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A FUNDAMENTAL SOLUTION OF THE PARABOLIC EQUATION ON HILBERT SPACE. II: THE SEMIGROUP PROPERTY

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Abstract. The existence of a family of solution operators $\{q_t : t>0\}$ corresponding to a fundamental solution of a second order infinite-dimensional differential equation of the form $\partial u/\partial t = Lu$ was previously established by the author. In the present paper, it is established that these operators are nonnegative, and satisfy the condition $q_s q_t = q_{s+t}$.

I. **Introduction.** This paper continues the study initiated in [12] of second order parabolic equations, with variable coefficients, on Hilbert space. In [12] we established a fundamental solution for the equation $\partial u/\partial t = Lu$, where L is a second order differential operator satisfying certain regularity hypotheses. This fundamental solution is given by a family of finite signed Borel measures $\{q_t(x, dy) : t>0, x \in B\}$ on a Banach space B (B will be defined later) or, equivalently, by a family of operators $\{q_t : t>0\}$ on the space of bounded Lip-1 functions on B. These operators were defined via infinite series, which made it difficult to determine either their nonnegativity or whether they satisfy a semigroup property ($q_s q_t = q_{t+s}$ for all $s, t>0$).

The technique developed in this paper for establishing both nonnegativity and the semigroup property is that of "semifinite" approximation. Basically, the differential operator L is approximated by a differential operator L^K acting in a finite-dimensional subspace K of our Hilbert space H plus the Laplacian Δ acting in $K^\perp$. Nonnegativity and the semigroup property are known for the fundamental solutions of $\partial u/\partial t = L^K u$ and $\partial u/\partial t = \Delta u$. Combining these fundamental solutions and passing to the limit as $K \to H$ in some suitable fashion, we obtain the desired properties for $\{q_t\}$.

II. **Preliminaries.** Most of the basic definitions and ideas necessary to the following work can be found in Gross [8], [9] and in the preliminaries of [12]. The notation is that of [12] to the extent to which that is possible.

Let H denote a real separable Hilbert space with norm $|\cdot|$ and inner product

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(,). Gauss measure on H with variance parameter t is denoted by ν_t, and is defined for a cylinder set $S\subset H$ by

$$\nu_t(S) \equiv (2\pi t)^{-n/2}\int_E \exp\left[-|x|^2/2t\right]dx$$

where $S=P^{-1}(E)$, P being an n-dimensional projection on H and E a Borel set in the range of P.

Let $\|\cdot\|$ denote a particular measurable norm on H, and let B be the completion of H with respect to $\|\cdot\|$. The triple (H, B, i), where i is the natural injection of H into B, is called an abstract Wiener space. Gauss measure ν_t on H induces a Borel measure p_t on B which is such that

$$p_t\{x\in B/(\langle y_1, x\rangle,\ldots,\langle y_n, x\rangle)\in E\} = \nu_t\{x\in H/(\langle y_1, x\rangle,\ldots,\langle y_n, x\rangle)\in E\}$$

for all finite subsets $y_1,\ldots,y_n$ of B^* and Borel sets $E\subset R^n$. (Here we identify B^* with a subset of H^*.) p_t is called Wiener measure on B with variance parameter t.

Certain functions f defined on H determine measurable functions on B. The manner in which this takes place is described in Gross [7] for tame functions on H and for functions which are uniformly continuous near zero in H_m (u.c.n. 0 in H_m), where H_m denotes H with the topology determined by the measurable seminorms. The measurable function on B determined in this fashion by f is denoted by $\tilde{f}$. We will generally omit the tilde whenever it is obvious that we are working on B—e.g. $\int_B f(y)p_t(dy)$. In this paper we assume that $\|y\|$ in is $L^p(p_t(dy))$ for all $1\leqq p<\infty$ and for all $t>0$.

Let W be any Banach space. If f is a W-valued function defined in a neighborhood of a point x of B, we will write $Df(x)$ for the Fréchet derivative of f at x, and will call f B-differentiable at x if $Df(x)$ exists. We may also regard f as a function g defined on a neighborhood of the origin of H by restricting f to the coset $x+H$ of B and defining $g(h)\equiv f(x+h)$. The Fréchet derivative of g at 0 is denoted by $f'(x)$, and we say that f is H-differentiable at x if $f'(x)$ exists. We write $\|Df(x)\|$ and $|f'(x)|$ for the $L(B, W)$ and $L(H, W)$ norms respectively.

We will now briefly sketch the results of [12]. Let $A(x)\equiv I-B(x)$, where $B(\cdot)$ is a map from B to the space of symmetric trace class operators on H. For a real-valued measurable function $f(x, t)$ on $B\times(0,\infty)$ we define

$$L_{x,t}f(x, t) \equiv \text{trace}\,[A(x)f''(x, t)]-(\partial/\partial t)f(x, t) \qquad (0<t<\infty)$$

whenever the right-hand side exists—that is, whenever $(\partial/\partial t)f(x, t)$ and $f''(x, t)$ exist and $[A(x)f''(x, t)]$ is trace class. When there is no danger of confusion, we will omit the subscripts on L. We assume that $B(x)$ satisfies the following hypotheses:

(a-1) $x\to B(x)$ is a bounded Lip-1 function from B to the space of symmetric trace class operators on H, with the trace class norm.

(a-2) There exists $\varepsilon_0>0$ such that $B(x)\leqq(1-\varepsilon_0)I$ for all $x\in B$.

(a-3) There exists a symmetric Hilbert-Schmidt class operator E on H and a family of operators $B_0(x) \in L(H, H)$ such that for all $x \in B$, $B(x) = EB_0(x)E$ and $|B_0(x)| \leqq 1$.

(a-4) $D^2B_0(x)$ exists and is a Lip-1 function from B to $L(B \to L(B \to L(H, H)))$.

(a-5) $\|DB_0(x)\|$, $\|D^2B_0(x)\|$ are uniformly bounded for all $x \in B$.

(a-6) There exists a constant c such that for any orthonormal basis $\{e_i : i = 1, 2, \ldots\}$ of H we have $\sum_{i=1}^{\infty} |DB_0(x)e_i|^2 < c$, independently of $x \in B$.

REMARKS. (1) Without loss of generality we may assume that $\varepsilon_0 < 1$.

(2) (a-6) is always satisfied if B is the completion of H with respect to a measurable norm of the form $\|y\| = |Sy|$ for all $y \in H$, where S is a Hilbert-Schmidt operator on H. For then $\sum_{i=1}^{\infty} |DB_0(x)e_i|^2$ is dominated for all x by a constant times $\sum_{i=1}^{\infty} \|e_i\|^2$ (by (a-5)), and we have

$$\sum_{i=1}^{\infty} \|e_i\|^2 = \sum_{i=1}^{\infty} |Se_i|^2 = (\text{the Hilbert-Schmidt norm of } S)^2.$$

(3) The argument given on p. 107 of [12] for the operator denoted there as $(C'(x)(\cdot)(\cdot), y)$ to be of trace class is incorrect. (a-6) is a sufficient, but by no means necessary, condition for this operator to be of trace class. We will show this in detail in the proof of (c-11) of §V.

Under the preceding hypotheses on $B(x)$ (and therefore on $A(x)$), there exists a family of finite real-valued signed Borel measures $\{q_t(x, dy) : 0 < t < \infty, x \in B\}$ on B such that if $q_t f(x) \equiv \int_B f(y) q_t(x, dy)$ then for each bounded Lip-1 function f from B to the reals we have $L_{x,t} q_t f(x) = 0$ (for all $0 < t < \infty$, $x \in B$) and $\lim_{t \to 0} q_t f(x) = f(x)$ uniformly in x.

III. **The semifinite approximation.** Consider a finite-dimensional subspace K of B of the following form: Let $y_1, \ldots, y_n$ be a set of orthonormal vectors in H^* which also lie in B^*. Let $K \equiv \operatorname{span}(y_1, \ldots, y_n)$. Then if P is the continuous extension to B of the orthogonal projection of H onto K, we have

$$Px = \sum_{i=1}^{n} \langle y_i, x \rangle y_i, \qquad (x \in B)$$

and P is a projection on B.

In order to carry out our approximations, we must make three further assumptions. They are as follows:

(a-7) There exists a sequence $\{P_n\}$ of commuting finite-dimensional projections on B, of the above form, such that $\{P_n\}$ converges strongly to the identity operator on B.

(a-8) E (see (a-3)) commutes with each P_n.

(a-9) For each $x \in B$ and P_n from (a-7), there exists a constant c_{x,P_n} such that

$$\sum_{i=1}^{\infty} |[DB_0(P_n x) - DB_0(x)]e_i|^2 < c_{x,P_n}$$

for every orthonormal basis $\{e_i\}$ of H, and $c_{x,P_n} \to 0$ as $n \to \infty$.

REMARKS. (1) By considering pairwise least upper bounds, we may assume, without loss of generality, that $P_{n+1} \supseteq P_n$.

(2) All projections which occur in this paper will be selected from this sequence, and the subscripts will be omitted—so that P will denote an arbitrary member of this sequence, corresponding to projection on the finite-dimensional subspace K (where we may consider K as a subspace of B^* or of B or of H).

(3) (a-7) is valid in the case of Wiener space. Let B be the space of real continuous functions on [0, 1] which vanish at zero and let H be that subset of B consisting of the absolutely continuous functions which have square integrable first derivatives. The inner product on H is given by

$$(x, y) \equiv \int_0^1 x'(t)\, y'(t)\, dt,$$

where $'$ denotes the first derivative with respect to t. B is the completion of H with respect to the sup norm ($\|\cdot\|_\infty$). We first construct a basis for H^* consisting of elements of B^*. For this purpose we use the Haar functions $\{\chi_n(t)\}$, which are defined by

$$\begin{aligned} \chi_1(t) &\equiv 1 && t \in [0, 1], \\ \chi_{2^n+k}(t) &\equiv \sqrt{2^n} && t \in [(k-1)/2^n, (k-\tfrac{1}{2})/2^n), \\ &\equiv -\sqrt{2^n} && t \in ((k-\tfrac{1}{2})/2^n, k/2^n], \\ &\equiv 0 && \text{otherwise in } [0, 1], \end{aligned}$$

for $n=0, 1, 2, \ldots, k=1, 2, \ldots, 2^n$. It is well known that $\{\chi_n(t)\}$ forms a complete orthonormal set in $L^2[0, 1]$ (using Lebesgue measure) ([15, p. 338]). Let $y_n(t) \equiv \int_0^t \chi_n(s)\, ds$. It is obvious that $\{y_n\}$ forms a complete orthonormal set in H^*. For $x \in H$, we have the formulas

$$\begin{aligned} \langle y_1, x\rangle &= \int_0^1 x'(t)\, dt = x(1), \\ \langle y_{2^n+k}, x\rangle &= \int_0^1 \chi_{2^n+k}(t)\, x'(t)\, dt \\ &= \int_{(k-1)/2^n}^{(k-1/2)/2^n} \sqrt{2^n} x'(t)\, dt - \int_{(k-1/2)/2^n}^{k/2^n} \sqrt{2^n} x'(t)\, dt \\ &= \sqrt{2^n}\left[2x\left(\frac{k-\frac{1}{2}}{2^n}\right) - x\left(\frac{k-1}{2^n}\right) - x\left(\frac{k}{2^n}\right)\right]. \end{aligned}$$

We may now use these formulas to define $\langle y_n, x\rangle$ for all $x \in B$. $y_n \in B^*$ since $|\langle y_n, x\rangle| \leqq 4\sqrt{2^n}\|x\|_\infty$. Moreover, for each $x \in B$,

$$x(t) = \sum_{n=1}^{\infty} \langle y_n, x\rangle y_n(t),$$

the convergence being uniform in t [1, Theorem 3]. If we now define

$$P_n x \equiv \sum_{i=1}^{n} \langle y_i, x \rangle y_i$$

we have a sequence satisfying (a-7).

