A CHARACTERIZATION OF INTEGRAL CURRENTS⁽¹⁾

BY
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1. Introduction. In [6] J. Král defined for each $r > 0$ a function v_r^C which for $z \in \mathbb{R}^n$ and $r = \infty$ gives a measure of the generalized solid angle under which the boundary A of a bounded Borel set C is visible from z . For smooth A , $v_r^C(z)$ is the integral over the space of lines λ passing through z of the number of points in $A \cap \lambda \cap \{x : 0 < |x - z| < r\}$. This function is used to give geometric conditions on an open set C which ensure the existence of solutions to the Neumann boundary value problem in C . In particular, Král shows that C has finite perimeter $P(C)$ (a definition is in [6, 2.8]) if and only if $v_\infty^C(z) < \infty$ for $z \notin A$.

On the other hand, v_r^C is related to the formulas of integral geometry. We show in 4.5 that there exists an absolute constant γ such that if $P(C) < \infty$, then

$$\int_{\mathbb{R}^n} v_r^C dH^n = \gamma r P(C).$$

In the present paper we seek to extend the results of Král relating the geometry of the boundary of C with properties of v_∞^C to analogous propositions concerning boundaries of $k+1$ dimensional objects in $\mathbb{R}^n$, $0 < k < n$. For smooth boundaries A the appropriate analogue of $v_\infty^C(z)$ is the integral $V^A(z)$ over the space of l planes B containing z , $k+l \geq n$, of the $k+l-n$ dimensional Hausdorff measure of $A \cap B \sim \{z\}$. We use the slicing theory developed by H. Federer in [3] to extend V^A to the case where A is a k dimensional flat chain T . (If S is the n current obtained by integration over C , then ∂S is flat; $P(C)$ equals the mass $M(\partial S)$ of ∂S , and $V^{\partial S} = v_\infty^C$. Rectifiable currents are flat.)

In §5 we derive for V^T generalizations of the properties of v_∞^C obtained by Král in [6, §§1,2]. We also obtain an absolute continuity result for T which extends [5, 8.5].

In §6 we show that $M(\partial S) < \infty$ if $V^{\partial S}(z) < \infty$ for sufficiently many points $z \in \mathbb{R}^n$. Combining this with results in §5 we obtain the extension of Král's characterization of sets of finite perimeter:

$M(\partial S) < \infty$ if and only if $V^{\partial S}(z) < \infty$ for $z \notin \text{spt } \partial S$. From this we obtain our characterization of integral currents by recalling from [5, 8.14] that if S is rectifiable and $M(\partial S) < \infty$, then S is an integral current.

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Now suppose S is rectifiable and $\partial S = T + U$ with $M(T) < \infty$. In [3, 2.2] Federer proved that if the support of U has Hausdorff measure zero in dimension k , then S is an integral current, thus extending [5, 8.14]. We further extend this in §6 by showing that S is an integral current if for sufficiently many $z \in \mathbb{R}^n$,

$$(\text{spt } U) \cap B \sim \{z\} = \emptyset$$

for almost all l planes B containing z .

Finally, we show in §7 that if T is obtained by integration over an oriented k dimensional submanifold T of class 1 and $z \in T$, then $V^T(z) < \infty$ for $k+l > n$, and also for $k+l=n$ if the tangent spaces of T satisfy a Hölder condition. (For $k=n-1$ such surfaces are the Lyapunov surfaces of classical potential theory.)

2. Preliminaries. The purpose of this section is to fix basic notation and terminology. We readopt the notation and terminology of [1] and [5].

Fix $n > 1$, and let $k < n$ and $l < n$ be positive integers such that $k+l \geq n$. Denote

$$B = \mathbb{R}^n \cap \{x : x^1 = \cdots = x^{n-l} = 0\} \sim \{0\}$$

and $\psi_B = dx^{n-l+1} \wedge \cdots \wedge dx^n$, and orient B so that $B^\rightarrow = e_{n-l+1} \wedge \cdots \wedge e_n$.

The map

$$f: \text{SO}(n) \times B \rightarrow \mathbb{R}^n \times \text{SO}(n), \quad f(g, b) = (g(b), g),$$

embeds $\text{SO}(n) \times B$ as a proper submanifold Φ of $\mathbb{R}^n \times \text{SO}(n)$ of dimension $m+l$, where $m = \frac{1}{2}n(n-1)$ is the dimension of $\text{SO}(n)$. Defining

$$p: \mathbb{R}^n \times \text{SO}(n) \rightarrow \mathbb{R}^n, \quad p(x, g) = x,$$

$$q: \mathbb{R}^n \times \text{SO}(n) \rightarrow \text{SO}(n), \quad q(x, g) = g,$$

we observe that $q^{-1}\{g\} \cap \Phi = g(B) \times \{g\}$ and, for $0 \neq x \in \mathbb{R}^n$,

$$F_x = p^{-1}\{x\} \cap \Phi = \{(x, g) : g^{-1}(x) \in B\}$$

is a compact submanifold of Φ of dimension $m+l-n$ which is orientable since Φ is orientable.

Fix $o \in B \cap S^{n-1} = B \cap \{x : |x| = 1\}$. Define

$$\pi: \text{SO}(n) \rightarrow S^{n-1}, \quad \pi(g) = g(o).$$

Set $I = \text{SO}(n) \cap \{g : g(o) = o\}$; I has dimension $\mu = \frac{1}{2}(n-1)(n-2)$. Assign a bi-invariant metric to $\text{SO}(n)$ so that $\pi_\#(g)$ is an orthogonal projection for $g \in \text{SO}(n)$. Orient $\text{SO}(n)$ and denote by ω the unit positively oriented m form on $\text{SO}(n)$. $e \in \text{SO}(n)$ is the identity map of $\mathbb{R}^n$.

Denote

$$F = \text{SO}(n) \cap \{g : g^{-1}(o) \in B\} = q(F_o) = \pi^{-1}(B)^{-1},$$

and observe that if $0 \neq x = |x|g_x(o)$, then

$$F_x = \{x\} \times g_x(F);$$

in particular, F_x is connected for $l > 1$. Furthermore,

$$\begin{aligned} H^{m+l-n}(F_x) &= H^{m+l-n}(F) = H^\mu(I)H^{l-1}(S^{l-1}) \\ &= \left[\prod_{i=2}^{n-1} i\alpha(i) \right] l\alpha(l). \end{aligned}$$

Whenever $z \in \mathbf{R}^n$ define $r_z(y) = |z - y|$ for each $y \in \mathbf{R}^n$; set $r_0 = r$. We also define

$$\rho_z: \mathbf{R}^n \sim \{z\} \rightarrow S^{n-1}, \quad \rho_z(x) = (x - z)/|x - z|;$$

set $\rho_0 = \rho$.

$\mathbf{R}^n \times \text{SO}(n)$ acts on $\mathbf{R}^n$ as the group of orientation preserving isometries of $\mathbf{R}^n$ through association of $(z, g) \in \mathbf{R}^n \times \text{SO}(n)$ with the isometry, also denoted by (z, g) , which maps x to $z + g(x)$. If $T \in E_k(\mathbf{R}^n)$, we denote

$$(z, g)_\# T = z + g_\# T.$$

If $v \in \bigwedge_k(\mathbf{R}^n)$, $w \in \bigwedge_l(\mathbf{R}^n)$, $*v \wedge *w \neq 0$, $\omega \in \bigwedge^k(\mathbf{R}^n)$, $\zeta \in \bigwedge^l(\mathbf{R}^n)$, define

$$v \cap w = (-1)^t \|*v \wedge *w\|^{-1} *(*v \wedge *w) \in \bigwedge_{k+l-n}(\mathbf{R}^n)$$

and

$$\omega \cap \zeta = (-1)^t *(*\omega \wedge *\zeta), \quad t = (k+l)(n+1).$$

Recalling the integralgeometric constants $\gamma(n, k, l)$ and $\gamma^2(n, k, l)$ defined in [1, 2.13], we also define

$$\delta(n, k, l) = \gamma^2(n-1, k, l-1)(k+l-n)l\alpha(l)(2k+l-n)^{-1}.$$

Let X be an open subset of $\mathbf{R}^n$. $F_k(X)$ is the closure of $N_k(X)$ in $E_k(X)$ with respect to the flat seminorm F [4, 4.1.12], which is complete relative to F . Elements of $F_k(X)$ are called k dimensional *flat chains*. Rectifiable currents are clearly flat; consequently, since the restriction of H^n to a bounded, open subset of $\mathbf{R}^n$ corresponds to a rectifiable n current, the restriction of H^n to a bounded Borel subset of $\mathbf{R}^n$ corresponds to a flat n current. Finally, we let

$$F_k^{\text{loc}}(X)$$

be the subset of $D_k(X)$ consisting of all T such that

$$T \wedge \gamma \in F_k(X) \quad \text{for every } \gamma \in D^0(U).$$

Suppose $z \in \mathbf{R}^n$ and consider $T \in E_k(\mathbf{R}^n)$. We shall denote by $T_z \in D_k(\mathbf{R}^n \sim \{z\})$ the restriction of T to $D^k(\mathbf{R}^n \sim \{z\})$. We recall from [4, 4.1.21] that if $T \in F_k(\mathbf{R}^n)$ and $M(T) < \infty$, then $M(T_z) = \|T\|(\mathbf{R}^n \sim \{z\}) = M(T)$.

3. Lifts of currents. Let I act to the left on F by means of left translation. Since I is connected, this action will preserve an orientation of F . Since $F = \pi^{-1}(B)^{-1}$,

$$T_e(F) = \pi_\#(e)^{-1}[T_o(B \cap S^{n-1})];$$

thus, if $0 \neq w \in \bigwedge_{m+l-n} [T_e(F)]$, and $v \in \bigwedge_{n-l} [T_e(\text{SO}(n))]$ is such that $\pi_{\#}(e)(v) = e_1 \wedge \cdots \wedge e_{n-l}$, then $v \wedge w \neq 0$. Choose w so that $\langle v \wedge w, \omega(e) \rangle > 0$, and for $l > 1$ orient F so that w is positively oriented. If $l=1$, then F has two components. Let $\sigma \in \text{SO}(n)$ be such that $\sigma(e_j) = e_j$ for $j < n-1$, and $\sigma(e_j) = -e_j$ for $j = n-1, n$. Right translation by σ leaves F invariant and permutes the components of F ; orient F so that w is positively oriented, and so that right translation by σ reverses orientation.

3.1. LEMMA. Φ is the bundle space of a fibre bundle $\mathcal{B}$ with fibre F , structure group I , projection $p| \Phi$ and base space $\mathbb{R}^n \sim \{0\}$.

Proof. Let U be an open subset of $\mathbb{R}^n \sim \{0\}$ such that on $U_1 = \{x/|x| : x \in U\}$ there is a cross-section $\sigma_U: U_1 \rightarrow \text{SO}(n)$ for π . Define a coordinate function $\varphi_U: U \times F \rightarrow \Phi$ by

$$\varphi_U(x, g) = (x, \sigma_U(x/|x|)g);$$

it is easy to verify that the set of φ_U provides Φ with the desired bundle structure.

