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A CHARACTERIZATION OF THE PEANO DERIVATIVE(1)

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Abstract. For each choice of parameters $\{a_i, b_i\}$, $i=0, 1, \ldots, n+e$, satisfying certain simple conditions, the expression

$$\lim_{h\to 0} h^{-n} \sum_{i=0}^{n+e} a_i f(x+b_i h)$$

yields a generalized nth derivative. A function f has an nth Peano derivative at x if and only if all the members of a certain subfamily of these nth derivatives exist at x. The result holds for the corresponding L^p derivatives. A uniformity lemma in the proof (Lemma 2) may be of independent interest.

Also, a new generalized second derivative is introduced which differentiates more functions than the ordinary second derivative but fewer than the second Peano derivative.

Introduction. There are several definitions of the nth derivative of a function of a real variable in addition to the classical one. The most important perhaps is that due to Peano: the function f has at a point x_0 a derivative if there is a polynomial $P(t)=a_0+a_1t+\cdots+a_nt^n$ of degree less than or equal to n such that $f(x_0+t)=P(t)+o(t^n)$ as $t\to 0$; the number $n!\,a_n$ is called the nth Peano derivative of f at x_0 and will subsequently be denoted by $f_n(x_0)$. Clearly, the existence of $f_n(x_0)$ implies that of $f_m(x_0)$, $0\leq m<n$.

Another definition of the nth derivative is called Riemann's nth derivative $D_nf(x_0)$, and is defined by

$$D_nf(x_0) = \lim_{h\to 0} h^{-n} \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} f\left(x_0+\left(i-\frac{n}{2}\right)h\right).$$

It is a familiar fact that the existence of $f_n(x_0)$ implies that of $D_nf(x_0)$ and both are then equal. The converse may be false at a point but it is known that it holds at almost every point where D_nf exists (and is finite) [5].

This is merely a special case of a more general situation where we define the nth derivative as

$$\lim_{h\to 0} h^{-n} \sum_{i=0}^{n+e} a_i f(x+b_i h) \tag{1}$$

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provided the numbers $\{a_i, b_i\}$ satisfy the (clearly necessary) conditions

$$(2) \qquad \sum_{i=0}^{n+e} a_i b_i^r = 0, \qquad 0 \leqq r \leqq n-1, \qquad \sum_{i=0}^{n+e} a_i b_i^n = n!.$$

The integer e (hereafter called the excess) is any nonnegative integer, but the case $e=0$ deserves special attention. This case was considered by Denjoy [2]. A study of the case of positive excess ($e>0$) is found in [1]; where it is shown, in particular, that if such an nth derivative exists at each point of a set E, then at almost every point of E the nth Peano derivative also exists.

Some special cases of these derivatives can be constructed by considering successive differences. Fix f and x. Let

$$\begin{aligned}
\Delta_1(h) &= f(x+h)-f(x),\\
\Delta_2(a_1; h) &= \Delta_1(a_1h)-a_1\Delta_1(h) = f(x+a_1h)-a_1f(x+h)+(a_1-1)f(x),\\
\Delta_3(a_1, a_2; h) &= \Delta_2(a_1; a_2h)-a_2^2\Delta_2(a_1; h) = f(x+a_1a_2h)-a_1f(x+a_2h)\\
&\quad -a_2^2f(x+a_1h)+a_1a_2^2f(x+h)+(a_1-1)(1-a_2^2)f(x), \ldots,\\
\Delta_n(a_1, \ldots, a_{n-1}; h) &= \Delta_{n-1}(a_1, \ldots, a_{n-2}; a_{n-1}h)-a_{n-1}^{n-1}\Delta_{n-1}(a_1, \ldots, a_{n-2}; h).
\end{aligned}$$

Let $D_n(\boldsymbol{a})=\lim_{h\to 0} h^{-n}\Delta_n(\boldsymbol{a}; h)$ where $\boldsymbol{a}=(a_1, \ldots, a_{n-1})$. If no a_i is 0 or 1 and if no a_{n-2i} is -1, say that $\boldsymbol{a}$ is *nondegenerate*(2). If $\boldsymbol{a}$ is nondegenerate, then after multiplication by a sutable constant $\lambda_n(\boldsymbol{a})$ the nth difference $\Delta_n(\boldsymbol{a}; h)$ satisfies conditions (2) so that $\lambda_n(\boldsymbol{a})D_n(\boldsymbol{a})$ is an nth derivative. The proof is given in Lemma 1 below.

In this paper we will show that $D_n(\boldsymbol{a})$ exists for *many* $\boldsymbol{a}$'s if and only if $f_n(x)$ exists also. (See Theorem 1.) This characterization provides a converse to the elementary fact that the nth Peano derivatives' existence implies the existence of each of the generalized derivatives given by (1). For a different type of converse, see Theorem 1 of [1].

A strengthening of the hypothesis in the characterization yields yet another derivative whose existence is implied by the ordinary second derivative's and the existence of which implies that of the second Peano derivative with neither implication being reversible. All functions mentioned in this paper will be Lebesgue measurable real valued functions of a real variable. Existence of a limit (in particular, of a derivative at a point) will always mean finite existence.

1. Let S be a set of real numbers with the following two properties:

(i) S contains an interval;

(ii) a negative number belongs to S.

Let $S^k=S\times S\times \cdots \times S$ be the cartesian product of k copies of S.

(2) The reason for the third exclusion is that in case some $a_{n-2i}=-1$, $\Delta_n(h)$ is either identically 0 if some $a_{n-2j+1}=-1$ or else corresponds to a $(n+1)$st derivative if no $a_{n-2j+1}=-1$. To see the former, note that if some a_{2k} (respectively, a_{2k+1}) is -1, then Δ_n is an odd (respectively, even) function of h. If a function is both even and odd, it must be identically 0. The latter is immediate from setting $r=n$ and $n+1$ in equation (4).

