COEFFICIENT ESTIMATES FOR DIRICHLET SERIES

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1. Introduction. The primary purpose of this paper is to study coefficient estimates of Dirichlet series

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x), \quad \lambda_n \rightarrow \infty, \quad 0 < \lambda_1 < \lambda_2 < \cdots,$$

for which the sequence $\{\lambda_n\}$ satisfies certain lacunarity conditions. D. Gaier [2] has used such estimates in one of his proofs of the high-indices theorem for Borel summability, in which results on the order of an entire function are inferred from its rate of growth along the real line. A by-product of his work is the estimate $|a_n| \leq 2\lambda_n p_n \|f\|_1$ when $\sum \lambda_n^{-1} < \infty$, $f \in L^1(0, \infty)$, and

$$p_n = \prod_{k \neq n} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right|.$$

Some subsequent papers in much the same vein are due to G. Halász [3] and to J. M. Anderson and K. G. Binmore [1]. Halász has obtained estimates for a_n , $\sum_1^N a_n$, and $\sum_1^N |a_n|$ when $\lambda_{n+1}/\lambda_n \geq q > 1$ and f satisfies appropriate conditions. The paper of Anderson and Binmore is concerned with the application of coefficient estimates to the study of entire functions. Their work includes the estimate $|a_n| \leq (2\lambda_n)^{1/2} p_n \|f\|_2$ where p_n is as above, $\sum \lambda_n^{-1} < \infty$, and $f \in L^2(0, \infty)$.

L. Schwartz [8] had earlier obtained results which essentially include both the estimates of Gaier and of Anderson and Binmore. Indeed, as the referee has pointed out, S. Mandelbrojt [5] had also established inequalities of the same kind and these appear in larger form in [6]. Later, Mandelbrojt's results were extended by F. Sunyer Belaguer [9], [10].

There is some question left to the sharpness of Schwartz's estimates and in §2 it is proved that if $1 \leq p \leq 2$ and $f \in L^p(0, \infty)$ then

$$|a_n| \leq (2\lambda_n)^{1/p} p_n \|f\|_p.$$

I do not know if the constant is sharp when $p < 2$, or if the order of the estimate still holds when $p > 2$ but in §2 there are estimates when $p > 2$ for functions which are subject to more stringent conditions. Also in §2 there are estimates for the sequence of partial sums.

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These results are applied in §3 to obtain some gap theorems. W. Rudin [7] showed that if $\liminf \lambda_{n+1}/\lambda_n = 1$ then there is a Dirichlet series so that $f' \in L^1(0, \infty)$ yet $\sum a_n$ diverges. In §3 it is shown that there are sequences $\{\lambda_n\}$ so that $\lim \lambda_{n+1}/\lambda_n = 1$, but so that if $f' \in L^p(0, \infty)$ when $p > 1$, or if $f'' \in L^1(0, \infty)$, then $\sum |a_n| < \infty$.

2. Coefficient estimates.

THEOREM 2.1. *If $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$, $\sum \lambda_n^{-1} < \infty$, and $f \in L^p(0, \infty)$, $1 \leq p \leq 2$, then $|a_n| \leq (2\lambda_n)^{1/p} p_n \|f\|_p$ where*

$$p_n = \prod_{k \neq n} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right|.$$

Proof. Let

$$f_N(x) = \sum_{k=1}^N a_k \exp(-\lambda_k x)$$

and

$$F(z) = \sum_{k=1}^N \frac{a_k \exp(-\lambda_k \varepsilon)}{\lambda_k - z}, \quad \varepsilon > 0.$$

Then if $\operatorname{Re} z < 0$

$$F(z) = \int_0^{\infty} e^{zt} f_N(t + \varepsilon) dt.$$

Let

$$B(z) = \prod_{k=1}^N \frac{\lambda_k - z}{\lambda_k + z}$$

so that

$$G(z) = F(z)B(z) = \sum_{k=1}^N \frac{a_k \exp(-\lambda_k \varepsilon)}{z + \lambda_k} B_k(z)$$

where

$$B_k(z) = \left(\prod_{j \neq k} \frac{\lambda_j - z}{\lambda_j + z} \right).$$

Now $B_k(z)/(z + \lambda_k) \in H^2$, a Hardy class of functions in the right half-plane, and so the same is true of G . Consequently, G may be represented by the Poisson integral of its values on the imaginary axis, and

$$\|G(x + iy)\|_q \leq \|G(iy)\|_q$$

if $x > 0$ and $q \geq 1$ [4, p. 124, and p. 128].