3.2. REMARKS. One proceeds as in [1, §3] to define the *lifting map*

$$L_{\mathcal{B}}: E_*(\mathbb{R}^n \sim \{0\}) \cap \{T : M(T) < \infty\} \rightarrow E_*(\Phi).$$

Inasmuch as $H^{m+l-n}(F) \neq 1$, suitable modifications must be made in the assertions of [1, §3] concerning the properties of $L_{\mathcal{B}}$.

Furthermore, [1, 3.5] can be used to show that if U is an open subset of $\mathbb{R}^n \sim \{0\}$ having compact closure, then there exists c such that $F[L_{\mathcal{B}}(T)] \leq cF(T)$ whenever $T \in E_*(U)$ and $M(T) < \infty$. One uses this and the F completeness of $F_*(\Phi)$ to extend $L_{\mathcal{B}}$ to an F continuous linear map of $F_*(\mathbb{R}^n \sim \{0\})$ into $F_*(\Phi)$; it is clear that [1, 3.3] remains valid. Moreover, since ∂ is F continuous, it follows from [1, 3.5(4)] that $\partial \circ L_{\mathcal{B}} = L_{\mathcal{B}} \circ \partial$.

Whenever $\gamma \in \text{SO}(n)$ we define

$$\gamma_{\Phi}: \Phi \rightarrow \mathbb{R}^n \times \text{SO}(n), \quad \gamma_{\Phi}(x, g) = (\gamma(x), \gamma g).$$

γ_{Φ} is an isometry of Φ onto Φ , and it is easy to verify that γ_{Φ} induces a bundle map of $\mathcal{B}$ onto $\mathcal{B}$, hence we conclude from [1, 3.3 and 3.5(2)] that if $T \in E_k(\mathbb{R}^n \sim \{0\})$ and $M(T) < \infty$, then

$$\gamma_{\Phi\#}[L_{\mathcal{B}}(T)] = L_{\mathcal{B}}[\gamma_{\#}(T)], \quad \gamma_{\Phi\#}[L_{\mathcal{B}}(T)^{\rightarrow}] = L_{\mathcal{B}}[\gamma_{\#}(T)^{\rightarrow}].$$

In case $l=1$ we define

$$\Sigma: \Phi \rightarrow \mathbb{R}^n \times \text{SO}(n), \quad \Sigma(x, g) = (x, g\sigma).$$

Σ is an isometry of Φ onto Φ . Suppose $T \in E_k(\mathbb{R}^n \sim \{0\})$, $M(T) < \infty$, and $\text{spt } T$ lies in a coordinate neighborhood U with associated coordinate function $\varphi_U: U \times F \rightarrow \Phi$ defined as in the proof of 3.1. Observing that $\Sigma \circ \varphi_U = \varphi_U \circ (e \times \sigma)$

and $L_{\mathcal{A}}(T) = \varphi_U(T \times F)$ by [1, 3.3], we use [1, 3.3] in conjunction with a suitable partition of unity to conclude that for T with arbitrary support,

$$\Sigma_{\#}[L_{\mathcal{A}}(T)] = -L_{\mathcal{A}}(T), \quad \Sigma_{\#}[L_{\mathcal{A}}(T)]^{\rightarrow} = -L_{\mathcal{A}}(T)^{\rightarrow}.$$

3.3 DEFINITION. For $(x, g) \in \Phi$, $\iota(x, g)$ is the linear right inverse of $p|\Phi_{\#}(x, g)$ whose range is orthogonal to $T_{(x, g)}(F_x)$.

Let v be a k vectorfield on $\mathbb{R}^n$. v^v is the function on Φ such that

$$v^v(x, g) = \|\iota(x, g)[v(x)]\|^{-1}.$$

3.4. LEMMA. Suppose $T \in E_k(\mathbb{R}^n \sim \{0\})$, $M(T) < \infty$.

(i) If h is a bounded Baire function on Φ , then

$$\|L_{\mathcal{A}}(T)\|(h\nu^{T^{\rightarrow}}) = \int_{\mathbb{R}^n} \int_{F_x} h(x, g) dH^{m+l-n}(x, g) d\|T\|_x.$$

(ii) If ψ is a bounded Baire $k+l-n$ form on Φ , then

$$\begin{aligned} & (-1)^t [L_{\mathcal{A}}(T) \wedge q^{\#}\omega](\psi) \\ &= \int \langle (*[T^{\rightarrow}(x) \wedge g_{\#}B^{\rightarrow}], 0), \psi(x, g) \rangle \nu^{T^{\rightarrow}}(x, g) |x|^{l-n} d\|L_{\mathcal{A}}(T)\|(x, g), \end{aligned}$$

where $t = (k+l-n)(m+l+1)$.

Proof. Using [1, 3.6(2)] and a suitable partition of unity we see that we can assume $\text{spt } T$ to lie in a coordinate neighborhood U with associated coordinate function $\varphi_U: U \times F \rightarrow \Phi$ defined as in the proof of 3.1. From [1, 3.5(5)] we infer that, for $\|L_{\mathcal{A}}(T)\|$ almost all $(x, g) \in \Phi$,

$$(1) \quad L_{\mathcal{A}}(T)^{\rightarrow}(x, g) = \nu^{T^{\rightarrow}}(x, g) \iota(x, g) [T^{\rightarrow}(x)] \wedge F_x^{\rightarrow}(x, g).$$

(i) Denoting the right member of our conclusion by $\lambda(h)$, we see that $\lambda = \varphi_{U\#}(\|T\| \times \|F\|)$. On the other hand, (1) implies that

$$L_{\mathcal{A}}(T)^{\rightarrow}(x, g) = [\nu^{T^{\rightarrow}} \varphi_{U\#}(T^{\rightarrow}, 0) \wedge F_x^{\rightarrow}](x, g)$$

for $\|L_{\mathcal{A}}(T)\|$ almost all $(x, g) \in \Phi$, hence by [1, 3.3 and 3.1],

$$\nu^{T^{\rightarrow}} L_{\mathcal{A}}(T) = \nu^{T^{\rightarrow}} \varphi_{U\#}(T \times F) = [\nu^{T^{\rightarrow}} \varphi_{U\#}(T^{\rightarrow}, 0) \wedge F_x^{\rightarrow}] \lambda = L_{\mathcal{A}}(T)^{\rightarrow} \lambda.$$

(ii) We must show that for $\|L_{\mathcal{A}}(T)\|$ almost all $(x, g) \in \Phi$,

$$\begin{aligned} (2) \quad & (-1)^t \langle L_{\mathcal{A}}(T)^{\rightarrow}, q^{\#}\omega \wedge \psi \rangle(x, g) \\ &= \langle (*[T^{\rightarrow}(x) \wedge g_{\#}B^{\rightarrow}], 0), \psi(x, g) \rangle \nu^{T^{\rightarrow}}(x, g) |x|^{l-n}. \end{aligned}$$

From [1, 3.5] it follows that for $\|L_{\mathcal{A}}(T)\|$ almost all $(x, g) \in \Phi$,

$$(3) \quad \|L_{\mathcal{A}}(T)^{\rightarrow}(x, g)\| = \|T^{\rightarrow}(x)\| = 1,$$

hence we can use the maps γ_{Φ} (and Σ for $l=1$) which were introduced in 3.2 to infer that we need only verify (2) for (x, g) such that $x = |x|o$, $g_{\#}(B) = B$, and (1) and (3) hold.

We compute $i = i(x, g)$ as follows: Choose the cross-section σ_U used in defining φ_U so that

$$\sigma_{U\#}(o) = \text{transpose } \pi_{\#}(e).$$

Define I on $\text{SO}(n)$ by $I(\gamma) = \gamma^{-1}$. Since $F_x = \{x\} \times F = \{x\} \times \pi^{-1}(B)^{-1}$, we have

$$\begin{aligned} T_{(x,g)}(F_x) &= \{0\} \oplus I_{\#}(g^{-1})[T_{g^{-1}}[\pi^{-1}(B)]] \\ &= \{0\} \oplus I_{\#}(g^{-1}) \circ L_{g^{-1}\#}(e)[T_e[\pi^{-1}(B)]] \\ &= \{0\} \oplus R_{g\#}(e)[T_e[\pi^{-1}(B)]]. \end{aligned}$$

On the other hand, it is easy to verify that for $1 \leq j \leq n-l$,

$$v_j = (e_j, |x|^{-1} R_{g\#}(e) \sigma_{U\#}(o)(e_j)) = \varphi_{U\#}(x, g)(e_j, 0);$$

v_j is clearly orthogonal to $T_{(x,g)}(F_x)$, hence

$$i(e_j) = v_j, \quad 1 \leq j \leq n-l.$$

Finally, since $T_{(x,g)}(q^{-1}\{g\} \cap \Phi) = (B \cup \{0\}) \oplus \{0\}$,

$$i(e_j) = (e_j, 0) \quad \text{for } n-l+1 \leq j \leq n.$$

We next observe that

$$\alpha = T^{\rightarrow}(x) - (-1)^{k(n+1)} *B^{\rightarrow} \wedge [*T^{\rightarrow}(x) \wedge *B^{\rightarrow}]$$

has the property $*\alpha \wedge *B^{\rightarrow} = 0$, hence use (1) to conclude

$$\begin{aligned} &\langle L_{\mathcal{A}}(T)^{\rightarrow}(x, g), q^{\#}\omega \wedge \psi(x, g) \rangle \\ &= \nu^{T^{\rightarrow}}(x, g) \langle [(-1)^{k(n+1)} i(*B^{\rightarrow}) \wedge (*[T^{\rightarrow}(x) \wedge *B^{\rightarrow}]), 0] + i(\alpha) \wedge F_x^{\rightarrow}(x, g), \\ &\quad q^{\#}\omega \wedge \psi(x, g) \rangle \\ &= (-1)^{k(n+1) + (k+l-n)(m+l-n)} \nu^{T^{\rightarrow}}(x, g) \langle i(*B^{\rightarrow}) \wedge F_x^{\rightarrow}(x, g), q^{\#}\omega(x, g) \rangle \\ &\quad \times \langle (*[T^{\rightarrow}(x) \wedge *B^{\rightarrow}], 0), \psi(x, g) \rangle. \end{aligned}$$

Finally,

$$\begin{aligned} &\langle i(*B^{\rightarrow}) \wedge F_x^{\rightarrow}(x, g), q^{\#}\omega(x, g) \rangle \\ &= \langle q_{\#}(x, g)[i(*B^{\rightarrow})] \wedge F^{\rightarrow}(g), \omega(g) \rangle \\ &= (-1)^{l(n+1)} |x|^{l-n} \langle R_{g\#}(e) \sigma_{U\#}(o)(e_1 \wedge \cdots \wedge e_{n-l}) \wedge F^{\rightarrow}(g), \omega(g) \rangle \\ &= (-1)^{l(n+1)} |x|^{l-n} \langle \sigma_{U\#}(o)(e_1 \wedge \cdots \wedge e_{n-l}) \wedge F^{\rightarrow}(e), \omega(e) \rangle \\ &= (-1)^{l(n+1)} |x|^{l-n}, \end{aligned}$$

whence follows (2).