(4) If B is itself a Hilbert space, with inner product [,], then $x, y \to [x, y]$ is a bilinear functional on H. Since $|[x, y]| \leqq \|x\| \cdot \|y\| \leqq c|x| \cdot |y|$ for some constant c, this bilinear form is bounded. Thus there exists a positive definite operator N on H such that $(Nx, y) = [x, y]$. $\sqrt{N}$ is completely continuous, since $|\sqrt{N}x| = [x, x]^{1/2} = \|x\|$ and the injection mapping from $H \to B$ is completely continuous [8]. Let $\{y_i\}$ be an orthonormal basis for H consisting of eigenvectors of $\sqrt{N}$, with $\{\lambda_i\}$ the corresponding sequence of eigenvalues. Each $\lambda_i > 0$ and $\lambda_i \to 0$. $\{\lambda_i^{-1} y_i\}$ forms an orthonormal basis for B. Considering y_i to be in H^*, we have

$$\langle y_i, x \rangle = (y_i, x) = (N(\lambda_i^{-2} y_i), x) = \lambda_i^{-1}[\lambda_i^{-1} y_i, x] \quad \text{for all } x \in H.$$

This formula makes sense for all $x \in B$, and so defines the unique extension of y_i to an element of B^*. Now for $x \in B$,

$$\sum_{i=1}^{n} \langle y_i, x \rangle y_i = \sum_{i=1}^{n} [\lambda_i^{-1} y_i, x] \lambda_i^{-1} y_i \to x$$

in the B-norm. Defining $P_n x \equiv \sum_{i=1}^{n} \langle y_i, x \rangle y_i$, we see that (a-7) is thus satisfied whenever B is a Hilbert space.

(5) (a-9) is satisfied whenever B is of the form defined in Remark (2) following (a-6), since in this case (a-5) gives

$$|[DB_0(Px) - DB_0(x)]y| \leqq \text{constant} \cdot \|Px - x\| \cdot \|y\|$$

for each $y \in H$.

Define

$$\begin{aligned} A^K(x) &\equiv I - PB(Px)P \qquad\qquad (x \in B) \\ &= (I-P) + (P - PB(Px)P) \\ &\equiv Q + A^P(x), \quad \text{say, where } PH = K \text{ as stated before.} \end{aligned}$$

Considering $K \subset B^*$, denote by $K^\perp$ the annihilator of K in B. Then if ν_t' denotes Gauss measure on K, and p_t'' denotes Wiener measure on $K^\perp$, we have [9, p. 131, Remark 2.2] $p_t = \nu_t' \times p_t''$ and ~~thus~~ $p_t(\cancel{x,}\ \cancel{dy}) = \nu_t'(x', dy') \times p_t''(x'', dy'')$ (for all $x \in B$, $t > 0$), where $x = x' + x''$, $y = y' + y''$, x' and $y' \in K$, x'' and $y'' \in K^\perp$.

NOTATION. If $\omega_t(x, dy)$ is a finite real-valued signed Borel measure on a space W, then for a Borel function f on W we define

$$(\omega_t f)(x) \equiv \int_W f(y)\, \omega_t(x, dy),$$

when this integral exists. (A finite Borel measure is one such that $|\omega_t(x, E)| < \infty$ for all Borel sets $E \subset W$. For a real-valued measure, finiteness is equivalent to bounded variation, by the Hahn Decomposition Theorem.)

If $f(x, t)$ is a real-valued Borel measurable function on $K \times (0, \infty)$, define

$$L^P_{x,t} f(x, t) \equiv \text{trace}_K\, [A^P(x) f_{xx}(x, t)] - \partial/\partial t\, f(x, t) \tag{1}$$

for all $t > 0$, $x \in K$, whenever the right-hand side exists. Here $f_{xx}(x, t)$ denotes the second Fréchet derivative of f with respect to x. $L^P_{x,t}$ is a parabolic operator in K. By the finite-dimensional theory [2], [3], [4], [5], [10], [11] there exists a family of functions $\{q'(t, x, y)\}$, where $t > 0$, x and $y \in K$, which satisfies

(i) $q'(t, x, y)$ is jointly continuous in t, x, and y;

(ii) $q'_{xx}(t, x, y)$ and $\partial/\partial t\, q'(t, x, y)$ exist and $L^P_{x,t} q'(t, x, y) = 0$ on $(0, \infty) \times K \times K$;

(iii) if $f(x)$ is bounded and continuous on K, then

$$\lim_{t \searrow 0} \int_K f(y) q'(t, x, y)\, dy = f(x) \qquad (x \in K)$$

where the convergence is uniform on compact subsets of K;

(iv) for any ε and $t_0 > 0$, $q'(t, x, y)$ is bounded on the set

$$\{t + |x - y| \geqq \varepsilon, 0 < t \leqq t_0\}.$$

Moreover, $q'(t, x, y)$ is unique among functions which satisfy (i)–(iv).

It is not difficult [3], [4], [5], [11] to show that the construction of a fundamental solution for the equation $L^P_{x,t} f = 0$ described in [12] (in this case K is the Hilbert space under consideration) produces a family of finite signed Borel measures $\{q'_t(x, dy)\}$ on K which are of the form $q'_t(x, dy) = q'(t, x, y)\, dy$ where $q'(t, x, y)$ satisfies properties (i)–(iv). (In $q'(t, x, y)\, dy$, the dy refers to Lebesgue measure on K.) It now follows from Dynkin [4, Chapter V] that the family $\{q'_t : t > 0\}$ forms a contraction semigroup of positive operators acting on the space $\mathscr{B}(K)$ of bounded Borel functions on K. Moreover,

$$q'(t, x, y) > 0 \qquad (t > 0,\ x \text{ and } y \in K),$$

and

$$\int_K q'(t, x, y)\, dy = 1 \qquad (t > 0,\ x \in K).$$

(The last property is found in [11].)

We define a family of finite Borel measures $\{q^K_t(x, dy)\}$ on B by

$$q^K_t(x, dy) \equiv q'_t(x', dy') \times p''_{2t}(x'', dy'') \qquad (t > 0,\ x \in B).$$

Remark. The family $\{p''_t(x, dy) : t > 0, x \in K^\perp\}$ is a fundamental solution of the heat equation $\partial/\partial t\, f(x, t) = \frac{1}{2}\, \text{trace}_{K^\perp} [f''(x, t)]$ in $K^\perp$ (see Gross [4, Theorem 3]). A straightforward change of variables shows that the factor of $\frac{1}{2}$ in the heat equation may be removed by considering the family $\{p''_{2t}(x, dy)\}$.

PROPOSITION 1. *$\{q_t^K : t>0\}$ is a contraction semigroup of positive operators acting in the space $\mathscr{B}(B)$ of bounded Borel functions on B with the* sup norm $\|\cdot\|_\infty$.

Proof. If E is a Borel set in B, of the form $E=E'\times E''$ where $E'\subset K$ and $E''\subset K^\perp$, then

$$(q_t^K\chi_E)(x) = \{(q_t'\chi_{E'})(x')\}\{(p_{2t}''\chi_{E''})(x'')\},$$

and each function on the right-hand side is a Borel function of x. The set $\mathscr{S}$ of all Borel sets E such that $q_t^K\chi_E$ is measurable is clearly closed under finite disjoint unions, and so contains the field generated by sets E of the above form $E'\times E''$. Since $\mathscr{S}$ is closed under monotone limits, it follows that $\mathscr{S}$ coincides with the σ-field of Borel sets. The set of all f for which $q_t^K f$ is a Borel function contains the characteristic functions of the Borel sets and is closed under bounded monotone limits, and thus contains all $f\in\mathscr{B}(B)$. Since $q_t^K(x, dy)$ is a probability measure for each $x\in B$ and $t>0$, we have $\|q_t^K f\|_\infty \leqq \|f\|_\infty$.

To prove the semigroup property, we note that for E of the above form

$$\begin{aligned}\int_{B;(z)}\int_{B;(y)} \chi_E(y)\cdot q_t^K(z, dy)\cdot q_s^K(x, dz) &= \int_B q_t'(z', E')\cdot p_{2t}''(z'', E'')\cdot q_s^K(x, dz)\\ &= \left\{\int_K q_t'(z', E')\cdot q_s'(x', dz')\right\}\cdot\left\{\int_{K^\perp} p_{2t}''(z'', E'')\cdot p_{2s}''(x'', dz'')\right\}\\ &= q_{s+t}'(x', E')\cdot p_{2(s+t)}''(x'', E'')\\ &= q_{s+t}^K(x, E)\\ &= \int_B \chi_E(y)\, q_{s+t}^K(x, dy).\end{aligned}$$

The set of all f for which

$$\int_{B;(z)}\int_{B;(y)} f(y)\cdot q_t^K(z, dy)\cdot q_s^K(x, dz) = \int_B f(y)\, q_{s+t}^K(x, dy)$$

is closed under finite linear combinations and under bounded monotone limits, and so by the preceding argument contains all $f\in\mathscr{B}(B)$. Thus we have established that $q_s^K(q_t^K)f=q_{s+t}^K f$ for all $f\in\mathscr{B}(B)$.

We next establish some notation and define some properties of measures. For any metric space W with metric d, let $\mathscr{B}(W)$ denote the space of all bounded real-valued Borel functions with the sup norm $\|\cdot\|_\infty$ and let $\mathscr{A}(W)$ be the space of all real bounded Lip-1 functions on W with norm $\|\cdot\|_1$ defined by

$$\|f\|_1 \equiv \|f\|_\infty+\inf\{c : |f(x)-f(y)| \leqq c\cdot d(x, y) \text{ for all } x, y\in W\}.$$

For a family $\{\omega_t(x, dy) : x\in W, t>0\}$ of finite real-valued signed Borel measures on W, we define the following properties:

(b-1) There exists a constant c, independent of t and x, such that

$$\int_W |\omega_t|(x, dy) \leqq ct^{-1/2} \quad \text{for all } x \in W,\ t > 0.$$

(Here $|\omega_t|(x, E)$ denotes the variation of $\omega_t(x, dy)$ over E for each Borel set E in W.)

(b-1(a)) Given $0 < t_0 < \infty$, there exists a constant c_{t_0}, independent of t and x, such that

$$\int_W |\omega_t|(x, dy) \leqq c_{t_0} t^{-1/2} \quad \text{for all } x \in W,\ 0 < t \leqq t_0.$$

(b-2) The map $f \to \omega_t f$ defined by $(\omega_t f)(x) \equiv \int_W f(y)\, \omega_t(x, dy)$ is a bounded linear operator on $\mathscr{B}(W)$ for each $t > 0$.

(b-3) (b-1) holds and $f \to \omega_t f$ is a bounded linear operator on $\mathscr{A}(W)$, with

$$\|\omega_t f\|_1 \leqq ct^{-1/2}\|f\|_1 \quad \text{for all } t > 0,$$

where c is given by (b-1).