THEOREM 1 (THE CHARACTERIZATION). *The existence of $D_n(\boldsymbol{a})$ for every $\boldsymbol{a} \in S^{n-1}$ is necessary and sufficient for the existence of $f_n(x)$.*

This holds in L^p, $1 \leqq p \leqq \infty$. All the L^p definitions are natural extensions of the L^∞ ones given above. For example, if $D_n^p(\boldsymbol{a})$ exists, it is the unique number such that

$$\left(\frac{1}{h}\int_0^h |\Delta_n(\boldsymbol{a};t) - D_n^p(\boldsymbol{a})t^n|^p\,dt\right)^{1/p} = o(h^n), \qquad h \to 0.$$

Henceforth by $\|f(h)\|$ we will mean

$$|f(h)| \quad \text{if } p = \infty \quad \text{or} \quad \left(\frac{1}{h}\int_0^h |f(t)|^p\,dt\right)^{1/p} \quad \text{if } p < \infty.$$

Statements such as $\|f(h)+g(h)\| \leqq \|f(h)\| + \|g(h)\|$ can be interpreted as the triangle inequality or as Minkowski's inequality; $D_n(\boldsymbol{a})$ will denote $D_n(\boldsymbol{a})$ or $D_n^p(\boldsymbol{a})$, etc.

We base the proof of Theorem 1 on four lemmas.

LEMMA 1. *If $f_n(x)$ exists, $D_n^p(\boldsymbol{a})$ exists: if $\boldsymbol{a}$ is nondegenerate, $f_n(x) = \lambda_n(\boldsymbol{a})D_n(\boldsymbol{a})$ where $\lambda_n(\boldsymbol{a}) = n!\,[(a_{n-1}^n - a_{n-1}^{n-1})\cdots(a_1^n - a_1)]^{-1}$ if $n \geqq 2$ $(\lambda_1 = 1)$; if $\boldsymbol{a}$ is degenerate, then $D_n(\boldsymbol{a}) = 0$.*

Proof. Since this proof is routine and tedious, for the duration of this proof only we introduce some notation to shorten the formulae. If $r \geqq 2$, let $\boldsymbol{a}^r = a_1 a_2^2 \cdots a_{r-1}^{r-1}$, $\boldsymbol{a}(r) = a_1 a_2 \cdots a_{r-1}$, $\boldsymbol{a}^{i(k,r)} = a_{i_1}^{i_1} a_{i_2}^{i_2} \cdots a_{i_k}^{i_k}$, and $\boldsymbol{a}(i(k,r)) = a_{i_1} a_{i_2} \cdots a_{i_k}$ for $1 \leqq k \leqq r-1$, where $i(k,r) = \{i_1, \ldots, i_k\}$, $1 \leqq i_1 < i_2 < \cdots < i_k \leqq r-1$; and $\boldsymbol{a}^1 = \boldsymbol{a}(1) = \boldsymbol{a}^{i(0,r)} = \boldsymbol{a}(i(0,r)) = 1$. Finally, let $\sum_r$ denote the sum over all possible $i(k,r)$ as k ranges from 0 to $r-1$.

We have the following identity.

$$(3) \qquad \Delta_n(\boldsymbol{a};h) = \boldsymbol{a}^n \sum_n (-1)^{n-k-1} \frac{f(x + \boldsymbol{a}(i(k,n))h)}{\boldsymbol{a}^{i(k,n)}} + \sum_n (-1)^{n-k} \frac{\boldsymbol{a}^n}{\boldsymbol{a}^{i(k,n)}} f(x).$$

The proof follows by induction. It is trivially true if $n=1$. Suppose it is true for $r-1$, $1 < r \leqq n$. Then applying this to the definition of Δ_r, we have

$$\begin{aligned}\Delta_r(\boldsymbol{a};h) = \Bigg\{ &\boldsymbol{a}^{r-1} \sum_{r-1} (-1)^{(r-1)-k-1} \frac{f(x + \boldsymbol{a}(i(k,r-1))a_{r-1}h)}{\boldsymbol{a}^{i(k,r-1)}} \\ &+ \sum_{r-1} (-1)^{r-1-k} \frac{\boldsymbol{a}^{r-1}}{\boldsymbol{a}^{i(k,r-1)}} f(x) \Bigg\} \\ &- \boldsymbol{a}^r \sum_{r-1} (-1)^{(r-1)-k-1} \frac{f(x + \boldsymbol{a}(i(k,r-1))h)}{\boldsymbol{a}^{i(k,r-1)}} \\ &- \sum_{r-1} (-1)^{r-1-k} \frac{\boldsymbol{a}^r}{\boldsymbol{a}^{i(k,r-1)}} f(x).\end{aligned}$$

Multiplying and dividing each term in the curly brackets by a_{r-1}^{r-1}, distributing the minus sign through the other terms, and reinterpreting each of the sums in the curly brackets as being taken over $1 \leqq i_1 < \cdots < i_k < i_{k+1} = r-1$, we see that $\Delta_r(\boldsymbol{a}; h)$ has the desired expansion (3).

By replacing each term of (3) with its nth order expansion and interchanging the order of summation, the proof is quickly seen to depend on the following n identities which are instances of equations (2) of the introduction.

$$\sum_n (-1)^{n-k-1} \frac{[\boldsymbol{a}(i(k,n))]^r}{\boldsymbol{a}^{i(k,n)}} = 0, \qquad r = 1, \ldots, n-1,$$

$$\boldsymbol{a}^n \sum_n (-1)^{n-k-1} \frac{[\boldsymbol{a}(i(k,n))]^n}{\boldsymbol{a}^{i(k,n)}} = (a_1^n - a_1) \cdots (a_{n-1}^n - a_{n-1}^{n-1}).$$

(Note that the equation for $r=0$, which is not displayed and which amounts to $\Delta_n(\boldsymbol{a}; 0) = 0$, is immediate from (3).)