3.5. DEFINITIONS. Orient $\mathbb{R}^n \times \text{SO}(n)$ so that $p^{\#}(dx^1 \wedge \cdots \wedge dx^n) \wedge q^{\#}\omega$ is positively oriented.

Let $\Omega = (0, \text{SO}(n)^{\rightarrow})$.

Let $P: \text{SO}(n) \times B \rightarrow B$ be the projection. Referring to the basis of $T_{(x,g)}(\Phi)$ constructed in the proof of 3.4 for $x = |x|o$, $g_\# B = B$, one infers that the intersection of $T_{(x,g)}(\Phi)$ with the orthogonal complement $V_{(x,g)}$ of $T_{(x,g)}[g(B) \times \{g\}]$ in $\mathbb{R}^n \oplus \{0\}$ is $\{0\}$. It follows from 3.2 that this remains true for arbitrary $(x, g) \in \Phi$. There therefore exists

$$\Psi \in E^l[(\mathbb{R}^n \sim \{0\}) \times \text{SO}(n)]$$

such that

$$f^\#(\Psi) = P^\#(\psi_B)$$

and, for $(x, g) \in \Phi$,

$$\langle v \wedge w, \Psi(x, g) \rangle = 0$$

whenever $v \in V_{(x,g)}$ and $w \in \bigwedge_{l-1}(T_{(x,g)}[\mathbb{R}^n \times \text{SO}(n)])$.

3.6. LEMMA. Let φ be a k form on $\mathbb{R}^n$ and $(x, g) \in \Phi$. If

$$v \in \bigwedge_{k+l-n} (T_{(x,g)}[g(B) \times \{g\}]),$$

then

$$\begin{aligned} (-1)^{m(n-l)} \langle v, p^\#[*(*\varphi \wedge g^{-1\#}\psi_B)](x, g) \rangle \\ = \langle v \wedge \Omega(x, g), *[(p^\#\varphi) \wedge *(\Psi \wedge q^\#\omega)](x, g) \rangle. \end{aligned}$$

Proof. One uses [1, 4.2(2)] to verify that the left member of our assertion is equal to

$$(-1)^{m(n-l)} \langle v \wedge \Omega(x, g), *p^\#(*\varphi \wedge g^{-1\#}\psi_B)(x, g) \rangle.$$

In order to complete the proof we need only compute

$$s = \langle w, (p^\#g^{-1\#}\psi_B)(x, g) \rangle$$

for $w \in \bigwedge_{n-l} [T_{(x,g)}(\mathbb{R}^n \times \{g\})]$. Denote by i the transpose of $p_\#(x, g)$ and by w_0 the orthogonal projection of $i[p_\#(x, g)(w)]$ on $T_{(x,g)}[g(B) \times \{g\}]$. Then, identifying $g(B) \times \{g\}$ with $g(B)$ and setting $y = g^{-1}(x)$, we have

$$f_\#(g, y)(0, g_\#^{-1}(w_0)) = w_0$$

and by [1, 4.2(1)],

$$\begin{aligned} (-1)^{l(n-l)} s &= \langle g_\#^{-1} p_\#(x, g)(w), \psi_B \rangle \\ &= \langle g_\#^{-1}(w_0), \psi_B \rangle \\ &= \langle P_\#(g, y) f_\#(g, y)^{-1}(w_0), \psi_B \rangle \\ &= \langle w_0, \Psi(x, g) \rangle \\ &= \langle i[*p_\#(x, g)(w)], \Psi(x, g) \rangle \\ &= \langle i[*p_\#(x, g)(w)] \wedge \Omega(x, g), (\Psi \wedge q^\#\omega)(x, g) \rangle \\ &= \langle *w, (\Psi \wedge q^\#\omega)(x, g) \rangle \\ &= (-1)^{(l+m)(n-l)} \langle w, *(\Psi \wedge q^\#\omega)(x, g) \rangle. \end{aligned}$$

4. **Plane intersections of currents.** If $S \in F_j(\Phi)$ and $j \geq m$, then for H^m almost all $g \in \text{SO}(n)$,

$$\langle S, q|\Phi, g \rangle \in F_{j-m}(\Phi)$$

is the slice of S by $q|\Phi$ over g as characterized in [4, §4.3].

4.1. DEFINITION. Suppose $T \in F_k(\mathbb{R}^n \sim \{z\})$. Whenever

$$\langle L_{\mathcal{A}}(-z+T), q|\Phi, g \rangle \in F_{k+l-n}(\Phi)$$

we define

$$T \cap (z+g_{\#}B) = (-1)^t [z+p_{\#}\langle L_{\mathcal{A}}(-z+T), q|\Phi, g \rangle],$$

where $t = (k+l-n)(m+l+1) + (k+l)(n+1)$.

4.2. THEOREM. (1) If $T \cap (z+g_{\#}B)$ exists, then

$$\text{spt}[T \cap (z+g_{\#}B)] \subset (\text{spt } T) \cap (z+gB).$$

(2) If U is an open subset of $\mathbb{R}^n \sim \{z\}$ having compact closure in $\mathbb{R}^n \sim \{z\}$, then there exists $c < \infty$ such that whenever $T \in F_k(U)$,

$$\int_{\text{SO}(n)} F[T \cap (z+g_{\#}B)] dH^m g \leq cF(T).$$

(3) If $M(T) < \infty$, then for H^m almost all $g \in \text{SO}(n)$ the following are true:

(i) $M[T \cap (z+g_{\#}B)] < \infty$.

(ii) For $\|T \cap (z+g_{\#}B)\|$ almost all $x \in \mathbb{R}^n$,

$$[T \cap (z+g_{\#}B)]^+(x) = T^-(x) \cap g_{\#}B^+.$$

(iii) If T is an oriented proper submanifold of class 1, then $T \cap (z+gB) = A \cup Z$ where A is an orientable, proper $k+l-n$ submanifold of class 1 and $H^{k+l-n}(Z) = 0$; if A is oriented according to (ii), then $A = T \cap (z+g_{\#}B)$.

(4) Whenever $\varphi \in E^l(\mathbb{R}^n)$, $i \leq k+l-n$, and $T \cap (z+g_{\#}B)$ exists, then so does

$$(T \wedge \varphi) \cap (z+g_{\#}B) = [T \cap (z+g_{\#}B)] \wedge \varphi.$$

(5) If $k+l > n$ and $T \cap (z+g_{\#}B)$ exists, then

$$\partial[T \cap (z+g_{\#}B)] = (-1)^{n-l} \partial T \cap (z+g_{\#}B).$$

Furthermore, if T is normal, then $T \cap (z+g_{\#}B)$ is normal for H^m almost all $g \in \text{SO}(n)$.

Proof. (1) follows from [1, 3.5(3)] and [4, 4.3.1].

(2) follows from 3.2, [4, 4.3.1 and 4.3.2(5)].

(3)(i) follows from [1, 3.5(2)] and [4, 4.3.2(2)].

(3)(ii) follows from 3.4(ii) and [4, 4.3.2(1) and (2)].

(3)(iii) follows from the coarea formula [4, 3.2.12], [1, 3.5(6)], [4, 4.3.8] and (ii).

(4) follows from [4, 4.3.1] and the second statement of [1, 3.6(2)].

(5) follows from 3.2, [4, 4.3.1] and (3)(i).

4.3. REMARK. 4.2(2) and (3) and [4, 4.1.23] show that the intersections $T \cap (z + g_{\#}B)$ are intrinsically determined by the action of $\text{SO}(n)$ on $\mathbf{R}^n$. In particular, if $w, z \in \mathbf{R}^n$ and $h \in \text{SO}(n)$, then

$$(w + h_{\#})[T \cap (z + g_{\#}B)] = (w + h_{\#}T) \cap [w + h(z) + hg_{\#}B]$$

for H^m almost all $g \in \text{SO}(n)$.

That our intersections are usually the same as those defined in [1, §4] will follow in §5.

Suppose $z \in \mathbf{R}^n$ and consider $T \in F_k(\mathbf{R}^n)$. Let $\mathcal{U}$ be a partition of unity subordinate to a locally finite open cover of $\mathbf{R}^n \sim \{z\}$. Then for H^m almost all $g \in \text{SO}(n)$,

$$(T \wedge u) \cap (z + g_{\#}B) \in F_{k+l-n}(\mathbf{R}^n)$$

for each $u \in \mathcal{U}$; for such g we define

$$T \cap (z + g_{\#}B) = \sum_{u \in \mathcal{U}} (T \wedge u) \cap (z + g_{\#}B) \in F_{k+l-n}^{\text{loc}}(\mathbf{R}^n \sim \{z\}).$$

One uses 4.2(4) to verify that this definition does not depend on the choice of $\mathcal{U}$, and that if $z \notin \text{spt } T$, then the definition reduces to the one in 4.1. With appropriate modifications the assertions of 4.2 hold for T .

4.4. DEFINITION. If $T \in F_k(\mathbf{R}^n)$ and $z \in \mathbf{R}^n$, then

$$V^T(z) = H^u(I)^{-1} \int_{\text{SO}(n)} M[T \cap (z + g_{\#}B)] dH^m g.$$

If $k = n - 1$, $l = 1$ and S corresponds to the restriction of H^n to a bounded Borel set C , then $\frac{1}{2}V^{\partial S}$ coincides with the function v_{∞}^C introduced in [6].

4.5. REMARK. If T is a rectifiable k current, then it follows from [5, 8.16], 4.2(3)(iii) and [1, 5.8 and 5.9] that for $r > 0$ and $T_{z,r} = T \cap \{x : |x - z| < r\}$,

$$\int_{\mathbf{R}^n} V^{T_{z,r}}(z) dH^n z = \gamma(n, k, l) \alpha(l) r^l M(T).$$

If S is the n current corresponding to the restriction of H^n to a bounded open set C , and if $M(\partial S) = P(C) < \infty$, then S is rectifiable, hence ∂S is rectifiable by [5, 8.14]. If $l = 1$, then $\frac{1}{2}V^{(\partial S)_{z,r}}(z)$ is equal to $v_r^C(z)$ as defined in [6], whence follows the integralgeometric formula mentioned in §1.