(b-3(a)) (b-1(a)) holds and $f \to \omega_t f$ is a bounded linear operator on $\mathscr{A}(W)$, with

$$\|\omega_t f\|_1 \leqq c_{t_0} t^{-1/2}\|f\|_1 \quad \text{for all } 0 < t \leqq t_0,$$

where c_{t_0} is given by (b-1(a)).

(b-4) Given $0 < \delta \leqq t_0 < \infty$, there exists a constant c_{δ,t_0}, independent of f and x, such that for $\delta \leqq t_1,\ t_2 \leqq t_0$ we have

$$|(\omega_{t_1} f)(x) - (\omega_{t_2} f)(x)| \leqq c_{\delta,t_0}|t_1 - t_2| \cdot \|f\|_1$$

for all $f \in \mathscr{A}(W)$ and $x \in W$.

It is a consequence of [13, Propositions 4 and 5] that if the family $\{\omega_t(x, dy)\}$ satisfies (b-3) or (b-3(a)), then it must satisfy (b-2).

Define the family $\{m_t^K(x, dy) : t > 0, x \in B\}$ of finite Borel measures on B by

$$m_t^K(x, dy) \equiv \exp\,[-(C^K(x)(x-y), x-y)/4t]^{\sim} p_{2t}(x, dy) \tag{2}$$

where $C^K(x) \equiv [A^K(x)]^{-1} - I = [I - PB(Px)P]^{-1}PB(Px)P$. On $K^\perp$, $C^K(x)$ acts as the zero operator. K is invariant under $C^K(x)$, and, if we define

$$C^P(x) \equiv [P - PB(Px)P]^{-1}PB(Px)P \in L(K, K)$$

then $C^K(x) = C^P(x)$ on K. Thus we can write

$$\begin{aligned} m_t^K(x, dy) &= \exp\,[-(C^P(x')(x'-y'), x'-y')/4t]\, \nu'_{2t}(x', dy') \times p''_{2t}(x'', dy'') \\ &\equiv m_t^P(x', dy') \times p''_{2t}(x'', dy'') \end{aligned}$$

where $x = x' + x''$, $y = y' + y''$, x' and $y' \in K$, x'' and $y'' \in K^\perp$. We may also define

$$\begin{aligned} \hat{m}_t^K(x, dy) &\equiv [\det A^K(y)]^{-1/2} m_t^K(x, dy) \\ &= [\det A^P(Py)]^{-1/2} m_t^K(x, dy) \\ &= [\det A^P(y')]^{-1/2} m_t^P(x', dy') \times p''_{2t}(x'', dy'') \\ &\equiv \hat{m}_t^P(x', dy') \times p''_{2t}(x'', dy''). \end{aligned} \tag{3}$$

All the measures which we have defined are finite Borel measures on the appropriate spaces.

We may now "apply" $L^K_{x,t}$ to $\hat{m}^K_t(x, dy)$ as in [12, Proposition 2], obtaining a family $\{M^K_t(x, dy)\}$ of Borel measures on B satisfying (b-1)–(b-3). We observe from equation (19) of [12] that we may write

$$M^K_t(x, dy) = M^P_t(x', dy') \times p''_{2t}(x'', dy'')$$

where $M^P_t(x', dy')$ acts in K and is given by

$$\begin{aligned} M^P_t(x, dy) \equiv [\det A^P(y)]^{-1/2}\{\mathrm{trace}_K\, [A^P(x)][&-4(t)^{-1}(C^{P''}(x)(\cdot)(\cdot)(x-y), x-y) \\ &-t^{-1}(C^{P'}(x)(\cdot)(\cdot), x-y) \\ &+(16t^2)^{-1}(C^{P'}(x)(\cdot)(x-y), x-y) \\ &\otimes (C^{P'}(x)(\cdot)(x-y), x-y)] \\ &+(4t)^{-1}(C^{P'}(x)(x-y)(x-y), x-y)\} \\ \cdot \exp\,[-(C^P(x)(x-y), x-y)/4t]\cdot \nu_{2t}(x, dy) \end{aligned} \tag{4}$$

for all $t > 0$, x and $y \in K$. The symmetric operator $T \in L(K, K)$ which is denoted by $(C^{P'}(x)(\cdot)(\cdot), x-y)$ is defined by

$$(Tk_1, k_2) \equiv \tfrac{1}{2}[(C^{P'}(x)k_1k_2, x-y)+(C^{P'}(x)k_2k_1, x-y)] \quad \text{for all } k_1, k_2 \in K.$$

If we replace P by I and ν_{2t} by p_{2t} in (4) ($A^I(y) \equiv A(y)$, $C^I(x) \equiv C(x)$), then we obtain the measures $\{M_t(x, dy)\}$ of [12]. We may also replace P by I in (2) and (3), obtaining $\{m_t(x, dy)\}$ and $\{\hat{m}_t(x, dy)\}$.

PROPOSITION 2. *The family $\{q^K_t(x, dy) : t > 0, x \in B\}$ coincides with the fundamental solution of*

$$L^K_{x,t} f(x, t) \equiv \mathrm{trace}_H\, [A^K(x) f''(x, t)] - \partial/\partial t f(x, t) = 0 \tag{5}$$

obtained by the method of [12].

LEMMA 2.1. *If $\{M_t : t > 0\}$ is any family of operators on $\mathscr{B}(B)$ which satisfies an inequality of the form $\|M_t f\|_\infty \leqq Q t^{-1/2}\|f\|_\infty$ for some constant Q independent of $t > 0$ and of $f \in \mathscr{B}(B)$, then any family $\{r_t(x, dy) : t > 0, x \in B\}$ of real-valued signed Borel measures on B which satisfies*

$$r_t f(x) = M_t f(x) + \int_0^t M_{t-u}[r_u f](x)\, du \tag{6}$$

for all $f \in \mathscr{B}(B)$ and property (b-1(a)) *is unique.*

Proof. Assume that $\{r_t(x, dy)\}$ and $\{\bar{r}_t(x, dy)\}$ each satisfy (6) and (b-1(a)). Without loss of generality we may assume that the constants c_{t_0} of (b-1(a)) are the same for both families. Then for $f \in \mathscr{B}(B)$

$$r_t f(x) - \bar{r}_t f(x) = \int_0^t M_{t-u}[r_u f - \bar{r}_u f](x)\, du.$$

Now

$$\|M_{t-u}[r_u f-\bar{r}_u f]\|_\infty \leqq Q(t-u)^{-1/2}\|r_u f-\bar{r}_u f\|_\infty \quad \text{for all } 0 < u < t < \infty.$$

Thus for all $0<t\leqq t_0$ we have

$$\begin{aligned}\|r_t f-\bar{r}_t f\|_\infty &\leqq Q\int_0^t (t-u)^{-1/2}\|r_u f-\bar{r}_u f\|_\infty\, du\\ &\leqq 2c_{t_0}\|f\|_\infty Q\int_0^t (t-u)^{-1/2}u^{-1/2}\, du\\ &= 2c_{t_0}\|f\|_\infty Q\pi^{2/2}t^{2/2-1}(\Gamma(2/2))^{-1}.\end{aligned}$$

Iterating, we get

$$\begin{aligned}\|r_t f-\bar{r}_t f\|_\infty &\leqq 2c_{t_0}\|f\|_\infty Q^2\pi^{2/2}(\Gamma(2/2))^{-1}\int_0^t (t-u)^{-1/2}u^{2/2-1}\, du\\ &= 2c_{t_0}\|f\|_\infty Q^2\pi^{3/2}t^{3/2-1}(\Gamma(3/2))^{-1},\end{aligned}$$

and eventually obtain

$$\|r_t f-\bar{r}_t f\|_\infty \leqq 2c_{t_0}\|f\|_\infty Q^{n-1}\pi^{n/2}t^{n/2-1}(\Gamma(n/2))^{-1}$$

for each $n=2, 3, \ldots$. But $Q^{n-1}\pi^{n/2}t^{n/2-1}(\Gamma(n/2))^{-1}$ goes to zero as $n\to\infty$, for each $t>0$. Therefore $\|r_t f-\bar{r}_t f\|_\infty=0$ for all $t>0$, and so $r_t f=\bar{r}_t f$ for all $f\in\mathscr{B}(B)$ and in particular $r_t(x, E)=\bar{r}_t(x, E)$ for all Borel sets $E\subset B$.

Proof of Proposition 2. From the construction of $\{q'_t(x, dy)\}$ described in [12], we have the existence of a family $\{r'_t(x, dy)\}$ of measures on K satisfying properties (b-2), (b-3(a)) and (b-4) and also

$$r'_t f(x) = M^P_t f(x) + \int_0^t M^P_{t-u}[r'_u f](x)\, du \quad \text{for all } f\in\mathscr{B}(K),\ x\in K.$$

Define $r^K_t(x, dy)\equiv r'_t(x', dy')\times p''_{2t}(x'', dy'')$. We will show that

$$r^K_t f(x) = M^K_t f(x) + \int_0^t M^K_{t-u}[r^K_u f](x)\, du \tag{7}$$

for all $f\in\mathscr{B}(B)$, $x\in B$.

If $f=\chi_E$, where $E=E'\times E''$, $E'\subset K$, $E''\subset K^\perp$, then

$$\begin{aligned}M^P_t(x', E')\cdot p''_{2t}(x'', E'')&+\int_0^t\left\{\int_K r'_u(y', E')\cdot M^P_{t-u}(x', dy')\right\}\\ &\quad\cdot\left\{\int_{K^\perp} p''_{2u}(y'', E'')\cdot p''_{2(t-u)}(x'', dy'')\right\} du\\ &= \left\{M^P_t(x', E')+\int_0^t\int_K r'_u(y', E')\cdot M^P_{t-u}(x', dy')\, du\right\}\cdot p''_{2t}(x'', E'')\\ &= r'_t(x', E')\cdot p''_{2t}(x'', E'')\\ &= r^K_t\chi_E(x).\end{aligned}$$

Since the set of all $f \in \mathscr{B}(B)$ which satisfies (7) is closed under finite linear combinations and under bounded monotone limits, it follows as in the proof of Proposition 1 that this set is exactly $\mathscr{B}(B)$. Moreover

$$\int_B |r_t^K|(x, dy) \leqq \int_K |r_t^P|(x, dy),$$

and since $\{r_t^P(x, dy)\}$ satisfies (b-1(a)), we conclude that $\{r_t^K(x, dy)\}$ satisfies (b-1(a)). Thus, by Lemma 2.1, $\{r_t^K(x, dy)\}$ coincides with the family of measures constructed via the technique of the proof of Proposition 3 of [12] during the construction of the fundamental solution of (5).

