NONCOMPACT SIMPLICES

BY
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Abstract. A bounded, but not necessarily closed, (Choquet) simplex in R^n with nonempty interior is the intersection of $n+1$ half-spaces. There is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff real linear topological space.

0. Introduction. We suppose that X is a nonzero real linear space. We use the symbols t, u, v, w, x, y, z to represent elements of X and $\alpha, \beta, \gamma, \delta, \eta$ to denote real numbers ≥ 0 . We say that $S (\subset X)$ is a *simplex* if S is nonempty and convex and whenever $(x + \alpha S) \cap (y + \beta S) \neq \emptyset$ then there exist z, γ such that $(x + \alpha S) \cap (y + \beta S) = z + \gamma S$. The main results of this paper are: *a bounded simplex in R^n with nonempty interior is the intersection of $n+1$ half-spaces (each of which can be open or closed) and there is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff linear topological space.* These results are in §§2 and 3, respectively. In §1, we discuss a nontopological boundedness condition.

1. The closure of a simplex is sometimes a simplex.

1. DEFINITION. If $\emptyset \neq A \subset X$ and $x \in X \setminus \{0\}$ we write

$$D(A, x) = \sup \{ \alpha : \alpha x \in A - A \}.$$

($D(A, x)$ is the “diameter of A in the direction of x ”.)

2. LEMMA. Let X be a linear topological space, $S (\subset X)$ be a simplex and $z \in \text{int } S$. If $t, t', u, u' \in \bar{S}$ are such that $t' - t = u' - u \neq 0$ and ϕ is a nonzero continuous linear functional on X such that $\phi(t) = \inf \phi(S)$, $\phi(u) = \sup \phi(S)$ then $D(S, t' - t) = \infty$.

Proof. We suppose $0 < \alpha < \frac{1}{2}$. Since $t, u \in \bar{S}$ and $z \in \text{int } S$,

$$(1) \quad (1 - \alpha)t + \alpha z \in S$$

and

$$\alpha t + (1 - 2\alpha)u + \alpha z \in S$$

from which

$$(2) \quad (1 - \alpha)t + \alpha z \in (1 - 2\alpha)(t - u) + S$$

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hence there exist w, β such that

$$(3) \quad S \cap [(1-2\alpha)(t-u) + S] = w + \beta S.$$

Now

$$S \supset w + \beta S \subset (1-2\alpha)(t-u) + S$$

hence

$$\begin{aligned} \inf \phi(S) &\leq \phi(w) + \beta \inf \phi(S) \leq \phi(w) + \beta \sup \phi(S) \\ &\leq (1-2\alpha)\phi(t-u) + \sup \phi(S) \end{aligned}$$

from which

$$\beta \text{ diam } \phi(S) \leq (1-2\alpha)[\phi(t) - \phi(u)] + \text{diam } \phi(S) = 2\alpha \text{ diam } \phi(S).$$

Since $\text{int } S \neq \emptyset$, $\text{diam } \phi(S) > 0$ and so

$$(4) \quad \beta \leq 2\alpha.$$

From (1), (2) and (3)

$$(1-\alpha)t + \alpha z \in w + \beta S.$$

Now $\phi(t') \geq \inf \phi(S) = \phi(t)$ and so $\phi(t' - t) \geq 0$. Similarly $\phi(u' - u) \leq 0$. Hence, since $t' - t = u' - u$, $\phi(t' - t) = \phi(u' - u) = 0$ and so $\phi(t') = \inf \phi(S)$ and $\phi(u') = \sup \phi(S)$. Further, $t' - u' = t - u$ and so, by an argument similar to that above,

$$(1-\alpha)t' + \alpha z \in w + \beta S.$$

Hence $\beta D(S, t' - t) \geq 1 - \alpha$ and so, from (4), $2\alpha D(S, t' - t) \geq 1 - \alpha$. Letting $\alpha \rightarrow 0$ gives the required result.

3. DEFINITION. If $\emptyset \neq A \subset X$ we say that A is D -bounded if, for all $x \in X \setminus \{0\}$, $D(A, x) < \infty$.

4. THEOREM. If X is a linear topological space, $S (\subset X)$ is a D -bounded simplex and $\text{int } S \neq \emptyset$ then $\bar{S}$ is a simplex.