This system of equations, in turn, follows from

$$\sum_n (-1)^{n-k-1} \frac{[\boldsymbol{a}(i(k,n))]^r}{\boldsymbol{a}^{i(k,n)}} = (a_1^{r-1}-1)(a_2^{r-2}-1)\cdots(a_{n-1}^{r-(n-1)}-1) \tag{4}$$

which is valid for any positive integral value of r. To prove (4), simply multiply out the right-hand side.

It is interesting to note that $\lambda_n(\boldsymbol{a})D_n(\boldsymbol{a})$ is a generalized derivative without excess, i.e., is based on only $n+1$ points, only if $\boldsymbol{a} = (a, a, \ldots, a)$ where $a \notin \{0, 1\}$ and, if $n \geqq 3$, $a \neq -1$. This is the only case which occurs if n is 1 or 2. For $n \geqq 3$ it is a specialization of $D_n(\boldsymbol{a})$ and yields a derivative very much like the derivative called $\tilde{D}_k f(x)$ in [5, p. 9–10].

LEMMA 2. *Suppose*

$$\lim_{h \to 0} \|g(ah) - g(h)\| = 0 \tag{5}$$

for every a in some interval $[\alpha, \beta]$, $\alpha < \beta$. *Then given any* $M > \varepsilon > 0$, *the above limit is uniform for* $a \in [\varepsilon, M]$, *i.e.*,

$$\lim_{h \to 0} \left(\sup_{a \in [\varepsilon, M]} \|g(ah) - g(h)\| \right) = 0.$$

Proof. By simply dropping a portion of $[\alpha, \beta]$ we may assume that either $\alpha < \beta < 0$ or $0 < \alpha < \beta$. Set $r = \frac{1}{2}(\alpha + \beta)$ and let $b = a/r$. As a ranges over the interval $[\alpha, \beta]$, b ranges over an interval whose interior contains 1. Replacing h by h/r in (5), we have

$$\lim_{h \to 0} \|g(bh) - g(h/r)\| = 0 \tag{6}$$

for all $b \in [\alpha', \beta']$ where $\alpha' < 1 < \beta'$. Specialize (6) to the case when $b = 1$, obtaining

$$\lim_{h \to 0} \|g(h) - g(h/r)\| = 0. \tag{7}$$

Combining (6) and (7) and applying the triangle inequality, we obtain

$$\lim_{h\to 0} \|g(bh)-g(h)\| = 0 \tag{5'}$$

for all $b \in [\alpha', \beta']$, $\alpha' < 1 < \beta'$. Choose $s > 1$ so small that $\alpha' \leqq s^{-2}$, $\beta' \geqq s$. It follows from (5′) that

$$\lim_{h\to 0} \|g(ah)-g(h)\| = 0, \qquad s^{-2} \leqq a \leqq s. \tag{5''}$$

Claim

$$\lim_{h\to 0} \left(\sup_{a\in[s^{-1},1]} \|g(ah)-g(h)\|\right) = 0. \tag{8}$$

Suppose (8) is false. Then there is a sequence $\{a_n\}$, $a_n \in [s^{-1}, 1]$, a sequence $\{h_n\}$, $h_n \to 0$, and a $\delta > 0$ such that

$$\|g(a_n h_n)-g(h_n)\| \geqq \delta. \tag{9}$$

Let $V_n = \{a \in [s^{-1}, s] \mid \|g(ah_k)-g(h_k)\| < \delta/2 \text{ for all } k \geqq n\}$. Then $V_n \subset V_{n+1}$ and $\bigcup_{n=1}^{\infty} V_n = [s^{-1}, s]$, so that $|V_n| \nearrow (s-s^{-1})$ as n increases. ($|V| =$ Lebesgue measure of V.) Let $W_n = \{b \in [s^{-1}, s] \mid \|g(ba_k h_k)-g(a_k h_k)\| < \delta/2 \text{ for all } k \geqq n\}$. As above, $|W_n| \nearrow (s-s^{-1})$ as n increases. Let $W_n' = \{c \mid c = a_n b, b \in W_n\}$. Since $a_n \geqq s^{-1}$, $\liminf_{n\to\infty} |W_n'| \geqq 1-s^{-2}$. Since $(s-s^{-1})+(1-s^{-2}) > s-s^{-2}$ and both W_n' and V_n are contained in $[s^{-2}, s]$, if n is sufficiently large $W_n' \cap V_n \neq \varnothing$. But if $c \in W_n'$, by (9) and the definitions of W_n and W_n', we have

$$\begin{array}{ccccccc} \|g(ch_n)-g(h_n)\| & \geqq & \|g(a_n h_n)-g(h_n)\| & - & \|g(ch_n)-g(a_n h_n)\| & & \\ & > & \delta & - & \delta/2 & = & \delta/2 \end{array}$$

so $c \notin V_n$. This contradiction establishes (8).

Finally, replace h by ah in (8) and combine the result with (8) to get

$$\lim_{h\to 0} \left(\sup_{a\in[s^{-1},1]} \|g(a^2h)-g(h)\|\right) = 0. \tag{8'}$$

Iterate this procedure k times, getting

$$\lim_{h\to 0} \left(\sup_{a\in[s^{-1},1]} \|g(a^k h)-g(h)\|\right) = 0, \tag{8''}$$

or, setting $b = a^k$,

$$\lim_{h\to 0} \left(\sup_{b\in[s^{-k},1]} \|g(bh)-g(h)\|\right) = 0. \tag{8'''}$$

Replacing h by h/b in (8‴) shows that b may be replaced by $1/b$, where $1/b$ ranges over $[1, s^k]$. If k is chosen so large that $s^{-k} < \varepsilon$ and $s^k > M$, the lemma is proved.