To complete the proof of the proposition, we need only establish that q_t^K satisfies

$$q_t^K f(x) = \hat{m}_t^K f(x) + \int_0^t \hat{m}_{t-u}^K [r_u^K f](x)\, du \tag{8}$$

for all $f \in \mathscr{B}(B)$, $x \in B$. If E is a Borel set in B of the form $E' \times E''$ with $E' \subset K$ and $E'' \subset K^{\perp}$, then

$$\begin{aligned}
&\hat{m}_t^K \chi_E(x) + \int_0^t \hat{m}_{t-u}^K [(r_u' \times p_{2u}'')\chi_E](x)\, du \\
&\quad = \hat{m}_t^P(x', E')\cdot p_{2t}''(x'', E'') + \int_0^t \int_B r_u'(y', E')\cdot p_{2u}''(y'', E'')\cdot \hat{m}_{t-u}^K(x, dy)\cdot du \\
&\quad = \hat{m}_t^P(x', E')\cdot p_{2t}''(x'', E'') \\
&\qquad + \int_0^t \left\{\int_K r_u'(y', E')\cdot \hat{m}_{t-u}^P(x', dy')\right\}\cdot\left\{\int_{K^\perp} p_{2u}''(y'', E'')\cdot p_{2(t-u)}''(x'', dy'')\right\} du \\
&\quad = \hat{m}_t^P(x', E')\cdot p_{2t}''(x'', E'') \\
&\qquad + \left\{\int_0^t \int_K r_u'(y', E')\cdot \hat{m}_t^P(x', dy')\cdot du\right\}\cdot p_{2t}''(x'', E'') \\
&\quad = q_t' \chi_{E'}(x')\cdot p_{2t}''\chi_{E''}(x'') \\
&\quad = q_t^K \chi_E(x).
\end{aligned}$$

Since the set of all $f \in \mathscr{B}(B)$ which satisfies (8) is closed under finite linear combinations and under bounded monotone limits, it again follows that this set is exactly $\mathscr{B}(B)$. This concludes the proof of Proposition 2.

IV. **Convergence of** $\{\hat{m}_t^K(x, dy)\}$. In the work that follows we will use c to represent a general constant whose dependence may only be on the coefficient operators $A(\cdot)$ and on the relationship of the space B to the space H. That is, c will always be independent of t for any $t > 0$, independent of any space variables x, y, etc., and independent of P. All estimates and all formulas will be valid for the case $P = I$ with the obvious modifications.

LEMMA 3.1. *Let ω be a finite positive measure on a space W, and $\{f_n\}$ be a sequence of real-valued functions on W which converge almost everywhere (a.e.) to f. If f_n and f belong to $L^{1+\lambda}(\omega)$ for some $\lambda > 0$, with $\|f_n\|_{1+\lambda}$ uniformly bounded, then $f_n \to f(L^1)$.*

Proof. Define

$$g_n(x) \equiv \frac{f_n(x)-f(x)}{|f_n(x)|+|f(x)|} \qquad \text{if } |f_n(x)|+|f(x)| \neq 0,$$

$$\equiv 0 \qquad\qquad \text{otherwise,}$$

$$\int_W |f_n-f|\, d\omega = \int_W |g_n|(|f_n|+|f|)\, d\omega \leqq \|g_n\|_\tau \cdot \|\, |f_n|+|f|\, \|_{1+\lambda}$$

where $(\tau)^{-1}+(1+\lambda)^{-1}=1$. Since $|g_n(x)| \leqq 1$ for all $x \in W$ and for all n, and since ω is a finite measure, $g_n \in L^\tau \cdot g_n \to 0$ a.e., and so, by Lebesgue's Dominated Convergence Theorem, $\|g_n\|_\tau \to 0$. $\|\, |f_n|+|f|\, \|_{1+\lambda} \leqq \|f_n\|_{1+\lambda}+\|f\|_{1+\lambda} \leqq c$ (independent of n). Thus $\int_W |f_n-f|\, d\omega \to 0$.

REMARK. Rather than assume that $f_n \to f$ a.e., it suffices to assume that $g_n \to 0$ in measure (defining g_n as in the above proof). For since $|g_n|^\tau \leqq 1$, it is a standard measure-theoretic result that again we have $\|g_n\|_\tau \to 0$ for each $1 \leqq \tau < \infty$.

PROPOSITION 3. *As P converges to the identity operator on B, $\hat{m}_t^K(x, dy) \to \hat{m}_t(x, dy)$ in variation, for each $x \in B$, $t>0$.*

Proof. We must show that

$$\text{(i)} \equiv \int_B |[\det A^K(y)]^{-1/2} \exp[-(C^K(x)(x-y), x-y)/4t]$$
$$-[\det A(y)]^{-1/2} \exp[-(C(x)(x-y), x-y)/4t]|\, p_{2t}(dy)$$

converges to zero as $P \to I$.

$$\text{(i)} \leqq \int_B [\det A^K(y)]^{-1/2} |\exp[-(C^K(x)(x-y), x-y)/4t]$$
$$-\exp[-(C(x)(x-y), x-y)/4t]|\, p_{2t}(x, dy)$$
$$+\int_B |[\det A^K(y)]^{-1/2}-[\det A(y)]^{-1/2}| \cdot \exp[(C(x)(x-y), x-y)/4t] \cdot p_{2t}(x, dy)$$
$$\equiv \text{(ii)}+\text{(iii)}, \quad \text{say.}$$

Treating (iii) first, we recall that $A^K(y)=I-PB(Py)P$. $PB(Py)P$ is uniformly (in P and y) bounded in trace norm. $A^K(y)$ is uniformly (in P and y) bounded away from zero in $L(H, H)$ norm. (Note that $A^K(y) \geqq \varepsilon_0 I$ for all P and y, where ε_0 is defined in (a-2).) Applying Lemma 4.1 of Gross [6] and noting the Remark on p. 98 of [12], we find that $\{\det A^K(y)\}$ is uniformly bounded both above and away from zero, and

$$|[\det A^K(y)]-[\det A(y)]| \leqq c\|PB(Py)P-B(y)\|_{\mathrm{tr}}. \tag{9}$$

(If $T \in L(H, H)$, then $\|T\|_{\mathrm{tr}} \equiv \mathrm{trace}\,[(T^*T)^{1/2}]$.) Since $x^{-1/2}$ is Lip-1 on subsets of $(0, \infty)$ which are both bounded above and bounded away from zero, we have

$$(10) \qquad |[\det A^K(y)]^{-1/2} - [\det A(y)]^{-1/2}| \leqq c|[\det A^K(y)] - [\det A(y)]|.$$

Now if we let $\|\cdot\|_{\mathrm{H-S}}$ denote the Hilbert-Schmidt norm, we have

$$\begin{aligned} \|PB(Py)P - B(y)\|_{\mathrm{tr}} &\leqq \|PB(Py)P - B(Py)\|_{\mathrm{tr}} + \|B(Py) - B(y)\|_{\mathrm{tr}} \\ &= \|(P-I)B(Py)P + B(Py)(P-I)\|_{\mathrm{tr}} + \|B(Py) - B(y)\|_{\mathrm{tr}} \\ &\leqq \|QEB_0(Py)EP\|_{\mathrm{tr}} + \|EB_0(Py)EQ\|_{\mathrm{tr}} + \|B(Py) - B(y)\|_{\mathrm{tr}} \\ &\leqq c\|QE\|_{\mathrm{H-S}}\|E\|_{\mathrm{H-S}} + \|B(Py) - B(y)\|_{\mathrm{tr}}. \end{aligned}$$

Since E is Hilbert-Schmidt, $\|QE\|_{\mathrm{H-S}} \to 0$ as $P \to I$. Since $B(\cdot)$ is Lip-1, $\|B(Py) - B(y)\|_{\mathrm{tr}} \leqq c\|Py - y\|_B \to 0$ pointwise in y as $P \to I$. Thus

$$(11) \qquad \|PB(Py)P - B(y)\|_{\mathrm{tr}} \to 0$$

as $P \to I$, the convergence being pointwise in y. Combining (9), (10) and (11), we conclude that

$$(12) \qquad |[\det A^K(y)]^{-1/2} - [\det A(y)]^{-1/2}| \to 0$$

as $P \to I$, the convergence being pointwise in y. It is shown in [12, p. 99] that

$$\exp\,[-(C(x)(x-y), x-y)/4t] \in L^{1+\lambda}(p_{2t}(x, \cdot))$$

for all positive λ which are sufficiently close to zero. For such a λ, the $L^{1+\lambda}$-norm is uniformly bounded with respect to x and t. Thus

$$(\mathrm{iii}) \leqq c\left\{\int_B |[\det A^K(y)]^{-1/2} - \det\,[A(y)]^{-1/2}|^{(1+\lambda)/\lambda} \cdot p_{2t}(x, dy)\right\}^{\lambda/(1+\lambda)}$$

The preceding integrand is uniformly (in P and y) bounded above, and converges pointwise in y to zero as $P \to I$. Thus, by Lebesgue's Dominated Convergence Theorem, (iii) $\to 0$ as $P \to I$. We note that the convergence is not necessarily uniform in x nor in t.

Turning now to (ii), we make the change of variables $y \to x + 2\sqrt{t}y$ and note that the determinant term is uniformly (in P, x and t) bounded above, obtaining

$$(\mathrm{ii}) \leqq c \int_B |\exp\,[-(C^K(x)y, y)] - \exp\,[-(C(x)y, y)]|\, p_{1/2}(dy).$$

Now

$$\begin{aligned} \|C^K(x) - C(x)\|_{\mathrm{tr}} &\leqq \|[(I - PB(Px)P)^{-1} - (I - B(x))^{-1}]B(x)\|_{\mathrm{tr}} \\ &\quad + \|(I - PB(Px)P)^{-1}[PB(Px)P - B(x)]\|_{\mathrm{tr}} \\ &\equiv (\mathrm{iv}) + (\mathrm{v}), \quad \text{say}. \end{aligned}$$

$(I-PB(Px)P)^{-1}$ is uniformly (in P and x) bounded in $L(H, H)$ norm. It now follows from (11) that (v) $\to 0$ as $P \to I$, pointwise in x. Writing

$$
\begin{aligned}
(I-PB(Px)P)^{-1} &- (I-B(x))^{-1} \\
&= (I-PB(Px)P)^{-1}(I-B(x))(I-B(x))^{-1} \\
&\quad -(I-PB(Px)P)^{-1}(I-PB(Px)P)(I-B(x))^{-1} \\
&= (I-PB(Px)P)^{-1}[(I-B(x))-(I-PB(Px)P)](I-B(x))^{-1} \\
&= (I-PB(Px)P)^{-1}[PB(Px)P-B(x)](I-B(x))^{-1},
\end{aligned} \tag{13}
$$

we find that

$$
\begin{aligned}
\text{(iv)} &\leqq c\|[PB(Px)P-B(x)]C(x)\|_{\mathrm{tr}} \\
&\leqq c\|PB(Px)P-B(x)\|_{\mathrm{tr}}|C(x)|_{L(H,H)} \\
&\leqq c\|PB(Px)P-B(x)\|_{\mathrm{tr}},
\end{aligned}
$$

and the right-hand side of this inequality converges pointwise to zero as $P \to I$. Thus $\|C^K(x)-C(x)\|_{\mathrm{tr}} \to 0$ as $P \to I$, pointwise in x. We use [6, Lemma 1.2] to evaluate

$$
\begin{aligned}
\int_B |(C^K(x)y, y)-(C(x)y, y)|p_{1/2}(dy) & \\
&\leqq \int_B |\,|C^K(x)-C(x)|^{1/2}y|^2 p_{1/2}(dy) \\
&= \tfrac{1}{2}(\text{Hilbert-Schmidt norm of } |C^K(x)-C(x)|^{1/2})^2 \\
&= \tfrac{1}{2}\|C^K(x)-C(x)\|_{\mathrm{tr}}^2.
\end{aligned}
$$