Proof. It is immediate that $\text{int } S$ is a D -bounded simplex. Further $\bar{S} = \overline{\text{int } S}$ so we can suppose, without loss of generality, that S is open. Suppose $T = (x + \alpha \bar{S}) \cap (y + \beta \bar{S}) \neq \emptyset$. If $T = \{z\}$ then $T = z + 0\bar{S}$ and the result follows. So we suppose that $T \ni v, T \ni w$ where $v \neq w$. This implies that $\alpha > 0, \beta > 0$. We shall show that

$$(5) \quad (x + \alpha S) \cap (y + \beta S) \neq \emptyset.$$

Indeed, if (5) is false then there exists a nonzero continuous linear functional ϕ on X such that

$$\inf \phi(x + \alpha S) = \phi(v) = \sup \phi(y + \beta S).$$

We suppose, without loss of generality, that $\beta \leq \alpha$. Then, from Lemma 2 with

$$t = \frac{v-x}{\alpha}, \quad u = \frac{v-y}{\beta}, \quad t' = \frac{w-x}{\alpha}, \quad u' = \frac{1}{\beta} \left\{ \left(1 - \frac{\beta}{\alpha}\right)v + \frac{\beta}{\alpha}w - y \right\},$$

$D(S, t' - t) = \infty$ contradicting our hypothesis that S is D -bounded. Hence (5) is true. Since S is a simplex, there exist z, γ such that

$$(6) \quad (x + \alpha S) \cap (y + \beta S) = z + \gamma S.$$

It follows by considering gauge functionals that if A, B are convex and open in X and $A \cap B \neq \emptyset$ then $\overline{A \cap B} = \overline{A} \cap \overline{B}$. Hence from (6),

$$(x + \alpha \bar{S}) \cap (y + \beta \bar{S}) = z + \gamma \bar{S}.$$

Thus $\bar{S}$ is a simplex.

5. REMARK. Let $X = R^2$ and $S = (0, 1) \times R$. Then S is a simplex but $\bar{S}$ is not a simplex. This example, suggested by D. Randtke, shows that some boundedness condition on S is essential in Theorem 4. In the next theorem we explore the properties of D -boundedness.

6. THEOREM. (a) If X is a Hausdorff linear topological space, $\emptyset \neq A \subset X$ and A is bounded then A is D -bounded.

(b) If $\emptyset \neq A \subset R^n$ and A is convex and D -bounded then A is bounded.

Proof. (a) Let $x \in X \setminus \{0\}$, U be a balanced neighborhood of 0 such that $U \not\ni x$ and V be a neighborhood of 0 such that $V - V \subset U$. If $\alpha x \in V - V$ then $\alpha x \in U$ and so $\alpha < 1$. Hence $D(V, x) \leq 1$. But A is absorbed by V hence $D(A, x) < \infty$.

(b) We first translate A to contain 0 and then replace R^n by the subspace spanned by A . We may, therefore, suppose that $\text{int } A \ni z$, say. If A is unbounded then, from the argument used in [2, p. 370], $\bar{A}$ contains an infinite half-line l . Then $A \supset \frac{1}{2}(z + l)$ and so A is not D -bounded.

7. REMARK. We give an example of a D -bounded open simplex in $C[0, 1]$. Let

$$S = \left\{ x : x \in C[0, 1], x(t) > 0 \text{ for all } t \in [0, 1], \int x < 1 \right\}.$$

S is clearly open. S is D -bounded, for if $x \in C[0, 1] \setminus \{0\}$ and $y, y + \alpha x \in S$ then

$$(7) \quad 0 \leq y + \alpha x$$

and

$$(8) \quad \int (y + \alpha x) \leq 1;$$

if, for some $t \in [0, 1]$, $x(t) < 0$ then, from (7), $\alpha |x(t)| \leq y(t)$ hence $D(S, x) < \infty$; if, on the other hand, for all $t \in [0, 1]$, $x(t) \geq 0$ then $\int x > 0$ and, from (8), $\alpha \int x \leq 1 - \int y$ and, again, $D(S, x) < \infty$. Finally, S is a simplex, for if $(x + \alpha S) \cap (y + \beta S) \neq \emptyset$ then

$$(x + \alpha S) \cap (y + \beta S) = x \vee y + \left[\left(\int x + \alpha \right) \wedge \left(\int y + \beta \right) - \int x \vee y \right] S.$$

We observe that S is unbounded. We shall see in Theorem 14 that this is, in fact, forced by the other conditions on S .

The above example emerged from a conversation with M. Rosenfeld.