The argument which derives (8) from (5″) is found in [4, pp. 81–82]. It was pointed out to me by Professor Lee Rubel of the University of Illinois who has also communicated to me an independently established, unpublished proof of Theorem 3 of [4]. By setting $h=e^{-x}$, $a=e^{-\lambda}$ and $k(x)=g(e^{-x})$, we may obtain an additive version of Lemma 2 which produces the conclusion of [4], Theorem 3, from a weaker hypothesis.

COROLLARY. *Let $k(x)$ be defined for $x\geqq 0$. Let $k(x)$ be real measurable on every interval $0\leqq x\leqq A$ and such that $|k(x+\lambda)-k(x)|\to 0$ as $x\to\infty$ for every λ in* **some** *interval. Then the limit is uniform in λ as λ varies over* **any** *finite interval.*

LEMMA 3. *If $D_n(a_1,\ldots,a_{n-1})$ exists for all $\boldsymbol{a}\in S^{n-1}$ $(n\geqq 2)$, then $D_{n-1}(\boldsymbol{a}')$ exists for all $\boldsymbol{a}'=(a_1,\ldots,a_{n-2})\in S^{n-2}$.*

Proof. Patrick O'Connor has shown that whenever two nth generalized derivatives given by (1) both exist for f at x, they are necessarily equal [6]. Hence, we may set D equal to the common value of all the $\lambda_n(\boldsymbol{a})D_n(\boldsymbol{a})$ for nondegenerate $\boldsymbol{a}$. By replacing $f(x)$ by $f(x)-D(x^n/n!)$, we may assume without loss of generality that $D=0$. Our hypothesis is now that for all $\boldsymbol{a}\in S^{n-1}$ and $\varepsilon>0$,

$$\|\Delta_n(\boldsymbol{a};h)\| \leqq \varepsilon|h|^n \quad \text{if } |h| < \delta(\boldsymbol{a},\varepsilon),$$

or, by the definition of Δ_n,

$$\|\Delta_{n-1}(\boldsymbol{a}';a_{n-1}h)-a_{n-1}^{n-1}\Delta_{n-1}(\boldsymbol{a}';h)\| \leqq \varepsilon|h|^n \quad \text{if } |h| < \delta(\boldsymbol{a},\varepsilon). \tag{10}$$

Fix any $\boldsymbol{a}'\in S^{n-2}$ and in (10) choose $a_{n-1}=b$ where $-1<b<0$. To see that this can be done, first observe that since by hypothesis S contains an interval, it contains numbers other than 0, 1, or -1. We consider several cases.

(i) There is a number $c\in S$, $-1<c<0$.

(ii) Although (i) fails, S contains an element $c<-1$.

In case (i), set $b=c$. In case (ii), an application of the identity

$$-c^{2n-1}[(ch)^{-n}\Delta_n(\boldsymbol{a}',c^{-1};ch)] = h^{-n}\Delta_n(\boldsymbol{a}',c;h)$$

(which is easily proved by expressing Δ_n in terms of Δ_{n-1} on both sides) shows that (10) holds with c replaced by c^{-1} and h by ch, so set $b=c^{-1}$.

If (i) and (ii) fail, then since by hypothesis S contains a negative number, we must have $-1\in S$. Also S contains a positive interval. Applying the identity of case (ii) if necessary, we may assume there is a number c, $0<c<1$ for which (10) holds. We may set $b=-c$. To see this let $\varepsilon>0$ be given. There is a $\delta_1=\delta_1(\boldsymbol{a}',c,\varepsilon)$ so that

$$\|\Delta_{n-1}(\boldsymbol{a}';c(-h))-c^{n-1}\Delta_{n-1}(\boldsymbol{a}';-h)\| \leqq \varepsilon|h|^n/2$$

provided $|h|<\delta_1$. There is a $\delta_2=\delta_2(\boldsymbol{a}',-1;\varepsilon)$ so that

$$\|\Delta_{n-1}(\boldsymbol{a}';(-1)h)-(-1)^{n-1}\Delta_{n-1}(\boldsymbol{a}';h)\| \leqq \varepsilon|h|^n/2c^{n-1}$$

provided $|h|<\delta_2$. Multiplying the second inequality by c^{n-1} and adding the two inequalities we obtain

$$\|\Delta_{n-1}(\boldsymbol{a}';(-c)h)-(-c)^{n-1}\Delta_{n-1}(\boldsymbol{a}';h)\| \leqq \varepsilon|h|^n$$

provided $|h| < \delta = \delta(\boldsymbol{a}', -c, h) = \min\{\delta_1, \delta_2\}$. Hence (10) holds with $-1 < a_{n-1} = b < 0$.

Now pick m_0 so large that $|b|^{m_0} < \delta(\boldsymbol{a}', b, \varepsilon)$. Let first $p = \infty$. Write out equation (10) for $h = b^k$, $k = m, m+1, \ldots, m+l-1$ where $m \geqq m_0$ is a positive integer. For the duration of the proof of this lemma, we will abbreviate $\Delta_{n-1}(\boldsymbol{a}'; h)$ by $\Delta(h)$.

$$|\Delta(b^{m+1}) - b^{n-1}\Delta(b^m)| \leqq \varepsilon|b|^{mn},$$
$$|\Delta(b^{m+2}) - b^{n-1}\Delta(b^{m+1})| \leqq \varepsilon|b|^{(m+1)n},$$
$$\cdot \quad \cdot \quad \cdot$$
$$|\Delta(b^{m+l}) - b^{n-1}\Delta(b^{m+l-1})| \leqq \varepsilon|b|^{(m+l-1)n}.$$