Thus $(C^K(x)y, y)^\sim \to (C(x)y, y)^\sim$ in mean $(p_{1/2})$. Since we are in a finite measure space we also have convergence in probability $(p_{1/2})$. Now for any two real numbers a and b,

$$
\left.
\begin{aligned}
\left|\frac{e^a-e^b}{e^a+e^b}\right| &= \left|\frac{e^d}{e^a+e^b}\right| \cdot |a-b| \quad \text{for some } d \text{ between } a \text{ and } b \\
&\leqq |a-b|.
\end{aligned}
\right\}
$$

Therefore if $|(e^a-e^b)/(e^a+e^b)| > \varepsilon$, then $|a-b| > \varepsilon$. Consequently,

$$
\begin{aligned}
p_{1/2}\left\{\left|\frac{\exp[-(C^K(x)y, y)]-\exp[-(C(x)y, y)]}{\exp[-(C^K(x)y, y)]+\exp[-(C(x)y, y)]}\right|^\sim > \varepsilon\right\} & \\
\leqq p_{1/2}\{|(C^K(x)y, y)^\sim-(C(x)y, y)^\sim| > \varepsilon\}, &
\end{aligned}
$$

showing that

$$
\left|\frac{\exp[-(C^K(x)y, y)]-\exp[-(C(x)y, y)]}{\exp[-(C^K(x)y, y)]+\exp[-(C(x)y, y)]}\right|^\sim
$$

converges to zero in probability for each $x \in B$. Since

$$
\|\exp[-(C^K(x)y, y)]\|_{1+\lambda} = \det[(I+(1+\lambda)C^K(x))^{-1/2}]
$$

for λ sufficiently small and positive, the calculation on p. 99 of [12] shows that $\{\|\exp[-(C^K(x)y, y)]\|_{1+\lambda}\}$ is uniformly bounded with respect to P and x. By Lemma 3.1 and the remark which follows it, we conclude that (ii) $\to 0$ as $P \to I$, the convergence being independent of t but not necessarily of x. This concludes the proof of Proposition 3.

V. **Estimates on the coefficients.** We again note that unless specified otherwise all estimates will be valid for the case $P=I$ $(K=H)$ with the obvious modifications.

(c-1) There exists a symmetric Hilbert-Schmidt class operator F on H and a family of operators $C_0^K(x) \in L(H, H)$ such that for all $x \in B$, $C^K(x)=FC_0^K(x)F$ and $|C_0^K(x)| \leqq 1$. F is independent of P (i.e. of K).

We follow the proof of c-2) of [12]. Since the operator E of (a-3) commutes with P it is easy to see that such an F exists for each P. However, to see that F can be chosen independently of P, we will go through the necessary calculations.

If P' is chosen from our special family of projections, and if $Q' \equiv I-P'$, then

$$\begin{aligned} I-PB(Px)P &= I-(P'+Q')PB(Px)P(P'+Q') \\ &= [I-P'PB(Px)PP']-[P'PB(Px)PQ'+Q'PB(Px)P]. \end{aligned}$$

Since $|P'PB(Px)PQ'+Q'PB(Px)P| \leqq c|EQ'|$, we may choose P' to satisfy

$$|P'PB(Px)PQ'+Q'PB(Px)P| \leqq (1-\varepsilon_0)\varepsilon_0.$$

The ε_0 used above is the ε_0 of hypothesis (a-2). Factoring out $[I-P'PB(Px)PP']$, we obtain

$$I-PB(Px)P = [I-P'PB(Px)PP'][I-D^K(x)] \tag{14}$$

where

$$D^K(x) \equiv [I-P'PB(Px)PP']^{-1}E[P'PB_0(Px)PQ'+Q'PB_0(Px)P]E.$$

$|D^K(x)| \leqq 1-\varepsilon_0$. Also, for $y_1, y_2 \in H$,

$$|(D^K(x)y_1, y_2)| \leqq 2|Ey_1|\cdot|E[I-P'PB(Px)PP']^{-1}y_2|.$$

Now

$$\begin{aligned} &|E[I-P'PB(Px)PP']^{-1}y_2|^2 \\ &\qquad = |E[I-P'PB(Px)PP']^{-1}P'y_2+E[I-P'PB(Px)PP']^{-1}Q'y_2|^2 \\ &\qquad \leqq |E|^2\varepsilon_0^{-2}|P'y_2|^2+|EQ'y_2|^2 \qquad (15) \\ &\qquad \leqq |E_1y_2|^2 \end{aligned}$$

where $E_1 \equiv 2[|E|\varepsilon_0^{-1}P'+EQ']$. Since E_1 is symmetric and of Hilbert-Schmidt class and $2|Ey_1| \leqq |E_1 y_1|$, it follows from [6, Lemma 4.2] that we may write $D^K(x) = E_1 D_0^K(x) E_1$, where $|D_0^K| \leqq 1$. Expanding,

$$\begin{aligned}
[I - D^K(x)]^{-1} &= I + \lim_{n\to\infty} \sum_{i=1}^{n} [D^K(x)]^i \\
&= I + E_1\left\{ D_0^K(x) + \lim_{n\to\infty} \sum_{i=0}^{n} D_0^K(x) E_1 [D^K(x)]^i E_1 D_0^K(x) \right\} E_1 \qquad (16)\\
&\equiv I + E_1 D_1^K(x) E_1, \quad \text{say,}
\end{aligned}$$

where $|D_1^K(x)| \leqq 1 + |E_1|^2 \varepsilon_0^{-1}$. Thus

$$\begin{aligned}
|(C^K(x)y_1, y_2)| &= |([I - D^K(x)]^{-1}[I - P'PB(Px)PP']^{-1}PB(Px)Py_1, y_2)| \\
&\leqq |(B_0(Px)PEy_1, PE[I - P'PB(Px)PP']^{-1}y_2)| \\
&\quad + |(D_1^K(x)E_1[I - P'PB(Px)PP']^{-1}PEB_0(Px)PEy_1, E_1 y_2)| \\
&\leqq |Ey_1| \cdot |E_1 y_2| + [1 + |E_1|^2 \varepsilon_0^{-1}] \cdot |E_1| \cdot \varepsilon_0^{-1} \cdot |E| \cdot |Ey_1| \cdot |E_1 y_2| \\
&\leqq |aEy_1| \cdot |E_1 y_2|
\end{aligned}$$

where $a \equiv 1 + [1 + |E_1|^2 \varepsilon_0^{-1}] \cdot |E_1| \cdot \varepsilon_0^{-1} \cdot |E|$. (c-1) now follows on applying Lemma 4.2 of [6] together with the argument following b-5) of [12].

(c-2) There exist families of operators $C_1^K(x) \in L(B \to L(H, H))$ and $C_2^K(x) \in L(B \to L(B \to L(H, H)))$ such that for all $x, z, z_1, z_2 \in B$ we have $DC^K(x)z = FC_1^K(x)zF$ and $D^2C^K(x)z_1z_2 = FC_2^K(x)z_1z_2F$ with $\|C_1^K(x)\| \leqq 1$ and $\|C_2^K(x)\| \leqq 1$.

$DC^K(x)z$ is given for all $x, z \in B$ by the formula

$$\begin{aligned}
DC^K(x)z &= [I - PB(Px)P]^{-1}P[DB(Px)z]P[I - PB(Px)P]^{-1}PB(Px)P \\
&\quad + [I - PB(Px)P]^{-1}P[DB(Px)z]P \\
&= [I - PB(Px)P]^{-1}P[DB(Px)z]P[(I - PB(Px)P]^{-1}.
\end{aligned}$$

We note for future reference that $DC^K(x)z$ depends only on Pz, since $B(Px)$ depends only on Px and so $DB(Px)z = 0$ for all $z \in K^\perp$. Moreover, $DC^K(x)z$ acts as the zero operator on $K^\perp$, and K is invariant under $DC^K(x)z$. For each $y_1, y_2 \in H$ we have

$$|(DC^K(x)zy_1, y_2)| \leqq |DB_0(Px)z| \cdot |E[I - PB(Px)P]^{-1}y_1| \cdot |E[I - PB(Px)P]^{-1}y_2|.$$

From (14) and (16) we obtain

$$\begin{aligned}
E[I - PB(Px)P]^{-1} &= E[I - D^K(x)]^{-1}[I - P'PB(Px)PP']^{-1} \\
&= E[I - P'PB(Px)PP']^{-1} + EE_1 D_1^K(x) E_1 [I - P'PB(Px)PP']^{-1}.
\end{aligned}$$

It now follows from (15) and from a similar estimate for $|E_1[I - P'PB(Px)PP']^{-1}y|$ that there exists a symmetric Hilbert-Schmidt class operator F_1 on H such that

$$|(DC^K(x)zy_1, y_2)| \leqq |F_1 y_1| \cdot |F_1 y_2| \cdot \|z\|.$$

Without loss of generality we may assume that $F_1 = F$. Lemma 4.2 of [6] now gives the desired result for $DC^K(x)z$.

The calculations for $D^2C^K(x)$ follow without difficulty from the above estimates.

(c-3) As $P \to I$, $|[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}| \to 0$ pointwise for all $x \in B$.

From (13) we have

$$[I-PB(Px)P]^{-1}-[I-B(x)]^{-1} = [I-PB(Px)P]^{-1}[PB(Px)P-B(x)][I-B(x)]^{-1}.$$

Since $|[I-PB(Px)P]^{-1}|$ is uniformly (in x and P) bounded, we have

$$\begin{aligned} |[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}| &\leqq c|PB(Px)P-B(x)| \\ &\leqq c\|PB(Px)P-B(x)\|_{\mathrm{tr}} \end{aligned}$$

and the right-hand side of the previous inequality converges to zero as $P \to I$, by (11).

(c-4) As $P \to I$, $|P[DB(Px)z]P-DB(x)z| \to 0$ pointwise for $x \in B$ and uniformly for z varying over a bounded set in B.

Let $Q \equiv I-P$. Then

$$\begin{aligned} &|P[DB(Px)z]P-DB(x)z| \\ &\quad \leqq |P[DB(Px)z]P-DB(Px)z|+|DB(Px)z-DB(x)z| \\ &\quad = |[P-I][DB(Px)z]P+[DB(Px)z][P-I]|+|E[DB_0(Px)z-DB_0(x)z]E| \\ &\quad \leqq |QE[DB_0(Px)z]EP|+|E[DB_0(Px)z]EQ|+c\|DB_0(Px)-DB_0(x)\|\cdot\|z\| \\ &\quad \leqq c\|z\|\cdot|EQ|+c\|Px-x\|\cdot\|z\| \end{aligned}$$

by (a-5), and the right-hand side of the above inequality converges to zero as $P \to I$, the convergence being pointwise in x and uniform on bounded sets of $\|z\|$.

(c-5) As $P \to I$, $|DC^K(x)z-DC(x)z| \to 0$ pointwise for $x \in B$ and uniformly for z varying over a bounded set in B.

Noting that

$$\begin{aligned} |DC^K(x)z-DC(x)z| = |&[I-PB(Px)P]^{-1}P[DB(Px)z]P[I-PB(Px)P]^{-1} \\ &-[I-B(x)]^{-1}DB(x)z[I-B(x)]^{-1}|, \end{aligned}$$

the result follows immediately from (c-3) and (c-4).