2. Simplices in R^n . We suppose throughout this section that $X = R^n$. If $S(\subset X)$ is a bounded simplex then, by using the argument sketched in the proof of Theorem 6 (b), we can suppose that $\text{int } S \neq \emptyset$ and, to avoid trivial cases, that $n > 1$.

If $A \subset X$ we write $\text{conv } A$ for the convex hull of A . If $A = \{x_1, \dots, x_k\}$ is a finite set we write

$$\text{conv}_+ A = \left\{ \sum_{i=1}^k \alpha_i x_i : \alpha_i > 0, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

If S is as above then, from Theorem 6 (a) and Theorem 4, $\bar{S}$ is a compact simplex hence [1, Remarks following Definition 5] and [3, Proposition 9.11, p. 75] there exists affinely independent $v_0, \dots, v_n \in X$ such that $\bar{S} = \text{conv} \{v_0, \dots, v_n\}$. (We shall return to this result in Theorem 12.) Since S is convex, $S \supset \text{conv}_+ \{v_0, \dots, v_n\}$.

8. LEMMA. *We suppose that $X, S, v_0, \dots, v_n$ are as above and, further, that $v_0 = 0$. Then $v_1, \dots, v_n$ form a basis of X . We write $\psi_1, \dots, \psi_n$ for the dual basis of X and ψ for $\psi_1 + \dots + \psi_n$. We define the lattice ordering $\leq$ (with lattice operations $\vee$ and $\wedge$) on X by " $x \leq y$ " means that "for all $i = 1, \dots, n$, $\psi_i(x) \leq \psi_i(y)$ ". We write $w = n^{-1}(v_1 + \dots + v_n) \in \bar{S}$.*

(a) *If $x + \alpha S \supset z + \delta S$ then $z \geq x$ and $\psi(z) + \delta \leq \psi(x) + \alpha$.*

(b) *If $\eta > 0$, $z \geq x$ and $\psi(z) + \eta < \psi(x) + \alpha$ then $z + \eta w \in x + \alpha S$.*

(c) *If $(x + \alpha S) \cap (y + \beta S) = T \neq \emptyset$ then $T = x \vee y + \gamma S$, where*

$$\gamma = \min \{ \psi(x) - \psi(x \vee y) + \alpha, \psi(y) - \psi(x \vee y) + \beta \}.$$

(d) *If $y \in S$, $z \geq 0$, $\psi(y) = \psi(z) = 1$ and*

$$\{i : 1 \leq i \leq n, \psi_i(z) = 0\} \subset \{i : 1 \leq i \leq n, \psi_i(y) = 0\}$$

then $z \in S$.

(e) *If $x \in \alpha S$ and $y \in \beta S$ then $x \wedge y \in \gamma S$, where*

$$\gamma = \min \{ \psi(x \wedge y) - \psi(x) + \alpha, \psi(x \wedge y) - \psi(y) + \beta \}.$$

Proofs. (a) follows by taking the images under ψ_i and ψ and the inf and sup, respectively. In (b) the conditions imply that $z - x + \eta w \in \alpha \text{ int } S \subset \alpha S$.

(c) Since S is a simplex; there exist z, δ such that $T = z + \delta S$. From (a),

$$(9) \quad z \geq x, \quad z \geq y, \quad \psi(z) + \delta \leq \psi(x) + \alpha, \quad \psi(z) + \delta \leq \psi(y) + \beta$$

and so $z \geq x \vee y$. If we had $z \neq x \vee y$ then, for some i such that $1 \leq i \leq n$ and for some $\eta > 0$, $\psi_i(x \vee y) + \eta < \psi_i(z)$ hence $\psi(x \vee y) + \eta < \psi(z)$. From (9),

$$\psi(z) \leq \min \{ \psi(x) + \alpha, \psi(y) + \beta \}$$

thus, from (b), $x \vee y + \eta w \in (x + \alpha S) \cap (y + \beta S) = T$ hence $x \vee y + \eta w \geq z$. Letting $\eta \rightarrow 0$ gives $x \vee y \geq z$, hence $z = x \vee y$, as required. Returning to (9) we now see that $0 \leq \delta \leq \gamma$. If $\gamma = 0$ then clearly $\gamma = \delta$. If $\gamma > 0$ then, for any η such that $0 < \eta < \gamma$,

$\psi(x \vee y) + \eta < \min \{\psi(x) + \alpha, \psi(y) + \beta\}$ and so, as above, $x \vee y + \eta w \in z + \delta S = x \vee y + \delta S$. Hence $\eta w \in \delta S$ which implies that $\eta \leq \delta$. Letting $\eta \rightarrow \gamma$ gives that $\gamma \leq \delta$. Hence $\gamma = \delta$, as required.