Next, let Γ be the boundary of a rectangle in the right half-plane with sides parallel to the real and imaginary axes, and enclosing λ_n .

Let $\zeta^{2/p} = \exp(2/p \log \zeta)$ for $|\arg \zeta| < \pi/2$, with $\log 1 = 0$. Then by the residue theorem

$$\frac{a_n \exp(-\lambda_n \varepsilon) B_n(\lambda_n)}{(2\lambda_n)^{2/p}} = \frac{-1}{2\pi i} \int_{\Gamma} \frac{F(z) B_n(z)}{(z + \lambda_n)^{2/p}} dz.$$

But since $G(z)$ goes to zero uniformly in any fixed half-plane $\operatorname{Re} z \geq \delta > 0$ [4, p. 125] it follows that

$$\begin{aligned} \frac{a_n \exp(-\lambda_n \varepsilon) B_n(\lambda_n)}{(2\lambda_n)^{2/p}} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x'_n + iy) B_n(x'_n + iy)}{(x'_n + iy + \lambda_n)^{2/p}} dy \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x''_n + iy) B_n(x''_n + iy)}{(x''_n + iy + \lambda_n)^{2/p}} dy \\ &= I_1 + I_2 \end{aligned}$$

where $x'_n < \lambda_n < x''_n$.

Then

$$|I_1| \leq \frac{1}{2\pi} \|G(x'_n + iy)\|_q \left[\int_{-\infty}^{\infty} \left| \frac{x'_n + iy + \lambda_n}{x'_n + iy - \lambda_n} \right|^p \frac{dy}{|x'_n + iy + \lambda_n|^2} \right]^{1/p}$$

where $1/p + 1/q = 1$. A similar estimate holds for I_2 . Since $\|G(x''_n + iy)\|_q \leq \|G(iy)\|_q$ we may let $x''_n \rightarrow \infty$ and $x'_n \rightarrow 0$ to conclude that

$$\begin{aligned} \frac{|B_n(\lambda_n) a_n \exp(-\lambda_n \varepsilon)|}{(2\lambda_n)^{2/p}} &\leq \frac{1}{2\pi} \|G(iy)\|_q \left(\int_{-\infty}^{\infty} \frac{dy}{y^2 + \lambda_n^2} \right)^{1/p} \\ &= \frac{1}{2\pi} \left(\frac{\pi}{\lambda_n} \right)^{1/p} \|G(iy)\|_q. \end{aligned}$$

Moreover, since $|B(iy)| = 1$, then $\|G(iy)\|_q = \|F(iy)\|_q$, but $F(iy)$ is the Fourier transform of $f_N(t + \varepsilon)$, so that by the Hausdorff-Young inequality [11, p. 96], $\|G(iy)\|_q \leq (2\pi)^{1/q} \|f_N(t + \varepsilon)\|_p$.

Combining these inequalities gives

$$|a_n \exp(-\lambda_n \varepsilon)| \leq \frac{(2\lambda_n)^{1/p}}{|B_n(\lambda_n)|} \|f_N(t + \varepsilon)\|_p.$$

If $\varepsilon > 0$, summation by parts shows that there is a constant C such that

$$|f_N(t + \varepsilon)| \leq C \exp(-\lambda_1 t)$$

and

$$|f(t + \varepsilon)| \leq C \exp(-\lambda_1 t).$$

Since $f_N(t) \rightarrow f(t)$ when $t > 0$, the Lebesgue dominated convergence theorem implies that

$$\|f_N(t + \varepsilon) - f(t + \varepsilon)\|_p \rightarrow 0.$$

But $\|f(t + \varepsilon)\|_p \leq \|f\|_p$ and so letting $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ in the above inequality completes the proof.

The next theorem deals with the case where $p > 2$. The technique is essentially that of Halász [3].