(c-6) As $P \to I$, $|P[D^2B(Px)z_1z_2]P-D^2B(x)z_1z_2| \to 0$ pointwise for $x \in B$ and uniformly for z_1 and z_2 varying over bounded sets in B.

Setting $Q \equiv I-P$, we have

$$\begin{aligned} &|P[D^2B(Px)z_1z_2]P-D^2B(x)z_1z_2| \\ &\quad \leqq |[P-I][D^2B(Px)z_1z_2]P+[D^2B(Px)z_1z_2][P-I]| \\ &\qquad +|D^2B(Px)z_1z_2-D^2B(x)z_1z_2| \\ &\quad \leqq |QE[D^2B_0(Px)z_1z_2]EP|+|E[D^2B_0(Px)z_1z_2]EQ| \\ &\qquad +|E[D^2B_0(Px)-D^2B_0(x)]z_1z_2E| \\ &\quad \leqq c|QE|\cdot\|z_1\|\cdot\|z_2\|+c\|D^2B_0(Px)-D^2B_0(x)\|\cdot\|z_1\|\cdot\|z_2\| \\ &\quad \leqq c\|z_1\|\cdot\|z_2\|\cdot[|QE|+\|Px-x\|]. \end{aligned}$$

$|QE| \to 0$ as $P \to I$, and $\|Px-x\| \to 0$, pointwise in x, as $P \to I$.

(c-7) As $P \to I$, $|D^2C^K(x)z_1z_2 - D^2C(x)z_1z_2| \to 0$ pointwise for $x \in B$ and uniformly for z_1 and z_2 varying over bounded sets in B.

We have the formula

$$\begin{aligned} D^2C^K(x)z_1z_2 - D^2C(x)z_1z_2 = {} & [I-PB(Px)P]^{-1} \\ & \cdot\{P[DB(Px)z_2]P[I-PB(Px)P]^{-1}P[DB(Px)z_1]P \\ & +P[D^2B(Px)z_1z_2]P \\ & +P[DB(Px)z_1]P[I-PB(Px)P]^{-1}P[DB(Px)z_2]P\}[I-PB(Px)P]^{-1} \\ & -[I-B(x)]^{-1}\{[DB(x)z_2][I-B(x)]^{-1}[DB(x)z_1]+[D^2B(x)z_1z_2] \\ & +[DB(x)z_1][I-B(x)]^{-1}[DB(x)z_2]\}[I-B(x)]^{-1}. \end{aligned}$$

(c-7) now follows on using estimates (c-3), (c-4) and (c-6).

The next set of estimates that we shall make will be integral estimates. The measurability of most of the functions which we will use follows from either [12, Lemma 1] or else from the following lemma.

LEMMA 4.1. *Let M be a normed linear space with norm $|\cdot|_M$ and let $H^{(N)} \equiv H \times H \times \cdots \times H$ (N times). Let $f(y_1, \ldots, y_N)$: $H^{(N)} \to M$ be linear in each y_i. If $|f(y_1, \ldots, y_N)|_M \leqq c|y_1|_1 \cdot |y_2|_2 \cdot \cdots \cdot |y_N|_N$ where $|\cdot|_i$ is a measurable seminorm on H ($i=1, \ldots, N$), then $G(y) \equiv |f(y, \ldots, y)|_M$ is u.c.n. 0 in H_m.*

The proof of this lemma is similar to that of Lemma 1 of [12], once we observe that

$$|G(y)-G(z)| \leqq |f(y, y, \ldots, y)-f(z, z, \ldots, z)|_M.$$

(c-8) As $P \to I$, $\int_B |([C^{K'}(x)-C'(x)]yy, y)|p_{2t}(dy) \to 0$ pointwise for $x \in B$ and $t>0$.

Set $Q \equiv I-P$ and write

$$\begin{aligned} (C'(x)yy, y) = {} & ([C'(x)](Py)(Py), Py)+([C'(x)](Py)(Py), Qy) \\ & +([C'(x)](Py)(Qy), y)+([C'(x)](Qy)y, y), \end{aligned}$$

valid for each $y \in H$. Noting that $(C^{K'}(x)yy, y)^{\sim} = ([C^{K'}(x)](Py)(Py), Py)$ for a.e. $y \in B$, we have

$$\begin{aligned} & \int_B |([C^{K'}(x)-C'(x)]yy, y)|p_{2t}(dy) \\ & \qquad \leqq \int_{PH} |([C^{K'}(x)-C'(x)](Py)(Py), Py)|\, \nu_{2t}(dy) \\ & \qquad\quad + \int_B (\|Py\| \cdot |FPy| \cdot |FQy| + \|Py\| \cdot |FQy| \cdot |Fy|)\, p_{2t}(dy) \\ & \qquad\quad + \int_B \|Qy\| \cdot |Fy|^2 \cdot p_{2t}(dy) \\ & \qquad \equiv \text{(i)}+\text{(ii)}+\text{(iii)}, \quad \text{say.} \end{aligned}$$

The integrand of (iii) is bounded by $c\|y\|\cdot|Fy|^{2^{\sim}}$ for a.e. $y\in B$ and moreover it converges to zero a.e. on B as $P\to I$. Thus (iii) $\to 0$ as $P\to I$ by Dominated Convergence.

$$\begin{aligned}
\text{(ii)} &\leqq c\int_B \|y\|\cdot|Fy|\cdot|FQy|\cdot p_{2t}(dy)\\
&\leqq c\left\{\int_B (\|y\|\cdot|Fy|)^2 p_{2t}(dy)\right\}^{1/2}\left\{\int_B |FQy|^2 p_{2t}(dy)\right\}^{1/2}\\
&\leqq c\{\text{trace }[(I-P)F]^2\}^{1/2}\\
&\to 0 \text{ as } P\to I.
\end{aligned}$$

Let S_n denote the sphere in H with center at the origin and with radius n. We have

$$\begin{aligned}
\int_{PH/PS_n} &|([C^{K'}(x)-C'(x)](Py)(Py), Py)|\,\nu_{2t}(dy)\\
&\leqq c\int_{PH/PS_n}\|Py\|\cdot|FPy|^2\cdot\nu_{2t}(dy)\\
&\leqq c\left\{\int_{PH}\|Py\|^2\nu_{2t}(dy)\right\}^{1/2}\left\{\int_{PH/PS_n}|FPy|^4\nu_{2t}(dy)\right\}^{1/2}\\
&= c\left\{\int_B\|Py\|^2 p_{2t}(dy)\right\}^{1/2}\left\{\int_{PH/PS_n}|FPy|^4\nu_{2t}(dy)\right\}^{1/2}\\
&\leqq c\left\{\int_B\|y\|^2 p_{2t}(dy)\right\}^{1/2}\left\{\int_{PH/PS_n}|FPy|^4\nu_{2t}(dy)\right\}^{1/2}\\
&\leqq c\left\{\int_{PH/PS_n}|FPy|^4\nu_{2t}(dy)\right\}^{1/2}.
\end{aligned}$$

We claim that the right-hand side of the previous inequality converges to zero as $n\to\infty$, the convergence being independent of P. Since PH is invariant under F (all P's commute with F), we can choose a basis $\{e_i : i=1,\ldots,m\}$ for PH consisting of eigenvectors of F. Let $\lambda_i\equiv Fe_i$ and $y\equiv\sum_{i=1}^m y_ie_i$. Then

$$\begin{aligned}
\int_{PH/PS_n} &|FPy|^4\nu_{2t}(dy)\\
&= \frac{1}{(4\pi t)^{m/2}}\int_{PH/PS_n}\left(\sum_{i=1}^m \lambda_i^2y_1^2\right)^2 \exp\left(-\sum_{i=1}^m y_i^2/4t\right)dy_1\cdots dy_m\\
&= (4\pi t)^{-m/2}\int_{PH/PS_n}\left(\sum_{i,j=1}^m \lambda_i^2\lambda_j^2y_i^2y_j^2\right)\exp\left(-\sum_{i=1}^m y_i^2/4t\right)dy_1\cdots dy_m.
\end{aligned}$$

Given $\varepsilon>0$, for each $i=1,\ldots,m$,

$$\begin{aligned}
(4\pi t)^{-m/2}&\int_{PH/PS_n}\lambda_i^4y_i^4\exp\left(-\sum_{j=1}^m y_j^2/4t\right)dy_1\cdots dy_m\\
&\leqq (4\pi t)^{-1/2}\lambda_i^4\int_{|y|\geqq n} y^4\exp(-y^2/4t)\,dy
\end{aligned}$$

and $\int_{|y|\geqq n} y^2 \exp(-y^2/4t)\,dy$ can be made $<\varepsilon(4\pi t)^{1/2}$ for n sufficiently large. Similarly,

$$(4\pi t)^{-m/2}\int_{PH/PS_n} \lambda_i^2\lambda_j^2 y_i^2 y_j^2 \exp\left(-\sum_{k=1}^{m} y_i^2/4t\right) dy_1\cdots dy_m$$
$$\leqq \lambda_i^2\lambda_j^2\left((4\pi t)^{-1/2}\int_{|y|\geqq n} y^2\exp(-y^2/4t)\,dy\right)^2$$

where $\int_{|y|\geqq n} y^2\exp(-y^2/4t)\,dy$ can be made $<(4\pi t\varepsilon)^{1/2}$ for n sufficiently large. Thus

$$\int_{PH/PS_n} |FPy|^4\nu_{2t}(dy) \leqq \varepsilon\sum_{i,j=1}^{m}\lambda_i^2\lambda_j^2$$
$$= \varepsilon\left(\sum_{i=1}^{m}\lambda_i^2\right)^2 \leqq \varepsilon(\text{trace } F^2)^2,$$

for all n sufficiently large and for all P.

Now for any fixed n we may make use of estimate (c-5) to choose P_0 so that $P\supseteq P_0$ implies that

$$|([C^{K'}(x)-C'(x)]yy, y)| < \varepsilon$$

for all $y\in S_n$. Then

$$\int_{PS_n} |([C^{K'}(x)-C'(x)](Py)(Py), Py)|\,\nu_{2t}(dy) < \varepsilon$$

and so (i) $\to 0$ as $P\to I$, pointwise for $x\in B$. This concludes the proof of (c-8).

(c-9) As $P\to I$,

$$\int_B \|([C^{K''}(x)-C''(x)](\cdot)(\cdot)y, y)\|_{\text{tr}}p_{2t}(dy)\to 0$$

pointwise for $x\in B$ and $t>0$.