5. Integralgeometric formulas. Whenever $0 \neq x \in \mathbf{R}^n$ define

$$I_x = \text{SO}(n) \cap \{g : g(x) = x\}$$

and choose g_x so that $x = |x|g_x(o)$. Since

$$I_x = g_x I g_x^{-1}, \quad H^u(I_x) = H^u(I) = H^{n-1}(S^{n-1})^{-1} H^u[\text{SO}(n)].$$

5.1. LEMMA. Suppose $T \in E_k(\mathbb{R}^n)$, $M(T) < \infty$, φ is a bounded Baire k form with support in $\mathbb{R}^n \sim \{0\}$, and h is the bounded Baire function on Φ such that

$$h(x, g) = \langle *T^\rightarrow(x) \wedge g_{\#} * B^\rightarrow, * \varphi(x) \wedge g^{-1\#} \psi_B \rangle.$$

Then

$$\begin{aligned} H^\mu(I)^{-1} \int_{\mathbb{R}^n} \int_{I_x g_x} h(x, g) dH^\mu g d\|T\|x \\ = \gamma^2(n-1, k, l-1)T(\varphi) \quad \text{for } \varphi = f_0 \rho^\# \varphi_0, \\ = \gamma^2(n-1, k-1, l-1)T(\varphi) \quad \text{for } \varphi = (f_1 \rho^\# \varphi_1) \wedge dr, k+l > n. \end{aligned}$$

Proof. Suppose $\varphi = f_0 \rho^\# \varphi_0$ and $\|T^\rightarrow(x)\| = 1$. Observing that we can assume $T^\rightarrow(x)$ to lie in the orthogonal complement X of $\mathbb{R}x$, we apply [1, 6.2, 9.3 and 2.13] to obtain

$$H^\mu(I)^{-1} \int_{I_x} h(x, gg_x) dH^\mu g = \gamma^2(n-1, k, l-1) \langle T^\rightarrow(x), \varphi(x) \rangle.$$

Now suppose $\varphi = (f_1 \rho^\# \varphi_1) \wedge dr$ and $\|T^\rightarrow(x)\| = 1$. Observing that we can assume $T^\rightarrow(x) = u \wedge x$, where u lies in X , we again apply [1, 6.2] in X to obtain our conclusion.

5.2. LEMMA. Suppose h is a bounded Baire function on Φ such that if $g_0 \in \text{SO}(n)$ and $g_0 \# B = B$, then $h(x, gg_0) = h(x, g)$ for $(x, g) \in \Phi$. Then

$$l\alpha(l) \int_{I_x g_x} h(x, g) dH^\mu g = \int_{F_x} h(x, g) dH^{m+l-n}(x, g).$$

Proof. First observe that

$$\begin{aligned} H^\mu(I) \int_{F_x} h(x, \gamma) dH^{m+l-n}(x, \gamma) &= \int_{I_x} \int_{F_x} h(x, g\gamma) dH^{m+l-n}(x, \gamma) dH^\mu g \\ &= \int_{F_x} \int_{I_x} h(x, g\gamma) dH^\mu g dH^{m+l-n}(x, \gamma). \end{aligned}$$

Fix $(x, \gamma) \in F_x$ and denote $\gamma^{-1}(x) = y \in B$. Choose g_0 so that $g_0(y) = |y|o$, $g_0 \# B = B$, and set $g'_x = \gamma g_0^{-1}$. Then for some $g_1 \in I_x$ we have $g_x = g_1^{-1} g'_x$ and

$$\begin{aligned} \int_{I_x} h(x, g\gamma) dH^\mu g &= \int_{I_x} h(x, gg'_x g_0) dH^\mu g \\ &= \int_{I_x} h(x, gg'_x) dH^\mu g = \int_{I_x g_x} h(x, g) dH^\mu g. \end{aligned}$$

5.3. THEOREM. If $T \in F_k(\mathbb{R}^n)$ and $\varphi \in E_k(\mathbb{R}^n \sim \{z\})$, then

$$\begin{aligned} H^\mu(I)^{-1} \int_{\text{SO}(n)} T \cap (z + g_{\#} B)(r_z^{n-l} \varphi \cap g^{-1\#} \psi_B) dH^m g \\ = l\alpha(l) \gamma^2(n-1, k, l-1)T(\varphi) \quad \text{for } \varphi = f_0 \rho_z^\# \varphi_0 \\ = l\alpha(l) \gamma^2(n-1, k-1, l-1)T(\varphi) \quad \text{for } \varphi = (f_1 \rho_z^\# \varphi_1) \wedge dr_z, k+l > n. \end{aligned}$$

Proof. With the assistance of 4.2(2) we see that we can assume that $z=0$, $0 \notin \text{spt } T$, and $M(T) < \infty$. Referring to 3.5 we let Ξ be the $k+l-n$ form on $\mathbf{R}^n \times \text{SO}(n)$ such that whenever

$$v \in \bigwedge_{k+l-n} [T_{(x,g)}(\mathbf{R}^n \times \text{SO}(n))],$$

$$(-1)^{m(n-l)} \langle v, \Xi(x, g) \rangle = \langle v \wedge \Omega(x, g), *[(p^\# * \varphi) \wedge *(\Psi \wedge q^\# \omega)](x, g) \rangle.$$

Suppose $\varphi = f_0 \rho^\# \varphi_0$. We apply 5.1, 5.2, 3.4(i), 3.6, 3.4(ii), [4, 4.3.2(1)], 3.6 and [1, 2.5] to verify that

$$\begin{aligned} H^{m+l-n}(F) \gamma^2(n-1, k, l-1) T(\varphi) &= \int_{\Phi} \langle (*[T^{\rightarrow}(x) \wedge g_\# * B^-], 0), \Xi(x, g) \rangle \nu^{T^{\rightarrow}}(x, g) d\|L_{\mathcal{A}}(T)\|(x, g) \\ &= (-1)^l L_{\mathcal{A}}(T) \wedge q^\# \omega(r^{n-l} \circ p\Xi) \\ &= (-1)^l \int_{\text{SO}(n)} \langle L_{\mathcal{A}}(T), q, g \rangle (r^{n-l} \circ p\Xi) dH^m g \\ &= (-1)^{(k+l)(n+1)} \int_{\text{SO}(n)} T \cap g_\# B[r^{n-l} * (*\varphi \wedge g^{-1\#} * \psi_B)] dH^m g \\ &= \int_{\text{SO}(n)} T \cap g_\# B(r^{n-l} \varphi \cap g^{-1\#} \psi_B) dH^m g. \end{aligned}$$

The proof for $\varphi = (f_1 \rho^\# \varphi_1) \wedge dr$ is analogous.

5.4. LEMMA. (1) If $z \in \mathbf{R}^n$, then

$$\Sigma_z = \rho_z^\# [E^k(S^{n-1})] \cup \rho_z^\# [E^{k-1}(S^{n-1})] \wedge dr_z$$

spans $E^k(\mathbf{R}^n \sim \{z\})$ over $E^0(\mathbf{R}^n \sim \{z\})$.

(2) If $z_1, \dots, z_{k+1} \in \mathbf{R}^n$ determine a k -plane Π , then

$$\Sigma = \bigcup_{i=1}^{k+1} \rho_{z_i}^\# [E^k(S^{n-1})]$$

spans $E^k(\mathbf{R}^n \sim \Pi)$ over $E^0(\mathbf{R}^n \sim \Pi)$.

Proof. (1) It is clear that for each $z \neq x \in \mathbf{R}^n$, $\Sigma_z(x)$ spans $\bigwedge^k(\mathbf{R}^n)$. Thus choose $\psi_i \in \Sigma_z$, $i=1, \dots, v=\binom{n}{k}$, such that $\psi_1(x), \dots, \psi_v(x)$ is a basis of $\bigwedge^k(\mathbf{R}^n)$. Then $\psi_1(y), \dots, \psi_v(y)$ are linearly independent for y in some neighborhood N of x , hence $\psi_1|N, \dots, \psi_v|N$ span $E^k(N)$ over $E^0(N)$, and our conclusion follows with use of a suitable partition of unity.

(2) Fix $x \in \mathbf{R}^n \sim \Pi$. Then $\bigwedge^k(\mathbf{R}^n)$ is spanned by $\Sigma(x)$. In fact, if this were not true there would exist $0 \neq w \in \bigwedge_k(\mathbf{R}^n)$ such that for $i=1, \dots, k+1$,

$$\langle w, \rho_{z_i}^\# [E^k(S^{n-1})](x) \rangle = \{0\},$$

or equivalently, $\rho_{z_i}^\#(x)(w) = 0$. But for each i ,

$$\dim \ker \left[\rho_{z_i}^\#(x) \mid \bigwedge_k(\mathbf{R}^n) \right] = \binom{n-1}{k-1},$$

hence we would have $w = (x - z_i) \wedge w_i$, and the linear independence of the $(x - z_i)$, $i = 1, \dots, k + 1$, would allow us to conclude that

$$w' = \alpha(x - z_1) \wedge \dots \wedge (x - z_k) \notin \ker \rho_{z_{k+1}}^\#(x).$$

Proceeding as in (1) we conclude that for some neighborhood N of x , $\Sigma|N$ spans $E^k(N)$ over $E^0(N)$, and our conclusion follows with use of a suitable partition of unity.

5.5. DEFINITION. If $z \in \mathbb{R}^n$, then $D^k(z)$ is the linear subspace of $D^k(\mathbb{R}^n \sim \{z\})$ consisting of sums of the form $\sum_{j=1}^N f_j \rho_z^\# \varphi_j$ with $f_j \in D^0(\mathbb{R}^n \sim \{z\}) = D^0(z)$ and $\varphi_j \in D^k(S^{n-1})$.

5.6. LEMMA. If $T \in F_k(\mathbb{R}^n)$ and $M(T) < \infty$, then

$$H^\mu(I)V^T(0) = l\alpha(I) \int_{\mathbb{R}^n \sim \{0\}} \int_{I_x g_x} \|*T^\rightarrow(x) \wedge g_\# *B^\rightarrow\| |x|^{l-n} dH^\mu g d\|T\|x.$$

Proof. Choosing $f_i \in D^0(0)$, $i = 1, 2, \dots$, such that $0 \leq f_1 \leq f_2 \leq \dots$ and

$$\lim_{i \rightarrow \infty} f_i(y) = 1 \quad \text{for } y \neq 0,$$

we apply 4.4, 4.1, [4, 4.3.2(2)], 3.4(ii), 3.4(i) and 5.2 to obtain

$$\begin{aligned} H^\mu(I)V^{T \wedge f_i}(0) &= M[L_{\mathcal{B}}(T \wedge f_i) \wedge q^\# \omega] \\ &= \int_{\Phi} \|*T^\rightarrow(x) \wedge g_\# *B^\rightarrow\| |x|^{l-n} d\|L_{\mathcal{B}}(T \wedge f_i)\|(x, g) \\ &= l\alpha(I) \int_{\mathbb{R}^n} \int_{I_x g_x} \|*T^\rightarrow(x) \wedge g_\# *B^\rightarrow\| dH^\mu g |x|^{l-n} f_i(x) d\|T\|x. \end{aligned}$$

Consequently, we infer from 4.2(4) that

$$\begin{aligned} l\alpha(I) \int_{\mathbb{R}^n \sim \{0\}} \int_{I_x g_x} \|*T^\rightarrow(x) \wedge g_\# *B^\rightarrow\| dH^\mu g |x|^{l-n} d\|T\|x \\ &= H^\mu(I) \lim_{i \rightarrow \infty} V^{T \wedge f_i}(0) \\ &\geq \int_{\text{SO}(n)} \liminf_{i \rightarrow \infty} M[(T \cap g_\# B) \wedge f_i] dH^m g \\ &\geq \int_{\text{SO}(n)} M(T \cap g_\# B) dH^m g \\ &= H^\mu(I)V^T(0), \end{aligned}$$

which implies our assertion for the case $V^T(0) = \infty$. On the other hand, if $V^T(0) < \infty$, then

$$\begin{aligned} H^\mu(I)V^{T \wedge f_i}(0) &= \int_{\text{SO}(n)} M[(T \cap g_\# B) \wedge f_i] dH^m g \\ &= \int_{\text{SO}(n)} \|T \cap g_\# B\|(f_i) dH^m g, \end{aligned}$$

hence

$$\lim_{i \rightarrow \infty} V^{T \wedge f_i}(0) = V^T(0),$$

which implies our assertion for such T .

