STRUCTURE SPACES OF SEMIGROUPS OF CONTINUOUS FUNCTIONS

BY

K. D. MAGILL, JR.

Abstract. In a previous paper, we associated a topological space with each left ideal of a semigroup. Here, we determine this space when the semigroup under consideration is the semigroup of all continuous selfmaps of any space belonging to a fairly extensive class of topological spaces and the left ideal is taken to be the kernel of the semigroup.

1. Introduction and statement of main theorem. In [3], we associated with each left ideal Z of a semigroup T , a structure space which we denoted by $\mathcal{U}(T, Z)$. This space is formed as follows: a nonempty subset A of $T \times Z$ is a *bond* if for any finite subset $\{(t_i, z_i)\}_{i=1}^N \subseteq A$, the system of equations $\{t_i x = z_i\}_{i=1}^N$ has a common solution x in Z . An *ultrabond* is a bond which is not properly contained in any other bond. $\mathcal{U}(T, Z)$ is then defined to be the family of all such ultrabonds. $\mathcal{U}(T, Z) \neq \emptyset$ since the existence of a bond is immediate and by Zorn's Lemma, every bond is contained in ultrabond. We topologize $\mathcal{U}(T, Z)$ in the following manner: for each $(t, z) \in T \times Z$, let

$$H(t, z) = \{A \in \mathcal{U}(T, Z) : (t, z) \in A\}$$

and take $\{H(t, z) : (t, z) \in T \times Z\}$ to be a subbasis for the closed subsets of $\mathcal{U}(T, Z)$. For some general facts about such spaces, one should consult [3, pp. 319–324]. In the event Z is the kernel (minimal two-sided ideal) of T_1 , we refer to $\mathcal{U}(T, Z)$ as the *$\mathcal{K}$ -structure space of $\mathcal{T}$* and denote it simply by $\mathcal{U}(T)$.

Before we can state the Main Theorem, we need to recall some facts about E -compact spaces which were introduced by Engleking and Mrówka in [1]. We use the terminology and notation adopted in [4] which gives a rather extensive account of the theory. Let E be any Hausdorff space. A space X is E -completely regular if it is homeomorphic to a subset of some cartesian product of copies of E and it is E -compact if it is homeomorphic to a closed subset of such a product. We will refer to an E -compact space Y which contains X as a dense subspace as an E -compactification of X . An important fact about E -compactifications is that if E is compact in the usual sense, then each E -completely regular space X has a

Received by the editors October 14, 1969.

AMS Subject Classifications. Primary 5453, 2092.

Key Words and Phrases. Structure spaces, extensions of topological spaces, compactifications, semigroups of continuous functions.

Copyright © 1970, American Mathematical Society

largest E -compactification $\beta_E X$ in the sense that any other E -compactification of X is a continuous image of $\beta_E X$ under a map which keeps the points of X fixed. This is an immediate consequence of Theorem 4.14 of [4, p. 177]. Before stating our Main Theorem, we introduce one additional concept.

DEFINITION. Let E be Hausdorff. A space X is E -separated if it is Hausdorff and for each pair H and K of disjoint nonempty closed subsets of X , there exists a continuous function f from X into E and two distinct points p and q of E such that $f(x)=p$ for $x \in H$ and $f(x)=q$ for $x \in K$.

Evidently, a space X is normal if and only if it is $\mathcal{I}$ -separated where $\mathcal{I}$ denotes the closed unit interval. Furthermore, it follows rather quickly from Theorem 2.1 [4, p. 165] that every E -separated space is also E -completely regular. We are now in a position to state our main result which concerns the $\mathcal{K}$ -structure space of the semigroup $S(X)$ of all continuous selfmaps of X where the binary operation is ordinary composition.

MAIN THEOREM. *Suppose that E is compact and that X is E -separated and contains a copy of E . Then the $\mathcal{K}$ -structure space of $S(X)$ is, in fact, $\beta_E X$ the largest E -compactification of X .*

2. Proof of the main theorem and two corollaries. Since the proof will rely rather heavily in several instances upon Theorem (1.10) of [3, p. 322], we begin by discussing some concepts which are relevant to that result. Once again T is a semigroup and Z is a left ideal of T . If T has a left identity, then for each $v \in Z$, $A_v = \{(t, tv) : t \in T\}$ is an ultrabond [3, Lemma (1.2), p. 320] and the set of all ultrabonds of this form is denoted by $\mathcal{R}(T, Z)$ and is referred to as the realization of Z . This is a subspace of $\mathcal{U}(T, Z)$ and may well be a proper subspace. Now to each element $a \in T$, one can associate in a very natural way an element f_a in $S(\mathcal{R}(T, Z))$ the semigroup, under composition, of all continuous selfmaps of $\mathcal{R}(T, Z)$. The mapping f_a is defined by $f_a(A_v) = A_{av}$ for each $A_v \in \mathcal{R}(T, Z)$. Theorem (1.8) of [3, p. 321] asserts, among other things, that f_a is continuous. Now we are in a position to extract the portion of Theorem (1.10) of [3, p. 322] which we need here. It is as follows:

(1) If the pair (T, Z) is admissible and T has a left identity then $\mathcal{U}(T, Z)$ is a Hausdorff compactification of $\mathcal{R}(T, Z)$ and each f_a in $S(\mathcal{R}(T, Z))$ has a unique extension to a function f_a^E in $S(\mathcal{U}(T, Z))$. The pair (T, Z) is defined to be admissible [3, Definition (1.6), p. 320] if

(2) A is an ultrabond and $A \in \mathcal{C}H(t_1, z_1)$ (the complement of $H(t_1, z_1)$ in $\mathcal{U}(T, Z)$) then there exist (t_2, z_2) and (t_3, z_3) in $T \times Z$ such that

$$A \in \mathcal{C}H(t_2, z_2) \subseteq H(t_3, z_3) \subseteq \mathcal{C}H(t_1, z_1).$$

Now suppose we turn our attention to the semigroup $S(X)$ where X satisfies the conditions stated in the Main Theorem. For any point $p \in X$, we denote by $\langle p \rangle$ the constant function which maps each point of X into p . One easily verifies

that the kernel $K(X)$ of $S(X)$ is precisely the family of all constant function on X . Furthermore, it is not difficult to show that

(3) For any $f \in S(X)$ and $\langle y \rangle, \langle z \rangle$ in $K(X)$, $f \circ \langle y \rangle = \langle z \rangle$ if and only if $f(y) = z$.

(4) A subset A of $S(X) \times K(X)$ is a bond if and only if $\{f^{-1}(z) : (f, \langle z \rangle) \in A\}$ has the finite intersection property.

(5) A is an ultrabond if and only if $(f, \langle z \rangle) \notin A$ implies $f^{-1}(z) \cap g_1^{-1}(y_1) \cap \dots \cap g_N^{-1}(y_N) = \emptyset$ for some finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of A .

We next want to observe that the pair $(S(X), K(X))$ is admissible. A space Y is defined in [3, Definition (2.5), p. 327] to be a strong S^* -space if it is Hausdorff and for each pair of disjoint closed subsets H and K of Y there exist distinct points p and q of Y and a continuous selfmap f of Y such that $f(x) = p$ for $x \in K$. Since X is E -separated and contains a copy of E , it follows readily that X is a strong S^* -space. Consequently, by Theorem (2.7) [3, p. 328] the pair $(S(X), K(X))$ is admissible and (1) now applies. We will use (1) in proving that

(6) $\mathcal{U}(S(X))$ is E -compact.

By (1), $\mathcal{U}(S(X))$ is a Hausdorff space which is compact in the usual sense. Thus, if an embedding into a cartesian product of copies of E exists, $\mathcal{U}(S(X))$ must necessarily be embedded as a closed subset. Consequently, we need only prove the existence of an embedding. According to Theorem (2.1) of [4, p. 165], it will be sufficient to show that for each closed subset W of $\mathcal{U}(S(X))$ and each $A \in \mathcal{U}(S(X))$ with $A \notin W$, there exists a continuous function f from $\mathcal{U}(S(X))$ into E and a point $q \in E$ such that $f(B) = q$ for $B \in W$ and $f(A) \neq q$.

Since $\{H(g, \langle y \rangle) : g \in S(X), y \in X\}$ is a subbasis for the closed subsets of $\mathcal{U}(S(X))$, there exists a finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^N$ of $S(X) \times K(X)$ such that

(7) $A \notin W^*, \quad W \subseteq W^*$

where $W^* = H(g_1, \langle y_1 \rangle) \cup \dots \cup H(g_N, \langle y_N \rangle)$. Since $A \notin W^*$, there exist by (5), finite subfamilies $\{(h_i, \langle v_i \rangle)\}_{i=1}^{N_i}$ of A with the property that $g_i^{-1}(y_i) \cap V_i = \emptyset$ where

$$V_i = h_{i_1}^{-1}(v_{i_1}) \cap \dots \cap h_{i_{N_i}}^{-1}(v_{i_{N_i}})$$

for $1 \leq i \leq N$. Now let $V^* = \bigcap \{V_i\}_{i=1}^N$ and let $H = \bigcup \{g_i^{-1}(y_i)\}_{i=1}^N$. Then $H \cap V^* = \emptyset$ and since X is E -separated and contains a copy E^* of E , there exists a continuous function f mapping X into E^* and two distinct points p and q of E^* such that