Set $Q\equiv I-P$ and write

$$(C''(x)(\cdot)(\cdot)y, y) = (C''(x)(\cdot)(\cdot)(Py), Py) + (C''(x)(\cdot)(\cdot)(Py), Qy)$$
$$+(C''(x)(\cdot)(\cdot)(Qy), y),$$

valid for each $y\in H$. Since

$$\|(C^{K''}(x)(\cdot)(\cdot)y, y)\|_{\tilde{\text{tr}}} = \|(C^{K''}(x)(\cdot)(\cdot)(Py), Py)\|_{\text{tr}}$$

for a.e. $y\in B$, we have

$$\int_B \|([C^{K''}(x)-C''(x)](\cdot)(\cdot)y, y)\|_{\text{tr}}p_{2t}(dy)$$
$$\leqq \int_{PH} \|([C^{K''}(x)-C''(x)](\cdot)(\cdot)(Py), Py)\|_{\text{tr}}\nu_{2t}(dy)$$
$$+\int_B (|FPy|\cdot|FQy|+|FQy|\cdot|Fy|)\,p_{2t}(dy)$$
$$\equiv \text{(i)}+\text{(ii)},\quad \text{say.}$$

$$\begin{aligned}
\text{(ii)} &\leqq c\int_B |Fy|\cdot|FQy|\cdot p_{2t}(dy)\\
&\leqq c\left\{\int_B |Fy|^2 p_{2t}(dy)\right\}^{1/2}\left\{\int_B |FQy|^2 p_{2t}(dy)\right\}^{1/2}\\
&\leqq c\{\text{trace }((I-P)F)^2\}^{1/2}\\
&\to 0 \text{ as } P\to I.
\end{aligned}$$

Let S_n denote the sphere in H with center at the origin and with radius n. We have

$$\int_{PS_n} \|([C^{K''}(x)-C''(x)](\cdot)(\cdot)(Py), Py)\|_{\mathrm{tr}}\nu_{2t}(dy)$$

$$\begin{aligned}
&\leqq c\int_{PS_n} \|D^2(C^K(x)-C(x))\|\cdot n^2\cdot \nu_{2t}(dy)\\
&\leqq cn^2 \sup_{\|z_1\|,\ \|z_2\|\leqq 1} |D^2(C^K(x)-C(x))z_1z_2|\\
&\to 0 \quad \text{as } P\to I, \qquad \text{by estimate (c-7).}
\end{aligned}$$

Now

$$\int_{PH/PS_n} \|([C^{K''}(x)-C''(x)](\cdot)(\cdot)(Py), Py)\|_{\mathrm{tr}}\nu_{2t}(dy) \leqq c\int_{PH/PS_n} |FPy|^2\nu_{2t}(dy).$$

As in the proof of (c-8) we can show that the right-hand side of the previous inequality converges to zero as $n\to\infty$, the convergence being independent of P. It follows that (i) $\to 0$ as $P\to I$.

(c-10) As $P\to I$

$$\int_B \|(C^{K'}(x)(\cdot)y, y)\otimes(C^{K'}(x)(\cdot)y, y)-(C'(x)(\cdot)y, y)\otimes(C'(x)(\cdot)y, y)\|_{\mathrm{tr}}\, p_{2t}(dy)$$

converges to zero, pointwise for $x\in B$ and $t>0$.

For $h\in H$ we will write h^2 for $h\otimes h$. $(C^{K'}(x)(\cdot)y, y)$ is, for each $x\in B$ and $y\in H$, a bounded linear functional on H. Thus it may be identified with an element $h\in H$ with $|h|\leqq c|Fy|^2$. Noting that the constant is independent of the choice of P and that $\|h\otimes h\|_{\mathrm{tr}}=|h|^2$ (see [14]), we obtain

$$\|(C^{K'}(x)(\cdot)y, y)^2-(C'(x)(\cdot)y, y)^2\|_{\mathrm{tr}} \leqq c|Fy|^4,$$

valid for each $y\in H$. This ensures that the integral in (c-10) exists, and also that the integrand arises from a function on H which is u.c.n. 0 in H_m.

Set $Q\equiv I-P$, and for $y\in H$ write

$$(C'(x)(\cdot)y, y) = (C'(x)(\cdot)(Py), Py)+(C'(x)(\cdot)(Py), Qy)+(C'(x)(\cdot)(Qy), y).$$

Noting that $(C^{K'}(x)(\cdot)y, y) = (C^{K'}(x)(\cdot)(Py), Py)$ for each $y \in H$, we have

$$\int_B \|(C^{K'}(x)(\cdot)y, y)^2 - (C'(x)(\cdot)y, y)^2\|_{\text{tr}} p_{2t}(dy)$$

$$\leqq \int_{PH} \|(C^{K'}(x)(\cdot)(Py), Py)^2 - (C'(x)(\cdot)(Py), Py)^2\|_{\text{tr}} \nu_{2t}(dy)$$

$$+ \int_B \|(C'(x)(\cdot)(Py), Py) \otimes [(C'(x)(\cdot)(Py), Qy) + (C'(x)(\cdot)(Qy), y)]$$

$$+ [(C'(x)(\cdot)(Py), Qy) + (C'(x)(\cdot)(Qy), y)]^2$$

$$+ [(C'(x)(\cdot)(Py), Qy) + (C'(x)(\cdot)(Qy), y)]$$

$$\otimes (C'(x)(\cdot)(Py), Py)\|_{\text{tr}} p_{2t}(dy)$$

$$\equiv \text{(i)} + \text{(ii)}, \quad \text{say.}$$

Using the fact that $\|h_1 \otimes h_2\|_{\text{tr}} = |h_1| \cdot |h_2|$ for each $h_1, h_2 \in H$, we obtain

$$\text{(ii)} \leqq c \int_B \{|FPy|^2[|FPy| \cdot |FQy| + |FQy| \cdot |Fy|]$$

$$+ [|FPy| \cdot |FQy| + |FQy| \cdot |Fy|]^2$$

$$+ [|FPy| \cdot |FQy| + |FQy| \cdot |Fy|] \cdot |FPy|^2\} p_{2t}(dy)$$

$$\leqq c \int_B [|Fy|^3 |FQy| + |Fy|^2 |FQy|^2] p_{2t}(dy)$$

$$\leqq c \int_B |Fy|^3 |FQy| p_{2t}(dy)$$

$$\leqq c \left\{\int_B |Fy|^6 p_{2t}(dy)\right\}^{1/2} \left\{\int_B |FQy|^2 p_{2t}(dy)\right\}^{1/2}$$

$$\leqq c\{\text{trace}\,((I-P)F)^2\}^{1/2}$$

$$\to 0 \text{ as } P \to I.$$

For $h_1, h_2 \in H$ we have

$$\|h_1^2 - h_2^2\|_{\text{tr}} = \|(h_1 - h_2) \otimes h_1 + h_2 \otimes (h_1 - h_2)\|_{\text{tr}} \leqq (|h_1| + |h_2|) \cdot |h_1 - h_2|.$$

Thus

$$\text{(i)} \leqq c \int_{PH} |FPy|^2 \cdot \sup_{h \in H;\, |h| = 1} \{[C^{K'}(x) - C'(x)]h(Py), Py)\} \cdot \nu_{2t}(dy).$$

Using (c-5) and proceeding in a manner similar to that used for (i) of (c-8) or (c-9), we find that (i) $\to 0$ as $P \to I$, the convergence being pointwise for $x \in B$.

(c-11) As $P \to I$,

$$\int_B \|(C^{K'}(x)(\cdot)(\cdot), y) - (C'(x)(\cdot)(\cdot), y)\|_{\text{tr}} p_{2t}(dy) \to 0,$$

pointwise for $x \in B$ and $t > 0$.

We recall that for each $y \in H$ the operators $T^K \equiv (C^{K'}(\cdot)(\cdot), y)$ are defined for all $h_1, h_2 \in H$ by

$$(T^K h_1, h_2) = \tfrac{1}{2}[(C^{K'}(x)h_1 h_2, y)+(C^{K'}(x)h_2 h_1, y)].$$

Each T^K is symmetric and of Hilbert-Schmidt class. The symmetry is obvious from the definition; to demonstrate the Hilbert-Schmidt property we will show that T^K is a sum of two Hilbert-Schmidt class operators. If $\{e_i\}$ is any orthonormal basis for H, then

$$\sum_{i=1}^{\infty} \sup_{h\in H;\,|h|=1} |(C^{K'}(x)he_i, y)|^2 = \sum_{i=1}^{\infty} \sup_{h\in H;\,|h|=1} |(FC_1^K(x)hFe_i, y)|^2$$
$$\leqq c|Fy|^2\cdot\sum_{i=1}^{\infty} |Fe_i|^2 < \infty$$

and

$$\sum_{i=1}^{\infty} \sup_{h\in H;\,|h|=1} |(C^{K'}(x)e_i h, y)|^2$$
$$= \sum_{i=1}^{\infty} \sup_{h\in H;\,|h|=1} |([I-PB(Px)P]^{-1}PDB(Px)e_iP[I-B(Px)P]^{-1}h, y)|^2$$
$$\leqq \sum_{i=1}^{\infty} |E[I-PB(Px)P]^{-1}y|^2\cdot|DB_0(Px)e_i|^2\cdot \sup_{h\in H;\,|h|=1} |E[I-PB(Px)P]^{-1}h|^2$$
$$\leqq c_y\cdot\sum_{i=1}^{\infty} |DB_0(Px)e_i|^2$$

where c_y is a constant depending only on y. The last sum is finite, by hypothesis (a-6).

To show now that each T^K is of trace class, it now suffices to show that $\sum_{i=1}^{\infty} |(T^K e_i, e_i)| < \infty$ for each $\{e_i\}$. We have

$$|(T^K(x)e_i, e_i)| = |([I-PB(Px)P]^{-1}EP[DB_0(Px)e_i]PE[I-PB(Px)P]^{-1}e_i, y)|$$
$$\leqq |DB_0(Px)e_i|\cdot|E[I-PB(Px)P]^{-1}e_i|\cdot|E[I-PB(Px)P]^{-1}y|$$

where

$$|E[I-PB(Px)P]^{-1}y| = |E[I-P'PB(Px)PP'][I-D^K(x)]y| \quad \text{by (14)}$$
$$\leqq |E[I-P'PB(Px)PP']y|+c|D^K(x)y|$$
$$\leqq |E_1 y|+c|Ey|$$

by (15) and the definition of $D^K(x)$. E_1 and c are independent of P, x and y, and both E_1 and E are of Hilbert-Schmidt class. Thus there exists a Hilbert-Schmidt class operator $E_2 \in L(H, H)$ such that for all $y \in H$ we have $|E[I-PB(Px)P]^{-1}y| \leqq |E_2 y|$. Then

$$\begin{aligned}\sum_{i=1}^{\infty} |(T^K(x)e_i, e_i)| &\leqq \sum_{i=1}^{\infty} |DB_0(Px)e_i| \cdot |E_2 e_i| \cdot |E_2 y| \\ &\leqq |E_2 y| \cdot \left\{\sum_{i=1}^{\infty} |DB_0(Px)e_i|^2\right\}^{1/2} \cdot \left\{\sum_{i=1}^{\infty} |E_2 e_i|^2\right\}^{1/2} \\ &\leqq c|E_2 y|\end{aligned}$$

by (a-6). It follows that $\|T^K(x)\|_{\mathrm{tr}} \leqq c|E_2 y|$. This ensures that the integrand in (c-11) exists and that it arises from a function which is u.c.n. 0 in H_m.