THEOREM 2.2. *If*

(a) $f(z) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n z)$ *converges when* $\operatorname{Re} z > 0$;

(b) $\|f(re^{i\theta})\|_p \leq M$ *whenever* $|\theta| \leq \eta = \delta\pi/2 < \pi/2$;

(c) $\sum \lambda_n^{-1/\beta} < \infty$, *where* $\beta + \delta = \gamma > 1$, *and* $\beta < 2$; *then*

$$|a_n| \leq \frac{2M\beta\lambda_n^{1/p}}{|q \cos(\gamma\pi/2)|^{1/q}} \prod_{k \neq n} \left| \frac{\lambda_k^{1/\beta} + \lambda_n^{1/\beta}}{\lambda_k^{1/\beta} - \lambda_n^{1/\beta}} \right|$$

where $1/p + 1/q = 1$.

Proof. Let

$$\phi_N(z) = \sum_{k=1}^N a_k \exp(-\lambda_k z)$$

and

$$F(z) = \sum_{k=1}^N \frac{a_k \exp(-\lambda_k \varepsilon)}{\lambda_k - z}, \quad \varepsilon > 0.$$

Then if $\operatorname{Re} ze^{i\alpha} < 0$, and $|\alpha| < \pi/2$

$$F(z) = \int_0^\infty \exp(zte^{i\alpha}) \phi_N(te^{i\alpha} + \varepsilon) e^{i\alpha} dt.$$

But if $z = re^{i\phi}$ and $\operatorname{Re} ze^{i\alpha} < 0$, then

$$\begin{aligned} |F(z)| &\leq \left[\int_0^\infty \exp(rtq \cos(\theta + \alpha)) dt \right]^{1/q} \|\phi_N(te^{i\alpha} + \varepsilon)\|_p \\ &= \|\phi_N(te^{i\alpha} + \varepsilon)\|_p \frac{1}{|qr \cos(\theta + \alpha)|^{1/q}}. \end{aligned}$$

Now let $(\zeta)^{1/\beta}$ be defined for $|\arg \zeta| < \pi$, $(1)^{1/\beta} = 1$, and set

$$B(z) = \prod_{k=1}^N \frac{1 - (z/\lambda_k)^{1/\beta}}{1 + (z/\lambda_k)^{1/\beta}}.$$

Similarly define $(\zeta)^{1/q}$, so that letting

$$\begin{aligned} \alpha &= \pm \eta, \quad z = r \exp(\pm i\pi\beta/2), \\ |z^{1/q} F(z) B(z)| &\leq \frac{\|\phi_N(te^{i\alpha} + \varepsilon)\|_p}{|q \cos(\gamma\pi/2)|^{1/q}} = A. \end{aligned}$$

Now $z^{1/q} F(z) B(z)$ is of relatively slow growth for $|\arg z| < \beta\pi/2$, so that by the Phragmén-Lindelöf theorems [12, p. 180]

$$|z^{1/q} F(z) B(z)| \leq A, \quad |\arg z| \leq \beta\pi/2.$$

In particular, if $z = \lambda_n$ then

$$\frac{\lambda_n^{1/q} |a_n| \exp(-\lambda_n \varepsilon)}{2\beta\lambda_n} \prod_{k \neq n} \left| \frac{\lambda_k^{1/\beta} - \lambda_n^{1/\beta}}{\lambda_k^{1/\beta} + \lambda_n^{1/\beta}} \right| \leq A.$$

As in the proof of Theorem 2.1, let $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ to obtain the desired inequality.

The next theorem deals with sequences of partial sums.

THEOREM 2.3. *If $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$, $\sum \lambda_n^{-1} < \infty$ and $f' \in L^p(0, \infty)$, $1 < p \leq 2$, then*

$$\left| \sum_{k=r}^s a_k \right| \leq A_p \|f'\|_p [C_{r-1} + C_s]$$

where A_p is a constant depending on p alone, and

$$C_n = \min_{\lambda_n < x < \lambda_{n+1}} \frac{1}{x^{1-1/p}} \prod_{k=1}^{\infty} \left| \frac{x + \lambda_k}{x - \lambda_k} \right|.$$

Proof. The proof is very much the same as that of Theorem 2.1 and hence many of the details are omitted.