(8) $f(x) = p \quad \text{for } x \in V^* \quad \text{and} \quad f(x) = q \quad \text{for } x \in H.$

As we observed in the discussion preceding (1), the mapping f_f defined by $f_f(A_{\langle x \rangle}) = A_{f \circ \langle x \rangle}$ is a continuous selfmap of $\mathcal{R}(S(X), K(X))$ which, by (1) has a unique extension to a continuous selfmap $\tilde{f}_f^E$ of $\mathcal{U}(S(X))$. Hereafter, we will denote the space $\mathcal{R}(S(X), K(X))$ more simply by $\mathcal{R}(S(X))$. Since X is an S^* -space (in fact, a strong S^* -space), the canonical map e which takes $x \in X$ into $A_{\langle x \rangle}$ in $\mathcal{R}(S(X))$

is a homeomorphism from X onto $\mathcal{R}(S(X))$ [3, Theorem (2.3), p. 325]. We assert that

$$(9) \quad \mathfrak{f}_f^E \text{ maps } \mathcal{U}(S(X)) \text{ into } e[E^*],$$

and

$$(10) \quad \mathfrak{f}_f^E(A) = e(p) \quad \text{and} \quad \mathfrak{f}_f^E(B) = e(q) \quad \text{for } B \in W^*.$$

We recall first of all that f maps all of X into E^* . Hence, for any $A_{\langle x \rangle} \in \mathcal{R}(S(X))$,

$$\mathfrak{f}_f^E(A_{\langle x \rangle}) = A_{\langle f(x) \rangle} = e(f(x)) \in e[E^*].$$

By (1), $\mathcal{R}(S(X))$ is dense in $\mathcal{U}(S(X))$ and since E^* is compact, we get

$$\mathfrak{f}_f^E[\mathcal{U}(S(X))] = \mathfrak{f}_f^E[\text{cl } \mathcal{R}(S(X))] \subseteq \text{cl } \mathfrak{f}_f^E[\mathcal{R}(S(X))] \subseteq \text{cl } e[E^*] = e[E^*]$$

where cl denotes closure. This verifies (9). Now we want to show that

$$(11) \quad A \in \text{cl } \{A_{\langle x \rangle} : x \in V^*\}.$$

Let $\mathcal{C}[H(k_1, \langle r_1 \rangle) \cup \dots \cup H(k_m, \langle r_m \rangle)]$ be any basic open subset of $\mathcal{U}(S(X))$ which contains A . Then by (5), there exist finite subfamilies $\{(t_i, \langle a_i \rangle)\}_{i=1}^{M_i}$ of A such that $k_i^{-1}(r_i) \cap U_i = \emptyset$ where

$$U_i = t_{i_1}^{-1}(a_{i_1}) \cap \dots \cap t_{i_{M_i}}^{-1}(a_{i_{M_i}}).$$

By (4), there exists a point x in $V^* \cap U_1 \cap \dots \cap U_M$. Thus, $x \notin k_i^{-1}(r_i)$, $i=1, 2, \dots, M$ from which it follows that $(k_i, \langle r_i \rangle) \notin A_{\langle x \rangle}$, $i=1, 2, \dots, M$. Therefore,

$$A_{\langle x \rangle} \in \mathcal{C}[H(k_1, \langle r_1 \rangle) \cup \dots \cup H(k_M, \langle r_M \rangle)]$$

and this proves (11). Now for any $x \in V^*$, $f(x)=p$ and we have

$$\mathfrak{f}_f^E(A_{\langle x \rangle}) = A_{\langle f(x) \rangle} = e(f(x)) = e(p).$$

This fact, together with (11) implies that $\mathfrak{f}_f^E(A)=e(p)$ which is the first half of (10). In much the same way that we verified (11), one can show that if $B \in W^*$, then $B \in \text{cl } \{A_{\langle x \rangle} : x \in H\}$ and since $f(x)=q$ for $x \in H$, it follows that $\mathfrak{f}_f^E(A_{\langle x \rangle})=e(q)$ for each x in H . Therefore, $\mathfrak{f}_f^E(B)=e(q)$ for each $B \in W^*$ and this completes the proof of statement (10). In view of the discussion immediately following statement (6), it is a consequence of (9) and (10) that (6) is valid, that is, $\mathcal{U}(S(X))$ is E -compact.