Multiplying these inequalities by $|b|^{-(m+1)(n-1)}, |b|^{-(m+2)(n-1)}, \ldots, |b|^{-(m+l)(n-1)}$ respectively, we obtain by addition,

$$\left|\frac{\Delta(b^{m+l})}{(b^{m+l})^{n-1}} - \frac{\Delta(b^m)}{(b^m)^{n-1}}\right| \leqq \frac{\varepsilon|b|^{m-n+1}}{1-|b|} \leqq \frac{\varepsilon|b|^{-n+1}}{1-|b|}.$$

Since ε was arbitrary, this inequality shows that the sequence

$$\{\Delta(b^m)/b^{m(n-1)}\}, \qquad m = 1, 2, \ldots, \tag{11}$$

is Cauchy and hence convergent. From (10) we also know that

$$|\Delta(a_{n-1}h)/(a_{n-1}h)^{n-1} - \Delta(h)/h^{n-1}| \to 0 \quad \text{as } h \to 0 \tag{12}$$

for each a_{n-1} in some interval. Applying Lemma 2, with $g(h) = \Delta(h)h^{-(n-1)}$, to (12) shows that (12) holds uniformly for a_{n-1} in the interval $[b^2, 1]$. Given any h, $0 < |h| < 1$, there is $a_{n-1} \in [b^2, 1]$ so $a_{n-1}h$ is a power of b. From (12) we conclude that the limit of the sequence (11) is also the limit of $\Delta(h)h^{-(n-1)}$ as $h \to 0$, in other words, that $D_{n-1}(\boldsymbol{a}')$ exists. Now suppose that $1 \leqq p < \infty$. For this proof we replace the somewhat vague notation $\|g(h)\|$ by $\|g(t)\|_h$ for the expression

$$\left(\frac{1}{h}\int_0^h |g(t)|^p\,dt\right)^{1/p}.$$

The following implication is well known.

$$\|g(t)\|_h \leqq A|h|^\alpha \quad \text{implies} \quad \|g(t)t^{-\beta}\|_h \leqq Ak|h|^{\alpha-\beta} \quad \text{if } \alpha \geqq \beta \geqq 0 \tag{13}$$

where k depends only on β(3). This can be proved by setting $G(t) = \int_0^t |g(s)|^p\,ds$ and integrating by parts. Divide equation (10) by $|a_{n-1}^{n-1}|$. Then apply (13) to the result with $\alpha = n$, $\beta = n-1$, obtaining

$$\left\|\frac{\Delta(a_{n-1}t)}{(a_{n-1}t)^{n-1}} - \frac{\Delta(t)}{t^{n-1}}\right\|_h < \frac{\varepsilon k|h|}{|a_{n-1}^{n-1}|} \quad \text{if } |h| < \delta(a_{n-1}, \varepsilon). \tag{14}$$

(3) This is true under the slightly more general hypothesis that $(\alpha-\beta) > -p^{-1}$; in that case, k will depend on α, β, and p.

Set $a_{n-1}=b$, change the variable of integration to $s=t/h$, and let $h=b^m,\ldots,$ b^{m+l-1}, where m is chosen as in the $p=\infty$ case, obtaining

$$\left\|\frac{\Delta(b^{m+1}s)}{(b^{m+1}s)^{n-1}}-\frac{\Delta(b^m s)}{(b^m s)^{n-1}}\right\|_1 \leqq \frac{\varepsilon k|b|^m}{|b|^{n-1}},$$

$$\left\|\frac{\Delta(b^{m+2}s)}{(b^{m+2}s)^{n-1}}-\frac{\Delta(b^{m+1}s)}{(b^{m+1}s)^{n-1}}\right\|_1 \leqq \varepsilon\frac{k|b|^{m+1}}{|b|^{n-1}},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left\|\frac{\Delta(b^{m+l}s)}{(b^{m+l}s)^{n-1}}-\frac{\Delta(b^{m+l-1}s)}{(b^{m+l-1}s)^{n-1}}\right\|_1 \leqq \frac{\varepsilon k|b|^{m+l-1}}{|b|^{n-1}}.$$

By Minkowski's inequality we obtain

$$\left\|\frac{\Delta(b^{m+l}s)}{(b^{m+l}s)^{n-1}}-\frac{\Delta(b^m s)}{(b^m s)^{n-1}}\right\|_1 \leqq \frac{\varepsilon k|b|^{1-n}}{1-|b|}.$$

It follows that the sequence of $L^p[0, 1]$ functions

$$\{\Delta(b^m s)/(b^m s)^{n-1}\}, \qquad m = 1, 2, \ldots, \tag{15}$$

is Cauchy in the L^p norm and, hence, there is an L^p function $G(s)$ defined (almost everywhere) on $[0, 1]$ such that the sequence (15) converges to G in the L^p norm. From (14) and Lemma 2 as in the $p=\infty$ case, it follows that

$$\|\Delta(hs)/(hs)^{n-1}-G(s)\|_1 \to 0 \quad \text{as } h\to 0. \tag{16}$$

If $G(s)$ is essentially constant in (16), say $G(s)=r$ (a.e.), then $D_{n-1}(\boldsymbol{a}')$ exists and equals r, since

$$\frac{1}{|h|^{n-1}}\|\Delta(t)-rt^{n-1}\|_h \leqq \left\|\frac{\Delta(t)}{t^{n-1}}-r\right\|_h = \left\|\frac{\Delta(hs)}{(hs)^{n-1}}-r\right\|_1 \to 0$$

as $h\to 0$. Suppose that $G(s)$ is not essentially constant. Then there are numbers $\alpha<\beta$ and subsets of $[0, 1]$, E_α, E_β of strictly positive measure such that $G(s)<\alpha$ for $s\in E_\alpha$ and $G(s)\geqq\beta$ for $s\in E_\beta$. For any set E, let $cE=\{cx \mid x\in E\}$. Note $|cE| =|c|\,|E|$. Let $x_0\in(0, 1)$ be a point of density of E_α and let $y_0\in(0, 1)$ be a point of density of E_β. Set $c=y_0/x_0$. Then y_0 is a point of density of both E_β and cE_α and, hence, there is a set B of strictly positive measure which is contained in both E_β and cE_α. Furthermore, B may be chosen in as small an interval about y_0 as one likes. If B is close to y_0, $c^{-1}B$ is close to x_0. Pick B so that both B and $c^{-1}B$ are contained in $[0, 1]$.