5.7. LEMMA. Suppose $r+s \leq n$, $v \in \bigwedge_r(\mathbb{R}^n)$, and $w \in \bigwedge_s(\mathbb{R}^n)$.

(1) If $r \neq n/2$ or $s \neq n/2$, then

$$H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\|v\| \|w\| \leq \binom{r+s}{r} \int_{\text{SO}(n)} \|v \wedge g_{\#}w\| dH^m g.$$

If v or w is simple, then this is true for arbitrary r and s , and the factor $\binom{r+s}{r}$ can be omitted.

$$(2) \quad \int_{\text{SO}(n)} \|v \wedge g_{\#}w\| dH^m g \leq H^m[\text{SO}(n)]\gamma(n, n-r, n-s)\|v\| \|w\|.$$

If v and w are simple, then equality holds. If v is not simple and the smallest linear subspace L of $\mathbb{R}^n$ containing v has dimension $\lambda \leq r+s+1$, or if an analogous condition holds for w , then the inequality is strict.

Proof. (1) Consider $\varphi \in \bigwedge^r(\mathbb{R}^n)$ and $\psi \in \bigwedge^s(\mathbb{R}^n)$ such that $\|\varphi\| \leq 1$ and $\|\psi\| \leq 1$. Then by [1, 6.2 and 9.3]

$$H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi \rangle \langle w, \psi \rangle \leq \binom{r+s}{r} \int_{\text{SO}(n)} \|v \wedge g_{\#}w\| dH^m g$$

if $r \neq n/2$ or $s \neq n/2$, whence follows the first statement. If w is simple, let ψ be the metric dual of $w/|w|$. Then $\langle w, \psi \rangle = |w|$ and

$$H^m[\text{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi \rangle |w| \leq \int_{\text{SO}(n)} \|v \wedge g_{\#}w\| dH^m g.$$

(2) If v and w are simple, equality follows from [1, 5.6 and 5.9]. On the other hand, suppose v is not simple and use [5, 2.2] to obtain simple r vectors $\alpha_1, \dots, \alpha_M$ and simple s vectors $\beta_1, \dots, \beta_N$ such that α_1 and α_2 are linearly independent,

$$v = \sum_{i=1}^M \alpha_i, \quad \|v\| = \sum_{i=1}^M |\alpha_i|,$$

and

$$w = \sum_{i=1}^N \beta_i, \quad \|w\| = \sum_{i=1}^N |\beta_i|.$$

Inasmuch as

$$\begin{aligned} \int_{\text{SO}(n)} \|v \wedge g_{\#}w\| dH^m g &\leq \sum_{j=1}^N \sum_{i=1}^M \int_{\text{SO}(n)} |\alpha_i \wedge g_{\#}\beta_j| dH^m g \\ &= H^m[\text{SO}(n)]\gamma(n, n-r, n-s) \sum_{j=1}^N \sum_{i=1}^M |\alpha_i| |\beta_j| \\ &= H^m[\text{SO}(n)]\gamma(n, n-r, n-s)\|v\| \|w\|, \end{aligned}$$

the remainder of (2) will follow if we can find a simple s vector $\beta \neq 0$ such that

$$(*) \quad \|(\alpha_1 + \alpha_2) \wedge \beta\| < |\alpha_1 \wedge \beta| + |\alpha_2 \wedge \beta|.$$

First assume $r+s=n$. Corresponding to $j=1, 2$ let P_j be the subset of

$$W = \bigwedge_s (\mathbf{R}^n) \cap \{\beta : \beta \text{ is simple and } |\beta| = 1\}$$

consisting of β for which $\alpha_j \wedge \beta$ is positively oriented, $Q_j = -P_j$, and $R_j = W \sim (P_j \cup Q_j)$. Then P_j and Q_j are open in W , R_j has no interior, and R_j is the boundary of P_j and of Q_j . Further, $R_1 \neq R_2$. In fact, if A_j is the r dimensional linear subspace of $\mathbf{R}^n$ containing α_j , C_j is the orthogonal complement of $A_1 \cap A_2$ in A_j , and D is the orthogonal complement of $A_1 + A_2$ in $\mathbf{R}^n$, then

$$\dim(D + C_j) = s.$$

If $0 \neq \beta_0 \in \bigwedge_s (D + C_1)$, then

$$\alpha_1 \wedge \beta_0 = 0, \quad \alpha_2 \wedge \beta_0 \neq 0.$$

Therefore, $P_1 \cap Q_2 \neq \emptyset$; any $\beta \in P_1 \cap Q_2$ will satisfy (*).

In case $\lambda \leq r+s$ we apply the result of the last paragraph with $\mathbf{R}^n$ replaced by L to obtain a simple $\beta_0 \in \bigwedge_{\lambda-r}(L)$ such that

$$\|(\alpha_1 + \alpha_2) \wedge \beta_0\| = \|(\alpha_1 + \alpha_2) \wedge \beta_0\| < |\alpha_1 \wedge \beta_0| + |\alpha_2 \wedge \beta_0|.$$

Thus if $\beta_1 \neq 0$ is a simple $r+s-\lambda$ vector lying in the orthogonal complement of L , $\beta = \beta_0 \wedge \beta_1$ will satisfy (*).

In case $\lambda = r+s+1$ we choose β_0 as above, choose a simple s vector β and $\beta_2 \in L$ such that $\beta_0 = \beta \wedge \beta_2$, and conclude that

$$\|(\alpha_1 + \alpha_2) \wedge \beta\| = \|(\alpha_1 + \alpha_2) \wedge \beta\| < |\alpha_1 \wedge \beta| + |\alpha_2 \wedge \beta|.$$

5.8. EXAMPLE. Let $n=6$, $r=3$, $s=1$, and

$$v = \frac{1}{2}[e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6].$$

One uses a procedure similar to the one used on [7, p. 54] to show that

$$\|v\| = \frac{1}{2}|e_1 \wedge e_2 \wedge e_3| + \frac{1}{2}|e_4 \wedge e_5 \wedge e_6| = 1.$$

Let w_1 be a linear combination of e_1, e_2, e_3 , and w_2 be a linear combination of e_4, e_5, e_6 such that $|w_1| = |w_2| = 1$. Then if $w = \alpha w_1 + \beta w_2$, one uses the fact that $\|e_1 \wedge e_2 + e_3 \wedge e_4\| = 2$ to verify that

$$\|v \wedge w\| = \frac{1}{2}(|\alpha| + |\beta|) = \frac{1}{2}|e_1 \wedge e_2 \wedge e_3 \wedge w| + \frac{1}{2}|e_4 \wedge e_5 \wedge e_6 \wedge w|;$$

consequently,

$$\int_{\text{SO}(6)} \|v \wedge g_{\#} w_1\| dH^{15}g = H^{15}[\text{SO}(6)]\gamma(6, 3, 5).$$

5.9. THEOREM. Assume $T \in F_k(\mathbf{R}^n)$ and $z \in \mathbf{R}^n$.

$$(1) \quad \delta(n, k, l)M(T_z \wedge r_z^{l-n}) \leq V^T(z) \leq l\alpha(l)M(T_z \wedge r_z^{l-n}).$$

Consequently,

$$V^T(z) \leq l\alpha(l)M(T_z)[\text{dist}(z, \text{spt } T)]^{l-n},$$

$$\delta(n, k, l)M(T_z) \leq V^T(z) \sup \{|x-z| : x \in \text{spt } T\}^{n-l}.$$

(2) Suppose $r > 0$ and T lies in the image of $E_k(z + rS^{n-1})$ under the map induced by the inclusion of $z + rS^{n-1}$ in $\mathbb{R}^n$. Then

$$l\alpha(l)\gamma^2(n-1, k, l-1)r^{l-n}M(T) \leq V^T(z) \leq l\alpha(l)\gamma(n-1, k, l-1)r^{l-n}M(T).$$

Suppose $M(T) < \infty$. The second inequality becomes equality when T^- is simple. If for x in a set of positive $\|T\|$ measure it is true that $T^-(x)$ is not simple and the smallest linear subspace of $\mathbb{R}^n$ containing $*T^-(x)$ has dimension not greater than $2n-k-l+1$, then the second inequality is strict.

(3) Suppose $l=n-k$ and T is rectifiable with k dimensional density Θ^k . Then

$$V^T(z) = (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_{S^{n-1}} \sum_{\rho_z^{-1}(\theta)} \Theta^k dH^k \theta.$$

(4) Suppose $l=n-k$ and $M(T) < \infty$. Whenever $z \neq x \in \mathbb{R}^n$ and $\|T^-(x)\| = 1$ let $\tau(x)$ be the orthogonal projection of $T^-(x)$ on $T_x(z + |x-z|S^{n-1})$. Define

$$\gamma_z(x) = H^\mu(I)^{-1} \|\tau(x)\|^{-1} \int_{I_{x-z}} |T^-(x) \wedge gg_{x-z} B^+| dH^\mu g$$

whenever $\|T^-(x)\| = 1$. Then

$$\begin{aligned} & (n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1) \sup \{T(r_z^{-k}\varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\} \\ & \leq V^T(z) = (n-k)\alpha(n-k) \sup \{T(\gamma_z r_z^{-k}\varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\} \\ & \leq (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \sup \{T(r_z^{-k}\varphi) : \varphi \in D^k(z), M(\varphi) \leq 1\}. \end{aligned}$$

The second inequality becomes equality if $\rho_{z\#}[T^-(x)]$ is simple for $\|T\|$ almost all $x \in \mathbb{R}^n \sim \{z\}$, in which case

$$\gamma_z = \gamma(n-1, k, n-k-1).$$

(5) If $l=n-k$, $M(T) < \infty$, and $\gamma_z(x)$ is a continuous function of z for $\|T\|$ almost all x , then V^T is lower semicontinuous.

Proof. We can assume $z=0$. Choose $f_i \in D^0(0)$, $i=1, 2, \dots$, such that $0 \leq f_1 \leq f_2 \leq \dots \leq 1$ and whenever K is a compact subset of $\mathbb{R}^n \sim \{0\}$ there exists i for which $f_i|_K = 1$.