Let

$$\begin{aligned} \phi_N(x) &= \sum_{k=1}^N \lambda_k a_k \exp(-\lambda_k x), \\ F(z) &= \sum_{k=1}^N \frac{\lambda_k a_k \exp(-\lambda_k(z+\varepsilon))}{\lambda_k - z}, \quad \varepsilon > 0, \end{aligned}$$

and

$$B(z) = \prod_{k=1}^N \frac{\lambda_k - z}{\lambda_k + z}.$$

If Γ is a rectangle in the right half-plane enclosing $\lambda_r, \dots, \lambda_s$ and no others then

$$\begin{aligned} \sum_{k=r}^s a_k \exp(-\lambda_k \varepsilon) &= -\frac{1}{2\pi i} \int \frac{F(z)}{z} dz \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_r + iy)}{x_r + iy} dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_s + iy)}{x_s + iy} dy \\ &= I_1 + I_2, \end{aligned}$$

where $\lambda_{r-1} < x_r < \lambda_r \leq \lambda_s < x_s < \lambda_{s+1}$.

If $G(z) = F(z)B(z)$ then

$$|I_1| \leq \frac{1}{2\pi} \|G(x_r + iy)\|_q D_r \left(\int_{-\infty}^{\infty} \frac{dy}{|x_r + iy|^p} \right)^{1/p}$$

where $1/p + 1/q = 1$ and

$$D_r = \max_y \frac{1}{|B(x_r + iy)|}.$$

A similar estimate holds for I_2 and since

$$\|G(x_r + iy)\|_q \leq \|G(iy)\|_q = \|F(iy)\|_q \leq (2\pi)^{1/q} \|\phi_N(t + \varepsilon)\|_p$$

it follows that

$$\left| \sum_{k=r}^s a_k \exp(-\lambda_k \varepsilon) \right| \leq A_p \|\phi_N(t + \varepsilon)\|_p \left(\frac{D_r}{x_r^q} + \frac{D_s}{x_s^q} \right).$$

Letting $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ completes the proof.

THEOREM 2.4. *If $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$, $\sum \lambda_n^{-1} < \infty$, and $f'' \in L^1(0, \infty)$ then*

$$\left| \sum_{k=r}^s a_k \right| \leq \frac{\|f''\|}{2} \left[\text{Min}_{\lambda_{r-1} < x < \lambda_r} \frac{1}{x} \prod_{k=1}^{\infty} \left| \frac{x + \lambda_k}{x - \lambda_k} \right| + \text{Min}_{\lambda_s < x < \lambda_{s+1}} \frac{1}{x} \prod_{k=1}^{\infty} \left| \frac{x + \lambda_k}{x - \lambda_k} \right| \right].$$

Proof. The proof is very similar to the proof of Theorem 2.3 and so will be omitted.

3. Some applications. Before applying the estimates of §2 it is necessary to establish a result about the special sequence $\lambda_n = \exp(n^\alpha)$.

THEOREM 3.1. *If $\lambda_n = \exp(n^\alpha)$ and $1 > \alpha > 0$ then*

$$P_n = \prod_{k \neq n} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right| \leq \exp C n^{2(1-\alpha)}$$

where C is a constant, dependent only on α .

Proof. Since $\lambda_k/\lambda_{k-1} \geq \lambda_{k+1}/\lambda_k$, it follows that

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n - \lambda_k} &= \sum_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n} \sum_{j=0}^{\infty} \left(\frac{\lambda_k}{\lambda_n} \right)^j = \sum_{j=1}^{\infty} \frac{1}{\lambda_n^j} \sum_{k=1}^{n-1} \lambda_k^j \\ &\leq \sum_{j=1}^{\infty} \frac{1}{\lambda_n^j} \lambda_{n-1}^{j-1} \sum_{k=1}^{n-1} \lambda_k \leq \sum_{j=1}^{\infty} \left(\frac{\lambda_{n-1}}{\lambda_n} \right)^j \frac{1}{1 - \lambda_{n-2}/\lambda_{n-1}} \\ &\leq \frac{1}{1 - \lambda_{n-1}/\lambda_n} \frac{1}{1 - \lambda_{n-2}/\lambda_{n-1}} \sim \frac{1}{\alpha^2 n^{2(\alpha-1)}}. \end{aligned}$$

And

$$\sum_{k=n+1}^{\infty} \frac{\lambda_n}{\lambda_k - \lambda_n} = \sum_{j=1}^{\infty} \lambda_n^j \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^j}.$$

