore $\tilde{M}$ is bicollared and so M has a closed normal 1-disk bundle neighborhood. Thus we have the following.

THEOREM 9. *Suppose that M is a closed (possibly one-sided) m -manifold in the interior of the n -manifold N , $n=m+1 \geq 5$. If M can be pointwise approximated by locally flat embeddings and $N \setminus M$ is 1-ULC, then M has a normal 1-disk bundle neighborhood.*

We remark in conclusion that it follows from the annulus conjecture [8], Connell's approximation theorem [6], and [7] that any locally flat $(n-1)$ -sphere in S^n , $n \geq 5$, is ε -tame; thus for the case of spheres, Theorems 6 and 9 can be strengthened to ε -taming results. In fact, since the locally flat side approximations become levels in the collar, there is an ε -taming result if M can be approximated by PL embeddings from both sides. Therefore it follows from the results here and in [4] that a closed PL m -manifold M in the interior of an n -manifold N is ε -tame if its complement is 1-ULC, it can be approximated by PL embeddings, $n \geq 5$, and $m \neq n-2$. Moreover, given one of these embeddings $h: M \rightarrow N$ and an $\varepsilon > 0$ there is a $\delta > 0$ such that if g is also one that is within δ of h then there is an ε -push p of $(N, h(M))$ such that $h = pg$.

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