If $M(T) < \infty$, then the second inequality of (1) follows from 5.6. In general, we use 4.2(4) to infer that

$$\begin{aligned} V^T(0) & \leq \liminf_{i \rightarrow \infty} V^{T \wedge f_i}(0) \\ & \leq l\alpha(l) \liminf_{i \rightarrow \infty} M(T_z \wedge f_i \wedge r_z^{l-n}) \\ & \leq l\alpha(l)M(T_z \wedge r_z^{l-n}). \end{aligned}$$

Since $\delta(n, k, n-k)=0$, we assume $k+l>n$. Fix $\varphi \in E^k(\mathbf{R}^n \sim \{0\})$ such that $M(\varphi) \leq 1$, and use 5.4 to write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D^k(0)$ and $\varphi_2 \in D^{k-1}(0) \wedge dr$. Noting that $M(\varphi_1) \leq 1$ and $M(\varphi_2) \leq 1$, we obtain from 5.3

$$\begin{aligned} l\alpha(l)H^u(I)T(r^{l-n}\varphi) &= \gamma^2(n-1, k, l-1)^{-1} \int T \cap g_{\#}B(\varphi_1 \cap g^{-1\#}\psi_B) dH^m g \\ &\quad + \gamma^2(n-1, k-1, l-1)^{-1} \int T \cap g_{\#}B(\varphi_2 \cap g^{-1\#}\psi_B) dH^m g \\ &\leq l\alpha(l)\delta(n, k, l)^{-1}H^u(I)V^T(0). \end{aligned}$$

(2) follows from 5.6 together with 5.7 with $\mathbf{R}^n$ replaced by $T_x(rS^{n-1})$.

Suppose $l=n-k$. Consider $0 \neq x \in \mathbf{R}^n$. Inasmuch as

$$T^{-}(x) \wedge g_{\#}B^{\rightarrow} = \tau(x) \wedge g_{\#}B^{\rightarrow}$$

for $g \in I_x g_x$, 5.7 implies that

$$(*) \quad \gamma^2(n-1, k, n-k-1) \leq \gamma_0(x) \leq \gamma(n-1, k, n-k-1),$$

with the second inequality becoming equality if $\tau(x)$ is simple. Moreover,

$$|x|^{-k}\tau(x) = \rho_{\#}(x)[T^{-}(x)],$$

and if T is rectifiable we infer from 5.6 and [5, 8.16] that

$$\begin{aligned} V^T(0) &= (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_{\mathbf{R}^n \sim \{0\}} |x|^{-k} |\tau(x)| d\|T\|x \\ (**) \quad &= (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_A |\rho_{\#}(x)[T^{-}(x)]| \Theta^k(x) dH^k x, \end{aligned}$$

where A is the union of a countable family of proper k dimensional submanifolds of class 1 of $\mathbf{R}^n$ and $T^{-}(x)$ is tangent to one of these submanifolds for $\|T\|$ almost all x . (3) now follows from application of the area formula [4, 3.2.3] to $\rho|A$.

We next observe that

$$\begin{aligned} \sup \{T(r^{-k}\varphi) : \varphi \in D^k(0), M(\varphi) \leq 1\} \\ &= \sup \left\{ \int r^{-k} \langle \tau, \varphi \rangle d\|T\| : \varphi \in D^k(0), M(\varphi) \leq 1 \right\} \\ &= \sup \left\{ \int r^{-k} \langle \tau, \psi \rangle d\|T\| : \psi \in D^k(\mathbf{R}^n \sim \{0\}), M(\psi) \leq 1 \right\} \\ &= \int r^{-k} \|\tau\| d\|T\|, \end{aligned}$$

and similarly,

$$\int r^{-k} \gamma_0 \|\tau\| d\|T\| = \sup \{T(\gamma_0 r^{-k}\varphi) : \varphi \in D^k(0), M(\varphi) \leq 1\}.$$

(4) now follows from 5.6 and (*).

In order to prove (5) it suffices to prove lower semicontinuity at 0. We can assume $V^T(0) > 0$. Fix $0 < \varepsilon < V^T(0)$ and choose $\varphi \in D^k(0)$ with

$$\varepsilon < (n-k)\alpha(n-k)T(\gamma_0 r^{-k}\varphi) \leq V^T(0).$$

Inasmuch as $\varphi_y = (-y, e)^\# \varphi \in D^k(y)$ for $y \in \mathbf{R}^n$,

$$\lim_{y \rightarrow 0} \gamma_y(x) r_y^{-k}(x) \varphi_y(x) = \gamma_0(x) r^{-k}(x) \varphi(x)$$

for $\|T\|$ almost all $x \in \mathbf{R}^n$, and $\|\gamma_y r_y^{-k} \varphi_y\| \leq M(r^{-k}\varphi)$, we can use Lebesgue's dominated convergence theorem to conclude that

$$\begin{aligned} \liminf_{y \rightarrow 0} V^T(y) &\geq \liminf_{y \rightarrow 0} (n-k)\alpha(n-k)T(\gamma_y r_y^{-k} \varphi_y) \\ &= (n-k)\alpha(n-k)T(\gamma_0 r^{-k} \varphi) > \varepsilon, \end{aligned}$$

whence

$$\liminf_{y \rightarrow 0} V^T(y) \geq V^T(0).$$

5.10. REMARK. Suppose S is the restriction of H^n to a bounded Borel set C and φ_0 is the unit $n-1$ form on S^{n-1} such that $d\mathbf{r} \wedge \varphi_0$ is positively oriented. Then for $h \in D^0(z)$, $\partial S(h\rho_z^\# \varphi_0)$ is the double layer potential $W_h(z)$ considered by Král in [6]. In this case 5.3 (with $T = \partial S$) reduces to the formula derived in [6, 2.5]. Furthermore, if $M(\partial S) = P(C) < \infty$, then 5.9(4) reduces to the formula given for v_∞^C in [6, 1.6].

5.11. COROLLARY. Consider $T \in F_k(\mathbf{R}^n)$.

(1) If there exist $z_1, \dots, z_{k+2} \in \mathbf{R}^n$ which do not lie on a k -plane, such that for $i = 1, \dots, k+2$,

$$(\text{spt } T) \cap (z_i + gB) = \emptyset$$

for H^m almost all $g \in \text{SO}(n)$, then

$$T = 0.$$

(2) If $k+l > n$ and there exists $z \in \mathbf{R}^n$ such that for H^m almost all $g \in \text{SO}(n)$,

$$(\text{spt } T) \cap (z + gB)$$

has measure 0 with respect to $k+l-n$ dimensional integralgeometric measure in $z + g(B)$ (in particular, if this set is $\emptyset$), then $T = 0$.

Proof. Consider (1) and assume $l = n - k$. By 4.2(1), $V^T(z_i) = 0$ for $i = 1, \dots, k+2$. Let Π_j denote the k -plane containing $\{z_k : k \neq j\}$, and fix $\varphi \in D^k(\mathbf{R}^n \sim \Pi_j)$. Recalling 5.4(2) and applying 5.3 we conclude that $T(\varphi) = 0$. Inasmuch as

$$\bigcup \{\mathbf{R}^n \sim \Pi_j : j = 1, \dots, k+2\} = \mathbf{R}^n,$$

the proof is complete.

Consider (2). From [4, 4.1.20] it follows that $T \cap (z + g_\# B) = 0$ for H^m almost all $g \in \text{SO}(n)$, hence $V^T(z) = 0$, and therefore $T = 0$ by 5.9(1).

5.12. COROLLARY. Suppose $T \in F_k(\mathbb{R}^n)$, $A \subset \mathbb{R}^n$, and the hypothesis of either 5.10(1) or (2) is satisfied with "spt T " replaced by " $(\text{spt } T) \cap A$ ".

(1) If A is open, then $\text{spt } T \subset \mathbb{R}^n \sim A$.

(2) If A is a Borel set and $M(T) < \infty$, then $\|T\|(A) = 0$.

Proof. One argues as in the proof of [4, 4.1.21], using 5.11 in place of [4, 4.1.20].

5.13. REMARK. Theorem 6.6 in [1] is erroneous. However, the following is true:

Let X , G and Ψ be as in [1, §6]. Consider $S \in N_k(X)$ and $T \in N_l(X)$.

(1) If $k > n/2$ or $l > n/2$, then

$$\binom{2n-k-l}{n-k}^{-1} \gamma^2(n, k, l) M(S)M(T) \leq \int_G M(S \cap g_{\#}T) d\Psi g.$$

If $S^{\rightarrow}$ or $T^{\rightarrow}$ is simple, then this is true for arbitrary k and l , and the factor $\binom{2n-k-l}{n-k}^{-1}$ can be omitted

$$(2) \quad \int_G M(S \cap g_{\#}T) d\Psi g \leq \gamma(n, k, l) M(S)M(T).$$

If $S^{\rightarrow}$ and $T^{\rightarrow}$ are simple, then equality holds. If for x in a set of positive $\|S\|$ measure, $S^{\rightarrow}(x)$ is not simple and the smallest linear subspace of $T_x(X)$ containing $*S^{\rightarrow}(x)$ has dimension not greater than $2n-k-l+1$, or if an analogous condition holds for $T^{\rightarrow}$, then the inequality is strict.

These statements follow from use of 5.7 in conjunction with [1, 5.5].

5.14. REMARK. We will now examine the relationship between the intersections defined in 4.1 and those defined in [1, §4].

First note that the slicing theory in [4, §4.3] can be used to extend the intersection theory in [1, §4] to flat chains having finite mass.