Now we are in a position to show that $\mathcal{U}(S(X))$ is $\beta_E X$. Actually, we show that $\mathcal{U}(S(X))$ is $\beta_E \mathcal{R}(S(X))$ but since the canonical map e maps X homeomorphically onto $\mathcal{R}(S(X))$ we identify the two spaces. In order to conclude that $\mathcal{U}(S(X))$ is $\beta_E \mathcal{R}(S(X))$ it is sufficient, according to Theorem 4.14 of [4, p. 177] to show that $\mathcal{U}(S(X))$ is E -compact and also that every continuous function from $\mathcal{R}(S(X))$ into E can be continuously extended to a function which maps $\mathcal{U}(S(X))$ into E . We have yet to verify the latter and for this, it will be sufficient to show that any continuous

function f from $\mathcal{R}(S(X))$ into $e[E^*]$ has a continuous extension to a function which maps $\mathcal{U}(S(X))$ into $e[E^*]$. To get this extension we note that $g = e^{-1} \circ f \circ e$ belongs to $S(X)$ and hence f_g^E is a continuous selfmap of $\mathcal{U}(S(X))$ by (1). For any $A_{\langle x \rangle} \in \mathcal{R}(S(X))$, we have

$$f_g^E(A_{\langle x \rangle}) = A_{g \circ \langle x \rangle} = A_{\langle g(x) \rangle} = e(g(x)) = f(e(x)) = f(A_{\langle x \rangle}).$$

Thus f_g^E is indeed an extension of f which (since $e[E^*]$ is compact and $\mathcal{R}(S(X))$ is dense in $\mathcal{U}(S(X))$) maps $\mathcal{U}(S(X))$ into $e[E^*]$. This completes the proof of the Main Theorem.

If, in the Main Theorem, we take E to be the closed unit interval, we immediately get the following result which first appeared in [3, p. 329] as Corollary (2.8).

COROLLARY 1. *Suppose X is normal, Hausdorff and contains an arc. Then the $\mathcal{K}$ -structure space of $S(X)$ is the Stone-Čech compactification of X .*

A partition of a space X is any finite collection of mutually disjoint subsets of X which are both closed and open and whose union is all of X . A 0-dimensional space here will mean a space whose Lebesgue dimension is zero, that is, one with the property that every open cover has a refinement by a partition of the space.

COROLLARY 2. *Let X be a normal 0-dimensional Hausdorff space. Then the $\mathcal{K}$ -structure space of $S(X)$ is the Stone-Čech compactification of X .*

Proof. Here again we apply the Main Theorem and in this case we take E to be the two-point discrete space $\mathcal{D}$. The conclusion is immediate if X has only one point so we assume that X has more than one point and, consequently contains a copy of $\mathcal{D}$. To show that X is $\mathcal{D}$ -separated, let H and K be two disjoint closed subsets of X . Then $\{\mathcal{C}H, \mathcal{C}K\}$ is a cover of X and hence has a refinement by a partition $\{V_i\}_{i=1}^N$ of X .

Let $W = \bigcup \{V_i : V_i \subset \mathcal{C}H\}$. Then W is a subset of X which is both closed and open. Furthermore, $H \subset \mathcal{C}W$ and since $K \subset \mathcal{C}H$ and $\{V_i\}_{i=1}^N$ is a refinement of $\{\mathcal{C}H, \mathcal{C}K\}$ it readily follows that $K \subset W$. Therefore, if p and q denote the two points of $\mathcal{D}$, the function which maps all of W into p and $\mathcal{C}W$ into q is continuous and we conclude that X is $\mathcal{D}$ -separated. Then by the Main Theorem, the $\mathcal{D}$ -structure space of $S(X)$ is $\beta_{\mathcal{D}}X$, the largest $\mathcal{D}$ -compactification of X . Now it is well known that the Stone-Čech compactification βX of X is the largest among all the compactifications of X . So, in order to conclude that $\beta_{\mathcal{D}}X = \beta X$, it is sufficient to observe that βX is a $\mathcal{D}$ -compactification of X . In [2, p. 243], a modified definition of Lebesgue dimension is used. However, the definition there agrees with the usual one for normal spaces. Consequently, it follows from Theorem 16.11 of [2, p. 245] that βX is 0-dimensional. Then βX is also 0-dimensional in the sense of [4], that is, it has a basis of sets which are both open and closed. But this implies that βX is $\mathcal{D}$ -compact [4, p. 176] and hence that βX is a $\mathcal{D}$ -compactification of X .

REFERENCES

1. R. Engleking and S. Mrówka, *On E-compact spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 492–436. MR **20** #3522.
2. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.
3. K. D. Magill, Jr., *Topological spaces determined by left ideals of semigroups*, Pacific J. Math. **24** (1968), 319–330. MR **36** #7125.
4. S. Mrówka, *Further results on E-compact spaces. I*, Acta Math. **120** (1968), 161–185. MR **37** #2165.

STATE UNIVERSITY OF NEW YORK AT BUFFALO,
BUFFALO, NEW YORK 14226