But

$$\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^j} \leq \frac{1}{\lambda_{n+1}^j} + \int_{n+1}^{\infty} \exp(-jx^\alpha) dx$$

so that

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_n^j \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^j} &\leq \sum_{j=1}^{\infty} \left(\frac{\lambda_n}{\lambda_{n+1}} \right)^j + \int_{n+1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\lambda_n}{\exp(x^\alpha)} \right)^j dx \\ &\leq \frac{\lambda_n}{\lambda_{n+1}} \frac{1}{1 - \lambda_n/\lambda_{n+1}} + \frac{1}{\alpha} \int_{(n+1)^\alpha}^{\infty} \frac{\lambda_n e^{-u}}{u^{1-1/\alpha}} \frac{du}{1 - \lambda_n e^{-u}} \\ &\leq \frac{\lambda_n}{\lambda_{n+1}} \frac{1}{1 - \lambda_n/\lambda_{n+1}} + \frac{1}{\alpha} \frac{\lambda_n}{1 - \lambda_n/\lambda_{n+1}} \int_{(n+1)^\alpha}^{\infty} u^{1/\alpha-1} e^{-u} du. \end{aligned}$$

But

$$\begin{aligned} \int_{(n+1)^\alpha}^{\infty} u^{1/\alpha-1} e^{-u} du &= (n+1)^{1-\alpha} \lambda_{n+1}^{-1} + \frac{1-\alpha}{\alpha} \int_{(n+1)^\alpha}^{\infty} \frac{e^{-u} u^{1/\alpha-1}}{u} du \\ &\leq (n+1)^{1-\alpha} \lambda_{n+1}^{-1} + \frac{1-\alpha}{\alpha} \frac{1}{(n+1)^\alpha} \int_{(n+1)^\alpha}^{\infty} e^{-u} u^{1/\alpha-1} du, \end{aligned}$$

so for large values of n

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^{\alpha}} &\leq \frac{1}{1-\lambda_n/\lambda_{n+1}} + \frac{1}{\alpha} \frac{(n+1)^{1-\alpha}}{1-\lambda_n/\lambda_{n+1}} \left(1 - \left(\frac{1-\alpha}{\alpha(n+1)^{\alpha}}\right)\right)^{-1} \\ &\sim \frac{1}{\alpha n^{\alpha-1}} + \frac{1}{\alpha^2} \frac{1}{(n+1)^{2(\alpha-1)}} \leq \frac{2}{\alpha^2} \frac{1}{(n+1)^{2(\alpha-1)}}. \end{aligned}$$

So

$$\begin{aligned} p_n &= \prod_{k=1}^{n-1} \frac{\lambda_k + \lambda_n}{\lambda_n - \lambda_k} \prod_{k=n+1}^{\infty} \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \\ &\leq \exp 2 \sum_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n - \lambda_k} \cdot \exp 2 \sum_{k=n+1}^{\infty} \frac{\lambda_n}{\lambda_k - \lambda_n} \\ &\leq \exp C n^{2(1-\alpha)} \end{aligned}$$

where C is some constant.

With this estimate the following theorems may be proved.

THEOREM 3.2. *If $p > 1$ then there is a sequence $\{\lambda_n\}$ such that if*

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$$

converges for each $x > 0$ and $f' \in L^p(0, \infty)$ then $\sum |a_n|^{\varepsilon} < \infty$ for each $\varepsilon > 0$.

Proof. Assume without loss of generality that $1 < p \leq 2$, for if $p > 2$ and $f' \in L^p(0, \infty)$ then $f' \in L^r(0, \infty)$ for each $r < p$. This is true since the integrability of f' depends only on its integrability over $(0, 1)$.

Now by Theorem 2.1 and Theorem 3.1, if $\lambda_n = \exp(n^a)$ and $2/3 < a < 1$ then

$$|a_n|^{\varepsilon} = O(\exp(\varepsilon C n^{2(1-\alpha)} - \varepsilon q^{-1} n^a)),$$

and this estimate shows that $\sum |a_n|^{\varepsilon}$ converges.

THEOREM 3.3. *There exists a sequence $\{\lambda_n\}$ such that if $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$ and $f'' \in L^1(0, \infty)$ then $\sum |a_n|^{\varepsilon}$ converges.*

Proof. This follows from Theorem 2.4. The proof is similar to that of Theorem 3.2.

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