Fix $T \in F_k(\mathbb{R}^n)$, $M(T) < \infty$, let U be a compact neighborhood of $\text{spt } T$, and use [4, 4.1.23] to obtain polyhedral chains $T_1, T_2, \dots$ lying in U such that $M(T_i) \leq 2M(T)$ and $T_i \rightarrow T$ as $i \rightarrow \infty$. Fix $u_1 \in D^0(\mathbb{R}^n \sim U)$, $M(u_1) \leq 1$, $u_2 \in D^0[\text{SO}(n)]$, $M(u_2) \leq 1$, and $\psi \in D^{k+l-n}(\mathbb{R}^n)$. Denoting intersection in the sense of [1] by $\cap_1$, we use [1, 4.4(4) and 4.9] to conclude that whenever $R > \text{diam}(U \cup \text{spt } u_1)$,

$$T_i \cap_1 (z + g_{\#}B) = T_i \cap_1 [z + g_{\#}(B \cap \{x : |x| < R\})]$$

for $H^n \times H^m$ almost all $(z, g) \in (\text{spt } u_1) \times \text{SO}(n)$, hence by [1, 4.4(3)]

$$T \cap_1 (z + g_{\#}B) = T \cap_1 [z + g_{\#}(B \cap \{x : |x| < R\})]$$

for $H^n \times H^m$ almost all $(z, g) \in (\text{spt } u_1) \times \text{SO}(n)$. Therefore, by [1, 4.4(2)]

$$\mathcal{J}_1 = \int_{\mathbb{R}^n \times \text{SO}(n)} T \cap_1 (z + g_{\#}B)(\psi)u_1(z)u_2(g) dH^n \times H^m(z, g) < \infty,$$

and Fubini's theorem applies to give

$$\mathcal{J}_1 = \int_{\mathbb{R}^n} \int_{\text{SO}(n)} T \cap_1 (z + g_{\#}B)(\psi)u_1(z)u_2(g) dH^m g dH^n z.$$

By [1, 4.4(4)] and 4.2(3) we have for H^n almost all $z \in \mathbb{R}^n \sim U$,

$$(-1)^c T_i \cap (z + g_{\#} B) = T_i \cap_1 (z + g_{\#} B)$$

for H^m almost all $g \in \text{SO}(n)$, where $c = n(k+l) + kl$. We use [1, 4.4(3), 4.4(2)] and Fubini's theorem to infer that

$$(-1)^c \mathcal{J}_1 = \lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} \int_{\text{SO}(n)} T_i \cap (z + g_{\#} B) (\psi) u_1(z) u_2(g) dH^m g dH^n z.$$

Now by 5.9(1) each of the inner integrals is not greater in absolute value than

$$l\alpha(l)2M(T)M(\psi) \text{dist}(\text{spt } u_1, U)^{l-n} |u_1(z)| H^m[\text{SO}(n)],$$

hence by 4.2(2) we can apply Lebesgue's dominated convergence theorem to obtain

$$(-1)^c \mathcal{J}_1 = \int_{\mathbb{R}^n} \int_{\text{SO}(n)} T \cap (z + g_{\#} B) (\psi) u_1(z) u_2(g) dH^m g dH^n z.$$

We therefore conclude that for H^n almost all $z \notin \text{spt } T$,

$$(-1)^c T \cap (z + g_{\#} B) = T \cap_1 (z + g_{\#} B)$$

for H^m almost all $g \in \text{SO}(n)$.

Next suppose T lies in the image of $E_k(S^{n-1})$ under the map induced by the inclusion of S^{n-1} in $\mathbb{R}^n$. Orient S^{n-1} so that for $x \in S^{n-1}$, $x \wedge (S^{n-1})^\rightarrow(x)$ is positively oriented in $\mathbb{R}^n$. Denote

$$B' = B \cap S^{n-1};$$

orient B' so that for $x \in B'$, $x \wedge B'^\rightarrow$ is positively oriented in B . Then for H^m almost all $g \in \text{SO}(n)$,

$$(-1)^c T \cap g_{\#} B = -T \cap g_{\#} B'$$

as defined in [1, §4] with $X = S^{n-1}$, $G = \text{SO}(n)$.

In order to show this, we infer from [4, 4.1.23], [1, 4.4(3) and (4)], and 4.2(2) and (3) that it is only necessary to show that for $x \in B'$ and $v \in \bigwedge_k [T_x(S^{n-1})]$,

$$(-1)^c v \cap B^\rightarrow = -v \cap_0 B'^\rightarrow,$$

where $\cap_0$ refers to use of the star operator in $\bigwedge_* [T_x(S^{n-1})]$. One verifies this directly from the definitions of the star operators.

6. Boundaries of flat chains of finite mass.

6.1. LEMMA. Suppose $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $T, U \in E_k(\mathbb{R}^n)$ with

$$\partial S = T + U, \quad M(T) < \infty,$$

and for some $z \in \mathbb{R}^n$,

$$(\text{spt } U) \cap (z + gB) = \emptyset$$

for H^m almost all $g \in \text{SO}(n)$. Then

$$\begin{aligned}\gamma^2(n-1, k, l-1)\partial S(\varphi) &\leq M(T)M(\varphi) \quad \text{for } \varphi \in D^k(z), \\ \gamma^2(n-1, k-1, l-1)\partial S(\varphi) &\leq M(T)M(\varphi) \quad \text{for } \varphi \in D^{k-1}(z) \wedge dr_z, k+l > n.\end{aligned}$$

Proof. Our proof was suggested by the proof of [3, 2.2].

We can assume $z=0 \notin (\text{spt } S) \cup (\text{spt } T)$.

Using local coordinates for the bundle $\mathcal{B}$ in conjunction with a suitable partition of unity for $R^n \sim \{0\}$, one applies [1, 3.3] to verify that

$$\text{spt} [\partial L_{\mathcal{B}}(S) - L_{\mathcal{B}}(T)] \subset C = p^{-1}(\text{spt } U) \cap \Phi.$$

Choose $h_1, h_2, \dots \in E^0(\text{SO}(n))$ such that for $j=1, 2, \dots$,

$$\text{spt } h_j \subset \text{SO}(n) \sim q(C), \quad 0 \leq h_j \leq 1,$$

and $\lim_{j \rightarrow \infty} h_j(g) = 1$ for $g \in \text{SO}(n) \sim q(C)$. Then for each $j=1, 2, \dots$,

$$\partial[L_{\mathcal{B}}(S) \wedge q^\#(h_j\omega)] = \partial L_{\mathcal{B}}(S) \wedge q^\#(h_j\omega) = L_{\mathcal{B}}(T) \wedge q^\#(h_j\omega),$$

because $d(h_j\omega)=0$ and

$$\text{spt } q^\#(h_j\omega) \cap \text{spt} [\partial L_{\mathcal{B}}(S) - L_{\mathcal{B}}(T)] = \emptyset.$$

Now $C' = q^{-1}[q(C)]$ is closed and $H^m[q(C')] = 0$, hence for H^m almost all $g \in \text{SO}(n)$, $\|\langle L_{\mathcal{B}}(S), q|\Phi, g \rangle\|(C') = 0$, and therefore by [4, 4.3.2(2)],

$$\|L_{\mathcal{B}}(S) \wedge q^\#\omega\|(C') = 0.$$

We apply Lebesgue's dominated convergence theorem and [1, 3.5(2)] to obtain for $\psi \in E^{k+l-n}(\Phi)$,

$$\begin{aligned}L_{\mathcal{B}}(S)[(q^\#\omega) \wedge d\psi] &= \lim_{j \rightarrow \infty} L_{\mathcal{B}}(S)[(q^\#h_j\omega) \wedge d\psi] \\ &= \lim_{j \rightarrow \infty} L_{\mathcal{B}}(T)[(q^\#h_j\omega) \wedge \psi] \\ &= L_{\mathcal{B}}(T) \cap (\Phi \sim C')[(q^\#\omega) \wedge \psi].\end{aligned}$$

Consider $\varphi \in D^k(0)$ and define $\Xi \in D^{k+l-n}(\Phi)$ as in the proof of 5.3. Then from [4, 4.3.2(1)], 3.6, 4.2(5) and 5.3 we have

$$\begin{aligned}(-1)^t \partial[L_{\mathcal{B}}(S) \wedge q^\#\omega](r^{n-l} \circ p\Xi) &= (-1)^t \int_{\text{SO}(n)} \partial \langle L_{\mathcal{B}}(S), q|\Phi, g \rangle (r^{n-l} \circ p\Xi) dH^m g \\ &= \int_{\text{SO}(n)} \partial S \cap g_\# B(r^{n-l}\varphi \cap g^{-1}\# \psi_B) dH^m g \\ &= l\alpha(l)H^\mu(I)\gamma^2(n-1, k, l-1)\partial S(\varphi),\end{aligned}$$

where $t = (k+l-n)(m+l+1) + m$. Finally, we conclude using 3.4(ii), 3.4(i), 5.2, 3.6 and 5.1 that this is equal to

$$(-1)^t (L_{\mathcal{B}}(T) \wedge q^\#\omega) \cap (\Phi \sim C')(r^{n-l} \circ p\Xi) \leq l\alpha(l)H^\mu(I)M(T)M(\varphi).$$

Verification of the inequality for $\varphi \in D^{k-1}(z) \wedge dr_z$, $k+l > n$, is analogous.

6.2. THEOREM. Assume $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $z \in \mathbb{R}^n$.

$$(1) \quad \delta(n, k, l)M(\partial S) \leq 4 \binom{n}{k} V^{\partial S}(z) \sup \{|x-z| : x \in \text{spt } \partial S\}^{n-l}.$$

(2) Under the hypothesis of 6.1,

$$\delta(n, k, l)M(\partial S) \leq 4 \binom{n}{k} l \alpha(l) M(T).$$

(3) Suppose $\|S\|(\mathbf{r}_z^{-k}) < \infty$, $h \in E^0(\mathbb{R}^n)$, $\varphi \in E^k(S^{n-1})$, $M(h) \leq 1$ and $M(\varphi) \leq 1$. Then

$$\begin{aligned} \gamma^2(n-1, k, n-k-1)S[d(h \wedge \rho_z^\# \varphi)] \\ \leq 4V_{n-k}^{\partial S}(z) + \gamma^2(n-1, k+1, n-k-2)^{-1}h(z)M(d\varphi)V_{n-k-1}^S(z). \end{aligned}$$

(V_l indicates use of l planes; the second term is 0 if $k+1=n$.)

Proof. We can assume that $z=0$.

Consider $h \in E^0(\mathbb{R}^n)$ such that $h(0)=0$. Corresponding to each $j=1, 2, \dots$ we choose $\varepsilon_j > 0$, $\delta_j > 0$ and $x_j \in \mathbb{R}^n$ such that

$$\lim_{j \rightarrow \infty} \varepsilon_j = \lim_{j \rightarrow \infty} \delta_j = 0, \quad |h(x_j)| = \sup \{|h(x)| : |x| \leq \delta_j\} = \varepsilon_j,$$

and $|x_j| = \delta_j$. Using the mean value theorem, we see that we can assume that $\lim_{j \rightarrow \infty} \varepsilon_j/\delta_j = |dh(0)|$ and $2\varepsilon_j \geq |dh(0)|\delta_j$. There exists $\eta_j \in E^0(\mathbb{R})$ such that

$$\begin{aligned} \eta_j(r) &= 0 \quad \text{for } r \leq \delta_j/j, \\ \eta_j(r) &= (1-j^{-1})\varepsilon_j^{-1}\delta_j|dh(0)| \quad \text{for } r \geq \delta_j, \\ 0 &\leq \eta_j'(r) \leq 2|dh(0)|/\varepsilon_j \quad \text{for } r \in \mathbb{R}; \end{aligned}$$

define $h_j = \eta_j \circ r$. Then for $0 \neq x \in \mathbb{R}^n$,

$$\begin{aligned} 0 \leq h_j(x) \leq 2, \quad |dh_j(x)| |h(x)| \leq 2|dh(0)|, \\ \lim_{j \rightarrow \infty} h_j(x) = 1, \quad \lim_{j \rightarrow \infty} dh_j(x) = 0. \end{aligned}$$

Recalling from [4, 4.1.21] that $\|S\|\{0\}=0$, we write $\varphi_0 = h \, dx^{t_1} \wedge \dots \wedge dx^{t_k}$,

$$\partial S(h_j \varphi_0) = S(dh_j \wedge \varphi_0) + S(h_j d\varphi_0),$$

and use Lebesgue's dominated convergence theorem to conclude that

$$\lim_{j \rightarrow \infty} \partial S(h_j \varphi_0) = \partial S(\varphi_0).$$

It follows that $\partial S(\varphi_0) \leq 2M(\varphi_0)M[(\partial S)_z]$, hence for $\varphi \in E^k(\mathbb{R}^n)$, $\varphi(0)=0$, $M(\varphi) \leq 2$,

$$\partial S(\varphi) \leq 4 \binom{n}{k} M[(\partial S)_z].$$

We obtain this relation for arbitrary φ with $M(\varphi) \leq 1$ by observing that $\partial S(\varphi) = \partial S[\varphi - \varphi(0)]$. (1) now follows from 5.9(1).