(d) If $\alpha > 0$ is sufficiently small then $z - \alpha y \geq 0$ and $z - \alpha y + \alpha \text{int } S \subset \text{int } S$ hence $(z - \alpha y + \alpha S) \cap S \neq \emptyset$. From (c), $(z - \alpha y + \alpha S) \cap S = (z - \alpha y + \alpha S)$, hence $z - \alpha y + \alpha S \subset S$. In particular, $z = z - \alpha y + \alpha y \in S$.

(e) We have $0 \in (-x + \alpha S) \cap (-y + \beta S)$. The result is immediate from (c).

9. LEMMA. We suppose that $X, S, v_0, \dots, v_n$ are as in the discussion preceding Lemma 8. We write $\mathcal{N} = \{F : \emptyset = F \subset \{0, \dots, n\}\}$ and, if $F \in \mathcal{N}$,

$$[F] = \text{conv}_+ \{v_i : i \in F\}.$$

Also we write $\mathcal{F} = \{F : F \in \mathcal{N}, [F] \cap S \neq \emptyset\}$.

(a) $S = \bigcup \{[F] : F \in \mathcal{F}\}$.

(b) $G \in \mathcal{F}, F \in \mathcal{N}$ and $G \subset F$ imply $F \in \mathcal{F}$.

(c) If $F, G \in \mathcal{F}$ and $F \cap G \neq \emptyset$ then $F \cap G \in \mathcal{F}$.

Proofs. (a) We first observe that $S \subset \bar{S} = \bigcup \{[F] : F \in \mathcal{N}\}$ hence

$$S \subset \bigcup \{[F] : F \in \mathcal{F}\}.$$

Now we suppose $F \in \mathcal{F}$. If $F = \{0, \dots, n\}$ then $[F] = \text{int } S \subset S$. If $F \subsetneq \{0, \dots, n\}$ we assume, without loss of generality, that $F \not\ni 0$ and we translate S so that $v_0 = 0$. There exists y , say, such that $y \in [F] \cap S$. It follows from Lemma 8 (d) that, if $z \in [F]$ then $z \in S$, i.e. $[F] \subset S$. Hence $\bigcup \{[F] : F \in \mathcal{N}\} \subset S$.

(b) As in (a) we suppose, without loss of generality, that $F \not\ni 0, v_0 = 0$ and $y \in [G] \cap S$. From Lemma 8 (d) again, $[F] \subset S$, hence $F \in \mathcal{F}$.

(c) This time we reduce the problem to the case $F \cap G \ni 0$ and $v_0 = 0$. We write $x = \sum \{v_i : i \in F\}$ and $y = \sum \{v_i : i \in G\}$. Then, from (a), $x \in (\psi(x) + 1)S$ and $y \in (\psi(y) + 1)S$. From Lemma 8 (e), $x \wedge y \in (\psi(x \wedge y) + 1)S$. But

$$x \wedge y = \sum \{v_i : i \in F \cap G\}$$

hence $F \cap G \in \mathcal{F}$.

10. THEOREM. If S is a bounded simplex in R^n and $\text{int } S \neq \emptyset$ then S is the intersection of $n+1$ half-spaces.

Proof. We use the notation of Lemma 9. If there exists $F, G \in \mathcal{F}$ such that $F \cap G = \emptyset$ then, using (b) and (c) of Lemma 9, for each $i \in \{0, \dots, n\}$, $\{i\} = (F \cup \{i\}) \cap (G \cup \{i\})$ hence $\{i\} \in \mathcal{F}$ and so $\mathcal{F} = \mathcal{N}$. Hence, from Lemma 9 (a), $S = \bar{S}$ which is the intersection of $n+1$ closed half-spaces.

If, on the other hand, for all $F, G \in \mathcal{F}$, $F \cap G \neq \emptyset$ then, from (b) and (c) of Lemma 9 again, $\mathcal{F}$ is a filter of subsets of $\{0, \dots, n\}$ and so there exists $F_0 \in \mathcal{N}$ such that $\mathcal{F} = \{F : F \in \mathcal{N}, F \supset F_0\}$. From Lemma 9 (a)

$$S = \{\lambda_0 v_0 + \dots + \lambda_n v_n : \lambda_i > 0 \text{ for all } i \in F_0,$$

$$\lambda_i \geq 0 \text{ for all } i \in \{0, \dots, n\} \setminus F_0, \lambda_0 + \dots + \lambda_n = 1\},$$

which is the intersection of m open and $n+1-m$ closed half-spaces, where m is the cardinality of F_0 .