If C is any subset of $[0, 1]$, it follows from (16) and Hölder's inequality, that

$$\left|\int_C\left\{\frac{\Delta(hs)}{(hs)^{n-1}}-G(s)\right\}ds\right| \leqq \int_0^1\left|\frac{\Delta(hs)}{(hs)^{n-1}}-G(s)\right|ds \leqq \left\|\frac{\Delta(hs)}{(hs)^{n-1}}-G(s)\right\|_1 \to 0 \quad \text{as } h\to 0$$

so that

$$\int_C\left\{\frac{\Delta(hs)}{(hs)^{n-1}}-G(s)\right\}ds = \eta(C, h)\to 0 \quad \text{as } h\to 0. \tag{17}$$

Apply (17), first with $C=B$, then with $C=c^{-1}B$. We get

$$k^{-1}\int_{kB}\Delta(t)t^{-(n-1)}\,dt = \eta(B,k)+\int_B G(s)\,ds \geqq \eta(B,k)+\beta|B|$$

and

$$(ck)^{-1}\int_{ck(c^{-1}B)}\Delta(t)t^{-(n-1)}\,dt = \eta(c^{-1}B,ck)+\int_{c^{-1}B} G(s)\,ds$$

$$\leqq \eta(c^{-1}B,ck)+\alpha|c^{-1}B|$$

since $B\subset E_\beta$ and $c^{-1}B\subset E_\alpha$. The former inequality implies

$$\liminf_{k\to 0} k^{-1}\int_{kB}\Delta(t)t^{-(n-1)}\,dt \geqq \beta|B|$$

and the latter implies

$$\limsup_{k\to 0} k^{-1}\int_{kB}\Delta(t)t^{-(n-1)}\,dt \leqq \alpha|B|,$$

a contradiction since $\alpha<\beta$.

LEMMA 4. *If the derivatives $D_n(a,a,\ldots,a)$, $a\notin\{-1,0,1\}$ and $f_{n-1}(x)$ both exist at a point x, so does $f_n(x)$.*

The proof is essentially that of Lemma 1 of [5].

Proof. Suppose that $f_i(x)=0$, $i=0,1,\ldots,n-1$, and that $D_n(a,\ldots,a)=0$. If $\|\Delta_n(h)\|=o(|h|^n)$, i.e., $\|\Delta_n(h)\|\leqq\varepsilon|h|^n$ for $|h|<\delta=\delta(\varepsilon)$, then $\|\Delta_{n-1}(ah)-a^{n-1}\Delta_{n-1}(h)\|\leqq\varepsilon|h|^n,\ldots,\|\Delta_{n-1}(a^kh)-a^{n-1}\Delta_{n-1}(a^{k-1}h)\|\leqq\varepsilon|a^{k-1}h|^n$. Multiplying these inequalities by $|a|^{-(n-1)}, |a|^{-2(n-1)},\ldots,|a|^{-k(n-1)}$ respectively, we obtain by addition

$$\|\Delta_{n-1}(h)-a^{-k(n-1)}\Delta_{n-1}(a^kh)\| \leqq \varepsilon|a|^{1-n}(1-|a|)^{-1}|h|^n.$$

To obtain this last inequality we assume $|a|<1$. If $|a|>1$, replace h successively by $ha^{-1}, ha^{-2},\ldots,ha^{-k}$ in the equation $\|\Delta_n(h)\|\leqq\varepsilon|h|$, obtaining virtually the same result. (Alternatively, note that the identity

$$(-1)^{m-1}a^{m(m-1)/2}\Delta_m(a^{-1},\ldots,a^{-1};a^{m-1}h) = \Delta_m(a,\ldots,a;h)$$

shows that there is no loss of generality in the original assumption that $|a|<1$.) By the triangle inequality,

$$\|\Delta_{n-1}(h)\| \leqq |a|^{-k(n-1)}\|\Delta_{n-1}(a^kh)\|+\varepsilon|a|^{1-n}(1-|a|)^{-1}|h|^n.$$

Hence, making $k\to\infty$, and observing that $f_{n-1}(x)=D_{n-1}(a,\ldots,a)=0$ (Lemma 1), we see that

$$\|\Delta_{n-1}(h)\| \leqq \varepsilon|a|^{1-n}(1-|a|)^{-1}|h|^n, \quad \text{i.e., } \|\Delta_{n-1}(h)\| = o(|h|^n).$$

From this we similarly deduce

$$\|\Delta_{n-2}(h)\| = o(|h|^n), \qquad \|\Delta_{n-3}(h)\| = o(|h|^n),\ldots,$$

and finally $\|\Delta_1(h)\|=o(|h|^n)$, i.e., that all n Peano derivatives exist and are equal to zero.