Since $\delta(n, k, n-k)=0$, we assume $k+l > n$ for the proof of (2). Consider $\varphi \in D^k(\mathbb{R}^n \sim \{0\})$, $M(\varphi) \leq 1$. Using 5.4 to write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D^k(0)$ and $\varphi_2 \in D^{k-1}(0) \wedge dr$, we apply 6.1 to φ_1 and to φ_2 to obtain $\delta(n, k, l)\partial S(\varphi) \leq l\alpha(l)M(T)$. Consequently, $\delta(n, k, l)M[(\partial S)_z] \leq l\alpha(l)M(T)$, whence follows (2).

Turning to (3), we choose h_j for $h_0 = h - h(0)$ and infer from 5.3 that since $\|\rho^\# \varphi\| \leq r^{-k}$,

$$(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1)S[d(h, h_0 \wedge \rho^\# \varphi)] \leq 4V_{n-k}^{2S}(0)$$

for $j=1, 2, \dots$. As we found previously,

$$S[d(h, h_0) \wedge \rho^\# \varphi] \rightarrow S(dh_0 \wedge \rho^\# \varphi) = S(dh \wedge \rho^\# \varphi) \quad \text{as } j \rightarrow \infty.$$

If $k+1=n$, then $d\varphi=0$ and the proof is complete. On the other hand, if $k+1 < n$, then we use 5.7, 5.6 and [4, 4.1.21] and proceed as in the derivation of equation (**) in the proof of 5.9 to infer that

$$\begin{aligned} \int_{\mathbb{R}^n} |\langle S^-, \rho^\# d\varphi \rangle| d\|S\| &\leq M(d\varphi) \int_{\mathbb{R}^n} \|\rho^\#(x)S^-(x)\| d\|S\| x \\ &\leq [(n-k-1)\alpha(n-k-1)\gamma^2(n-1, k+1, n-k-2)]^{-1} M(d\varphi) V_{n-k-1}^S(0). \end{aligned}$$

Consequently, Lebesgue's dominated convergence theorem implies that if $V_{n-k-1}^S(0) < \infty$, then

$$S(h, h_0 \wedge d\rho^\# \varphi) \rightarrow S(h_0 \wedge d\rho^\# \varphi) \quad \text{as } j \rightarrow \infty,$$

whence

$$(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1)S[d(h \wedge \rho^\# \varphi)] \leq 4V_{n-k}^{2S}(0) + |h(0)S(d\rho^\# \varphi)|,$$

and our assertion is clear.

6.3. THEOREM. Suppose $S \in F_{k+1}(\mathbb{R}^n)$, $M(S) < \infty$, and $z_1, \dots, z_{k+2} \in \mathbb{R}^n$ do not lie on a k -plane. If either $V^{\partial S}(z_i) < \infty$ or the hypothesis of 6.1 holds for z_i for each $i=1, \dots, k+2$, then

$$S \in N_{k+1}(\mathbb{R}^n).$$

Proof. The proof was suggested by the proof of [6, 2.10]. Let $\varphi = dx^{i_1} \wedge \dots \wedge dx^{i_k}$. It suffices to show that

$$M(\partial S \wedge \varphi) < \infty \quad \text{for } 1 \leq i_1 < \dots < i_k \leq n.$$

Let Π_j denote the k -plane containing $\{z_k : k \neq j\}$. Observing that

$$\{\mathbb{R}^n \sim \Pi_j : j = 1, \dots, k+2\}$$

covers $\mathbb{R}^n$, we choose a subordinate partition of unity $\alpha_1, \dots, \alpha_{k+2}$ with $\Pi_j \cap \text{spt } \alpha_j = \emptyset$. Thus it is sufficient to show that

$$M(\partial S \wedge \alpha_j \varphi) < \infty \quad \text{for } j = 1, \dots, k+2.$$

Consider, for instance, $j=k+2$. By 5.4 there exist $\beta_1, \dots, \beta_{k+1} \in E^0(\mathbf{R}^n)$, $\text{spt } \beta_i \subset \text{spt } \alpha_{k+2}$, and $\varphi_i \in \rho_{z_i}^\# [E^k(S^{n-1})]$ such that

$$\alpha_{k+2} \varphi = \sum_{i=1}^{k+1} \beta_i \varphi_i.$$

Thus it is sufficient to show that

$$M(\partial S \wedge \beta_i \varphi_i) < \infty \quad \text{for } i = 1, \dots, k+1.$$

Fix $h \in E^0(\mathbf{R}^n)$, $M(h) \leq 1$. Then from 5.3 and 4.2 we infer that

$$l\alpha(l)\gamma^2(n-1, k, l-1)\partial S(h\beta_i\varphi_i) \leq V^{\partial S}(z_i)M(\beta_i\varphi_i) \sup \{|x-z_i| : x \in \text{spt } \partial S\}^{n-l}.$$

Moreover, if the hypothesis of 6.1 is satisfied, then

$$\gamma^2(n-1, k, l-1)\partial S(h\beta_i\varphi_i) \leq M(T)M(\beta_i\varphi_i).$$

Therefore, $M(\partial S \wedge \beta_i \varphi_i) < \infty$.

6.4. REMARK. Let S be a rectifiable $k+1$ current. If one of the following conditions is satisfied, then $M(\partial S) < \infty$, hence S is an integral current by [4, 4.2.16]:

(1) The hypothesis of 6.3.

(2) $k+l > n$ and for some $z \in \mathbf{R}^n$ either $V^{\partial S}(z) < \infty$ or the hypothesis of 6.1 is satisfied.

Conversely, if $M(\partial S) < \infty$, then from 5.9(1) it follows that $V^{\partial S}(z) < \infty$ for $z \notin \text{spt } \partial S$.

7. V^T for T a manifold. Let $T \in F_k(\mathbf{R}^n)$, $M(T) < \infty$, be obtained by integration over an oriented k dimensional proper submanifold of class 1, which is also denoted by T . Assume $z \in T$.

7.1. THEOREM. $\|T\|(\mathbf{r}_z^{-k+1}) = \int_T \mathbf{r}_z^{-k+1} dH^k < \infty$. Consequently, if $k+l > n$, then by 5.9(1), $V^T(z) < \infty$.

Proof. This is easily verified by introducing a bi-Lipschitzian coordinate system for T in a neighborhood of z .

For the remainder of this section we assume $k+l=n$.

7.2. LEMMA.

$$V^T(0) = (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int |T^{-}(x) \wedge x| |x|^{-k-1} dH^k x.$$

Proof. Recalling equation (**) in the proof of 5.9(3), we observe that for $x \neq 0$,

$$|\rho_\#(x)[T^{-}(x)]| = |T^{-}(x) \wedge x| |x|^{-k-1}.$$

7.3. EXAMPLE. Let $n=2$ and $k=1$. Whenever $|t| < \frac{1}{2}$ define

$$g(t) = \int_0^t \int_0^u -[\ln|v|]^{-1} dH^1 v dH^1 u.$$

Let T be the graph of the function F defined by

$$F(t) = g(t) \sin(1/t),$$

oriented so that

$$T^{-1}(t, F(t)) = [1 + F'(t)^2]^{-1/2} [e_1 + F'(t)e_2].$$

Then T is of class 1, but not of class 2, and by 7.2

$$V^T(0) \geq \int_{-\varepsilon}^{\varepsilon} \left| \frac{F'(t)}{t} - \frac{F(t)}{t^2} \right| dH^1 t = \infty$$

for sufficiently small positive numbers ε .

7.4. THEOREM. Suppose there exist an open subset $U \subset \mathbb{R}^k$, a diffeomorphism $F: U \rightarrow T$ with $z \in F(U)$, and constants $A > 0$ and $0 < \alpha \leq 1$ for which

$$|F_{\#}(x)(v) - F_{\#}(y)(v)| < A|x - y|^{\alpha}$$

whenever $x, y \in U$ and $|v| \leq 1$. Then $V^T(z) < \infty$.

Proof. We can assume that $z=0$, that $T_0(T)$ is spanned by $e_1, \dots, e_k$ and, by 5.9(1), that $T=F(U)$. Let $P: \mathbb{R}^n \rightarrow T_0(T)$ be the orthogonal projection. It is easy to see that the condition on $F_{\#}$ holds (with a different A) for $(F \circ \varphi)_{\#}$ where φ is a Lipschitzian diffeomorphism of an open subset V of $\mathbb{R}^k$ onto U and

$$\sup \{ |\varphi_{\#}(x)(v)| : x \in V, |v| \leq 1 \} < \infty.$$

Thus we can assume that $U = \mathbb{R}^k \cap \{x : |x| < \varepsilon\}$,

$$F^{-1} = P_0 = P|T$$

and the Jacobian $J_k P_0 > \frac{1}{2}$. Observe that

$$\sigma = \sup \{ |F_{\#}(x)(v)| : x \in U \text{ and } |v| \leq 1 \} < \infty.$$

Then by 7.2 and the area formula [4, 3.2.3],

$$\begin{aligned} & [(n-k)\alpha(n-k)]^{-1} V^T(0) \\ & \leq 2 \int_U |T^{-1}[F(x)] \wedge F(x)| |F(x)|^{-2} |x|^{-k+1} dH^k x \\ & = 2 \int_{S^{k-1}} \int_0^{\varepsilon} |T^{-1}[F(t\theta)] \wedge F(t\theta)| |F(t\theta)|^{-2} dH^1 t dH^{k-1} \theta \\ & \leq 2 \int_{S^{k-1}} \int_0^{\varepsilon} |F'_{\theta}(t) \wedge F_{\theta}(t)| |F_{\theta}(t)|^{-2} dH^1 t dH^{k-1} \theta, \end{aligned}$$

where $F_{\theta}(t) = F(t\theta)$. Thus fix θ and observe that in evaluating the inner integral we can assume that

$$F_{\theta}(t) = te_1 + \varphi(t),$$

where $\varphi(t)$ involves only $e_{k+1}, \dots, e_n$ and $\varphi'(0)=0$. It is a routine matter to verify with the help of the mean value theorem that for $0 < t < \varepsilon$,

$$\begin{aligned} |F'_\theta(t) \wedge F_\theta(t)| |F_\theta(t)|^{-2} &\leq (1+\sigma)t^{-2}|\varphi(t)| + At^{\alpha-1} \\ &\leq [A + (1+\sigma)(n-k)A]t^{\alpha-1}, \end{aligned}$$

whence $V^T(0) < \infty$.

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