11. REMARKS. We leave to the reader the proof of the converse of Theorem 10, that any *bounded* set with nonempty interior that is the intersection of $n+1$ half-spaces is a simplex. We observe that there are exactly $n+2$ affinely different bounded simplices with nonempty interior in R^n , distinguished by the minimum dimension of an "open face" (i.e., a set $[F]$ for $F \in \mathcal{N}$). On the other hand there are just 3 topologically different such simplices. Finally, any such simplex contains exactly 0, 1 or $n+1$ of its vertices.

3. **Bounded simplices with nonempty interior.** In this section we return to the general notation of §1.

12. THEOREM. *If S is a linearly compact (see [2]) simplex in X , $x \in S$, $\eta > 0$ and $x - \eta(S - x) \subset S$, then S is the convex hull of a finite affinely independent set containing at most $1 + 1/\eta$ points.*

Proof. Let C be the cone $\{(\alpha y, \alpha) : \alpha \geq 0, y \in S\}$ in $E \times R$. From [2], C induces a lattice ordering on $L = C - C$. The map $\phi: (x, \alpha) \rightarrow \alpha$ is a positive linear functional on L , and $f \in L, f \geq 0$ and $\phi(f) = 0$ imply that $f = 0$. If $f \in L$ we write $\|f\| = \phi(|f|)$. $\|\cdot\|$ is a norm on L with respect to which L is a normed lattice and $\|\cdot\|$ is additive on the positive elements of L . Hence the completion, $\tilde{L}$, of L is an abstract L -space.

We write $g = (1 + 1/\eta)(x, 1)$. If $f \in L, f \geq 0$ then there exists $\alpha \geq 0, y \in S$ such that $f = (\alpha y, \alpha)$. Then

$$\begin{aligned} \phi(f)g - f &= \alpha(1 + 1/\eta)(x, 1) - (\alpha y, \alpha) \\ (10) \qquad &= (\alpha/\eta)(x - \eta(y - x), 1) \geq 0 \end{aligned}$$

by hypothesis, from which g is an order unit for $\tilde{L}$. From [4, V. 8.6, Corollary 1, p. 249], $\tilde{L}$ is finite dimensional and hence, for some $n \geq 1$, $L = \tilde{L} \cong l_n^1$.

We suppose that $f_1, \dots, f_n$ are the elements of L that correspond to the basic unit vectors of l_n^1 . For each $i = 1, \dots, n$ there exists $x_i \in S$ such that $f_i = (x_i, 1)$. It follows from the linear independence of $\{f_i\}$ that $\{x_i\}$ are affinely independent and from $L = \text{lin}\{f_i\}$ that $S = \text{conv}\{x_i\}$.

There exist $\alpha_1, \dots, \alpha_n \geq 0, \sum \alpha_i = 1$ such that $(x, 1) = \sum \alpha_i f_i$. We suppose that $i \in \{1, \dots, n\}$ has been chosen so that $\alpha_i \leq 1/k$. From (10), $f_i \leq \phi(f_i)g = \|f_i\|g = g = (1 + 1/\eta) \sum \alpha_i f_i$ hence, projecting along the vector f_i , $1 \leq (1 + 1/\eta)\alpha_i \leq (1 + 1/\eta)1/k$. Thus $k \leq 1 + 1/\eta$, as required.

13. REMARKS. The constant $1 + 1/\eta$ above is the best possible in the sense that if S is the convex hull of $n+1$ affinely independent points and x is the barycenter of S then $x - n^{-1}(S - x) \subset S$. This observation is due to D. Randtke. Theorem 12 generalizes the results used at the beginning of §2.

14. THEOREM. *There is no bounded simplex with nonempty interior in an infinite dimensional Hausdorff linear topological space.*

Proof. If X is a Hausdorff linear topological space and $S (\subset X)$ is a bounded simplex with $x \in \text{int } S$ then, from Theorem 6 (a) and Theorem 4, $\bar{S}$ is a simplex. $\bar{S}$ is linearly compact and, for some $\eta > 0$, satisfies the condition of Theorem 12. It now follows from Theorem 12 that $\dim X < \infty$.

15. REMARK. The above result should be compared with Remark 7.

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