Proof of Theorem 1. Lemma 1 shows that if $f_n(x)$ exists, so does $D_n(\boldsymbol{a})$ for each $\boldsymbol{a}\in S^{n-1}$. Conversely, suppose that $D_n(\boldsymbol{a})$ exists for each $\boldsymbol{a}\in S^{n-1}$. If $n=1$, the definitions of $D_1(\boldsymbol{a})$ and $f_1(x)$ coincide, so trivially $f_1(x)$ exists. We proceed by induction. Since $D_n(\boldsymbol{a})$ exists for all $\boldsymbol{a}$ in S^{n-1}, by Lemma 3, $D_{n-1}(\boldsymbol{a})$ exists for all $\boldsymbol{a}$ in S^{n-2}. By the induction hypothesis, $f_{n-1}(x)$ exists. Since $D_n(\boldsymbol{a})$ exists for every $\boldsymbol{a}\in S^{n-1}$, in particular, it exists for an $\boldsymbol{a}$ of the form $(a,\ldots,a)$. Finally, by Lemma 4, from the existence of $D_n(a,\ldots,a)$ and $f_{n-1}(x)$, we conclude the existence of $f_n(x)$, completing the characterization.

2. Let $d_2f(x)$ be the unique number, if it exists, with the property that

$$\lim_{h\to 0}\left(\sup_{a\in T}|\lambda_2(a)\Delta_2(a;h)h^{-2}-d_2f(x)|\right)=0$$

where T is a set of the form $[-A,0)\cup(0,1)\cup(1,A]$, $A>1$. To see that $d_2f(x)$ is indeed a reasonable second derivative, we prove the following theorem.

THEOREM 2. *If $f''(x)$ exists, then $d_2f(x)$ exists and is equal to $f''(x)$. If $d_2f(x)$ exists, then $f_2(x)$ exists and is equal to $d_2f(x)$.*

Conversely, there is a set E of positive Lebesgue measure and functions $u(x)$, $v(x)$ such that

(i) *$u_2(x)$ exists for all x in E, but $d_2u(x)$ exists for no x in E, and*

(ii) *$d_2v(x)$ exists for all x in E, but $v''(x)$ exists for no x in E.*

Proof. Let $f''(x)$ exist. Since it is easy to calculate that $\lambda_2(a)\Delta_2(a;h)=h^2p''(x)$ when p is a quadratic polynomial, we may assume that $x=0$ and $f(0)=f'(0)=f''(0)=0$. Let h be so small that $f'(t)$ exists in the neighborhood $[-A|h|, A|h|]$ of $x=0$. Let $g(x)=f(x)/x$ if $x\neq 0$, $g(0)=0$. By the mean value theorem,

$$\frac{\lambda_2(a)\Delta_2(a;h)}{h^2}=2\frac{f(ah)/ah-f(h)/h}{ah-h}=2g'(x)\Big|_{x=h+\theta(ah-h)}$$

where $0<\theta<1$. Noting that $g'(0)=0$, we have

$$|\lambda_2(a)\Delta_2(a;h)h^{-2}|\leqq 2\sup_{x\in[-A|h|,A|h|]}|f'(x)x^{-1}|+|f(x)x^{-2}|\to 0\quad\text{as } h\to 0.$$

The above proof was shown to me by Professor Antoni Zygmund.

The second part of the theorem is an immediate consequence of the characterization of $f_2(x)$ given in §1, since T contains both an interval and a negative number.

Remove from $[0, 1]$ its open middle half—the interval $(\frac{1}{4},\frac{3}{4})$. Remove from both of the remaining closed intervals their open middle quarters. The four remaining closed intervals have total measure $(1-\frac{1}{2})(1-\frac{1}{4})$. After repeating the process infinitely often, we are left with a closed "fat Cantor set" C of measure

$$c=\prod_{k=1}^{\infty}(1-2^{-k})>0.$$

Let E be the set of all points of C which are points of density of C. $|E|=|C|=c>0$.

(i) Define u, a function on $[0, 1]$, by $u(t)=0$ if $t \in C$; and if t belongs to one of the disjoint open intervals that makes up the complement of C in $[0, 1]$, then set $u(t)=l^3$ where l is the length of that interval. Let x be any point of E. Given any $\delta>0$ we may find d and l such that $\delta>d+l>d>0$ and such that the interval $I=(x+d, x+d+l)$ is one of the intervals of the complement of C. In particular, the set S of endpoints of complementary intervals is dense in E. Setting $h=d$ and picking $a>1$ very close to 1 makes

$$\lambda_2(a)\Delta_2(a; h)h^{-2} = 2l^3a^{-1}(a-1)^{-1}h^{-2} = O((a-1)^{-1})$$

arbitrarily large, whereas setting $h=d+l$, picking $a<1$ very close to 1, and using the same estimate makes $\lambda_2(a)\Delta_2(a; h)h^{-2}$ arbitrarily negative so that

$$\lim_{h\to 0}\left(\sup_a \lambda_2(a)\Delta_2(a; h)h^{-2}\right) = +\infty, \qquad \lim_{h\to 0}\left(\inf_a \lambda_2(a)\Delta_2(a; h)h^{-2}\right) = -\infty.$$

In particular, $d_2u(x)$ does not exist.

Since x is a point of density of C, we may find $\varepsilon>0$ such that

$$|C \cap [x-\eta/2, x+\eta/2]| > \tfrac{3}{4}\eta$$

whenever $0<\eta\leqq\varepsilon$. If $0<k\leqq\varepsilon/2$ and if an interval I of distance d from x and of length l meets $[x-k/2, x+k/2]$, then $d>l$ because $d\leqq l$ would imply

$$|C \cap [x-2d, x+2d]| \leqq 3d.$$

Hence, if $|h|<k/2$, $|u(x+h)|\leqq l^3\leqq d^3\leqq |h|^3=o(h^2)$ so that $u_2(x)$ exists and is zero.

(ii) Set $v(t)=0$ if $t \in C$. If $I=(a, a+l)$ is an interval contained in the complement of C, and if $0<t<l$, set $w(a+t)=t^2(l-t)^2$.

$$K = \{a+l2^{-k}, a+l-l2^{-k}\}, \qquad k = 2, 3, \ldots,$$

is a sequence of points of I which has a and $a+l$ as limit points. Set $v(t)=w(t)$ whenever $t \in K$ and make $v(t)$ linear and continuous on each closed subinterval of I whose endpoints are consecutive elements of K. This defines v on all of $[0, 1]$. Again let x be any point of E. Since the set S of endpoints of complementary intervals is dense in C, and since points of an appropriate K may be found arbitrarily close to any point of S, it follows that the union of all the K's is dense in C and, hence, also in E. v' does not exist at any point of any K and so every neighborhood of x contains points of nondifferentiability of v. Hence, $v''(x)$ does not exist.

However, $d_2v(x)$ does exist; in fact, $d_2v(x)=0$. To see this set $g(t)=v(x+t)t^{-1}$ if $t\neq 0$, $g(0)=0$. Let $t \in (-\delta, 0) \cup (0, \delta)$ where $\delta<\varepsilon/4$, ε having been chosen as in part (i) above. If $x+t \in I$, then $I=(x+d, x+d+l)$ or $I=(x-d-l, x-d)$ where $d>l$. Then $x+t=x\pm(d+s)$ where $0<s<l<d$. Hence, $w(x+t)=w(x\pm(d+s))$ $=s^2(l-s)^2$. The right-hand derivative of g at t has modulus less than

$$\left|\frac{dw}{ds}(s_0)t^{-1}\right|+|w(s_1)t^{-2}| \leqq \frac{\sqrt{3}}{9}l^3l^{-1}+\frac{1}{16}l^4l^{-2} \leqq l^2 \leqq \delta^2$$

where $0<s_0<l$ and $0<s_1<l$ are produced by the mean value and the intermediate value theorems respectively. If $x+t\in C$, then

$$\left|\frac{v(x+t+s)/(t+s)-v(x+t)/t}{s}\right| = \left|\frac{v(x+t+s)}{s(t+s)}\right| = O\left(\frac{s^2(l-s)^2}{s}\right) = O(s) = o(1)$$

as $s\to 0$, so that $g'(t)=0$. Finally,

$$g(h)h^{-1} = v(x+h)h^{-2} = O(l^4h^{-2}) = O(h^2) = o(1)$$

as $h\to 0$ implies $g'(0)=0$.

Since g is continuous with a right-hand derivative that exists throughout $(-\delta,\delta)$ and tends to 0 uniformly there as $\delta\to 0$, g is Lipschitz on $(-\delta,\delta)$ with Lipschitz constant tending to 0 as $\delta\to 0$ (see (v) on p. 355 of [7]); it follows that

$$|\lambda_2(a)\Delta_2(a;h)h^{-2}| = 2\left|\frac{g(ah)-g(h)}{ah-h}\right| \to 0$$

as $h\to 0$, i.e., that $d_2v(x)=0$.

3. **Remarks.** The inclusion of a negative number in the set S of the characterization is certainly necessary, since for the function $f(x)=|x|$, $D_2(a)=0$ at $x=0$ for every positive a; but $f'(0)$, and hence also $f_2(0)$, does not exist. If we drop the demand for a negative number, we obtain the following result. The condition $D_n(\boldsymbol{a})$ exists for all $\boldsymbol{a}\in[\alpha,\beta]^{n-1}$ where $0\leqq\alpha<\beta$ characterizes functions having at x both one-sided nth Peano derivatives existing and equal; i.e., for $t>0$,

$$f(x+t) = f(x)+\sum_{i=1}^{n-1} f_i^+(x)(i!)^{-1}t^i+f_n(x)(n!)^{-1}t^n+o(t^n)$$

and

$$f(x-t) = f(x)+\sum_{i=1}^{n-1} f_i^-(x)(i!)^{-1}(-t)^i+f_n(x)(n!)^{-1}(-t)^n+o(t^n)$$

where $f_i^+(x)$ does not necessarily equal $f_i^-(x)$, $i=1,2,\ldots,n-1$. The proof is similar to that of Theorem 1.

The example $g(x)=x$, x rational, $g(x)=0$, x irrational, has at the point 0 for any rational r, $\Delta_2(r;h)=0$, for all h, so that

$$\lim_{h\to 0}\left(\sup_{r\text{ rational}}|\lambda_2(r)\Delta_2(r;h)h^{-2}|\right) = 0,$$

although $g'(0)$ does not exist. This gives some indication why the sets S and T of §§1 and 2 have to be fairly "thick".

Although A. Denjoy, in [3], has given another characterization of $f_2(x)$ in terms of the existence of a double limit([4]), our condition given in §1 (with $n=2$, $p=\infty$) is

([4]) Denjoy's condition is that the double limit of

$$(2/h+k)\{(f(x+h)-f(x))/h-(f(x)-f(x-k))/k\}$$

must exist as h and k tend independently to 0 through positive values.

easier to verify in practice. Denjoy's condition remains of interest as a property enjoyed by functions possessing two Peano derivatives. An easy consequence of Denjoy's characterization and the characterization of §1 is: If, in the definition of $d_2f(x)$, the set T is replaced by a set of the form $[-\varepsilon, 0)$, the resultant condition, with $d_2f(x)$ replaced by $f_2(x)$, is necessary and sufficient for the existence of the second Peano derivative.

We close by listing some questions which may be worth consideration.

(a) How can Theorem 2 be generalized to higher order derivatives and/or to L^p, $1 \leq p < \infty$?

(b) Can the sets S and/or T be replaced by "thinner" ones; for example, by demanding only that $|S| > 0$ and S contain a negative number?

(c) Is it necessary to demand that the function be measurable in the characterization? But keep in mind that Lemma 2 is not true if g is not measurable. An example is given in [4] and another example is due to Professor Lee Rubel of the University of Illinois.

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