We now estimate the integrand by

$$\begin{aligned}\|(C^{K'}(x)(\cdot)(\cdot), y)-(C'(x)(\cdot)(\cdot), y)\|_{\mathrm{tr}} &\leqq \|(C^{K'}(x)(\cdot)(\cdot), Py)-(C'(x)(\cdot)(\cdot), Py)\|_{\mathrm{tr}} \\ &\quad + \|(C'(x)(\cdot)(\cdot), Qy)\|_{\mathrm{tr}} \\ &\equiv \text{(i)}+\text{(ii)}, \quad \text{say.}\end{aligned}$$

If we can show that (i) $\to 0$ as $P \to I$, pointwise for each $x \in B$ and uniformly for y in a bounded set in H, then the techniques employed in proving (c-8)–(c-10) will give us (c-11). Note that for any orthonormal basis $\{e_i\}$ of H

$$\begin{aligned}&\sum_{i=1}^{\infty} |(C^{K'}(x)e_i e_i, Py)-(C'(x)e_i e_i, Py)| \\ &\quad \leqq |Py| \cdot \sum_{i=1}^{\infty} |[[I-PB(Px)P]^{-1}P[DB(Px)e_i]P[I-PB(Px)P]^{-1} \\ &\qquad\qquad -[I-B(x)]^{-1}DB(x)e_i[I-B(x)]^{-1}]e_i|.\end{aligned}$$

It suffices to show that the latter sum converges (for each $x \in B$) to zero as $P \to I$, and that the convergence is independent of the choice of basis $\{e_i\}$. This sum may be written as

$$\begin{aligned}&\sum_{i=1}^{\infty} \Big|\{[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}\}P[DB(Px)e_i]P[I-PB(Px)P]^{-1}e_i \\ &\qquad +[I-B(x)]^{-1}\{P[DB(Px)e_i]P-DB(x)e_i\}[I-PB(Px)P]^{-1}e_i \\ &\qquad\qquad +[I-B(x)]^{-1}DB(x)e_i\{[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}\}e_i\Big| \\ &\leqq c|[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}| \cdot \sum_{i=1}^{\infty} |DB_0(Px)e_i| \cdot |E[I-PB(Px)P]^{-1}e_i| \\ &\quad +c\sum_{i=1}^{\infty} |[DB_0(Px)-DB_0(x)]e_i| \cdot |E[I-PB(Px)P]^{-1}e_i| \\ &\quad +c\sum_{i=1}^{\infty} |DB_0(x)e_i| \cdot |E\{[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}\}e_i| \\ &\equiv \text{(iii)}+\text{(iv)}+\text{(v)}, \quad \text{say.}\end{aligned}$$

(c-3), (a-6) and the fact that E is Hilbert-Schmidt give us the result that (iii) $\to 0$ as $P \to I$, independent of the choice of $\{e_i\}$; (a-9) gives us the same result for (iv); and (v) follows from (a-6) and the calculation

$$\left\{\sum_{i=1}^{\infty} |E\{[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}\}e_i|^2\right\}^{1/2}$$
$$\leqq |[I-PB(Px)P]^{-1}-[I-B(x)]^{-1}|_{L(H,H)}\|E\|_{\text{H-S}}$$
$$\to 0 \quad \text{as } P \to I, \qquad \text{by (c-3).}$$

VI. **Convergence of** $\{M_t^K(x, dy)\}$, $\{r_t^K(x, dy)\}$ **and** $\{q_t^K(x, dy)\}$.

PROPOSITION 4. *As $P \to I$, $M_t^K(x, dy) \to M_t(x, dy)$ in variation norm, for each $x \in B$, $t>0$.*

Proof. It follows from (4) that we may write

$$\begin{aligned} M_t^K(x, dy) = [\det A^K(x+y)]^{-1/2}\cdot\{\text{trace } [A^K(x)]\cdot[(-4t)^{-1}(C^{K''}(x)(\cdot)(\cdot)y, y) \\ +t^{-1}(C^{K'}(x)(\cdot)(\cdot), y) \\ +(16t^2)^{-1}(C^{K'}(x)(\cdot)y, y)^2] \\ +(-4t)^{-1}(C^{K'}(x)yy, y)\}^{\sim} \\ \cdot\exp[-(C^K(x)y, y)/4t]^{\sim}\cdot p_{2t}(dy). \end{aligned}$$

This formula, with the obvious modifications, also holds for $P=I$. To simplify the notation in this proof, we will fix x and t and write

$$a(y) \equiv [\det A(x+y)]^{-1/2}, \qquad b \equiv A(x),$$
$$d(y) \equiv (-4t)^{-1}(C''(x)(\cdot)(\cdot)y, y)+t^{-1}(C'(x)(\cdot)(\cdot), y)+(16t^2)^{-1}(C'(x)(\cdot)y, y)^2,$$
$$e(y) \equiv (-4t)^{-1}(C'(x)yy, y), \qquad f(y) \equiv \exp[-(C(x)y, y)/4t],$$

and $a^K, \ldots, f^K$ for the above functions with A and C replaced by A^K and C^K respectively. We must show that

$$\int_B |a^K[\text{trace }(b^Kd^K)+e^K]f^K-a[\text{trace }(bd)+e]f|\, p_{2t}(dy) \to 0$$

as $P \to I$. The integrand may be written as

$$\begin{aligned} |[a^K-a][\text{trace }(bd)+e]f+a^K\{\text{trace }[(b^K-b)d]+\text{trace }[b^K(d^K-d)]\}f^K \\ +a^K(e^K-e)f^K+a^K[\text{trace }(bd)+e][f^K-f]|^{\sim} \\ \leqq \{|a^K-a|\cdot|\text{trace }(bd)+e|\cdot|f|+|a^K|\cdot|b^K-b|_{L(H,H)}\cdot\|d\|_{\text{tr}}\cdot|f^K| \\ +|a^K|\cdot|b^K|_{L(H,H)}\cdot\|d^K-d\|_{\text{tr}}\cdot|f^K|+|a^K|\cdot|e^K-e|\cdot|f^K| \\ +|a^K|\cdot|\text{trace }(bd)+e|\cdot|f^K-f|\}^{\sim}. \end{aligned}$$

From [12, pp. 98–99] we find that for any $\lambda>0$

$$\int_B |f^K|^{1+\lambda}p_{2t}(dy) = \det[(I+(1+\lambda)C^K(x))^{-1/2}].$$

Moreover there exists an $\varepsilon_1 > 0$, independent of P and x, such that for $\lambda < \varepsilon_1$ we have $[I+(1+\lambda)C^K(x)] > 0$ and $|[I+(1+\lambda)C(x)]^{-1}|$ is uniformly bounded for all $x \in B$. Thus, by Lemma 4.1 of [6], for any fixed $\lambda < \varepsilon_1$, $\int_B |f^K|^{1+\lambda} p_{2t}(dy)$ is uniformly bounded for $x \in B$, for all P, and for all $t > 0$. For $\lambda < \varepsilon_1$ we claim that $|f^K - f| \to 0$ in $L^{1+\lambda}(p_{2t})$ as $P \to I$. This is equivalent to saying that $|g^K - g| \to 0$ in $L^{1+\lambda}(p_{1/2})$, where $g^K \equiv \exp[-(C^K(x)y, y)]$ and $g \equiv g^H$. To prove the latter, we write

$$\begin{aligned} \int_B |g^K - g|^{1+\lambda} p_{1/2}(dy) \\ &= \int_B \left| \frac{g^K - g}{|g^K| + |g|} \right|^{1+\lambda} (|g^K| + |g|)^{1+\lambda} p_{1/2}(dy) \\ &\leqq \left\{ \int_B \left| \frac{g^K - g}{|g^K| + |g|} \right|^{(1+\lambda)\tau} p_{1/2}(dy) \right\}^{1/\tau} \left\{ \int_B (|g^K| + |g|)^{(1+\lambda)(1+\rho)} p_{1/2}(dy) \right\}^{1/(1+\rho)} \end{aligned}$$

where $(\tau)^{-1} + (1+\rho)^{-1} = 1$. $(|g^K| + |g|) \in L^{(1+\lambda)(1+\rho)}$ if $(1+\lambda)(1+\rho) < 1 + \varepsilon_1$. This condition is satisfied iff $\lambda + \rho(1+\lambda) < \varepsilon_1$ or, equivalently, iff $\rho < (\varepsilon_1 - \lambda)(1+\lambda)^{-1}$. Since $(\varepsilon_1 - \lambda) > 0$, we can always find such a ρ. The result now follows once we observe that $|(g^K - g)/(|g^K| + |g|)|^{\sim}$ is bounded by 1 a.e. on B and converges to zero in probability by a previous calculation for the proof of Proposition 3.

$a^K \to a$ pointwise for all $y \in B$. Since $|a^K| \leqq c$, it follows that $|a^K - a| \to 0$ in $L^q(p_{2t})$ for all $1 \leqq q < \infty$ and for all $t > 0$. $b^K \to b$ in $L(H, H)$ norm, $|b^K|_{L(H,H)}$ is uniformly bounded with respect to P, and thus, since b^K is independent of $y \in B$ we have $|b^K - b|_{L(H,H)} \to 0$ in $L^q(p_{2t})$ for all $1 \leqq q < \infty$ and for all $t > 0$.

$\|d^K - d\|_{\text{tr}} \to 0$ in $L^1(p_{2t})$ by (c-9), (c-10) and (c-11), and $|e^K - e| \to 0$ in $L^1(p_{2t})$ by (c-8). The L^1 convergence of these functions to zero implies their L^q convergence to zero for any $1 \leqq q < \infty$. For if we have a sequence of functions $\{g^K\}$, say, such that $|g^K|$ is dominated a.e. for each K by a function h which is in $L^q(p_{2t})$ for each $1 \leqq q < \infty$, then

$$\int_B |g^K - g| p_{2t}(dy) \to 0$$

is equivalent to $|g^K - g| \to 0$ in probability (p_{2t}) which, in turn, is equivalent to $|g^K - g|^q \to 0$ in probability (p_{2t}) and this is equivalent to

$$\int_B |g^K - g|^q p_{2t}(dy) \to 0.$$

It remains to check that d^K and e^K are dominated by the correct type of function. This follows immediately from

$$\|d^K\|_{\text{tr}} \leqq (4t)^{-1}|Fy|^2 + ct^{-1}|E_2 y| + (16t^2)^{-1}|Fy|^4,$$

and

$$|e^K| \leqq (4t)^{-1}\|y\| \cdot |Fy|^2.$$

Proposition 4 follows from the above estimates and several applications of Hölder's inequality.

PROPOSITION 5. *As $P \to I$, $r_t^K(x, dy) \to r_t(x, dy)$ in variation norm, for each $x \in B$, $t>0$.*

Proof. We may write

$$r_t^K(x, dy) = \sum_{n=1}^{\infty} r_{t,n}^K(x, dy)$$

where $r_{t,1}^K(x, dy) \equiv M_t^K(x, dy)$ and $r_{t,n}^K(x, dy)$ is defined for $n \geqq 2$ by

$$r_{t,n}^K(x, dy) \equiv \int_0^t \int_B \int_B f(z) \cdot r_{u,n-1}^K(y, dz) \cdot M_{t-u}^K(x, dy) \cdot du$$

for all $f \in \mathscr{B}(B)$, $x \in B$ and $t>0$. From the estimates of [12] it is easy to see that there exists a constant Q, independent of P, x and t, such that

$$\int_B |M_t^K|(x, dy) \leqq Qt^{-1/2}$$

and

$$\int_B |r_{t,n}^K|(x, dy) \leqq Q^n \pi^{n/2} t^{n/2-1}/\Gamma(n/2). \tag{17}$$

Thus, given $\varepsilon>0$, there exists an $N(\varepsilon)$, independent of P and x (but dependent on t) such that

$$\int_B |r_t^K - r_t|(x, dy) \leqq \sum_{n=1}^{N(\varepsilon)} \int_B |r_{t,n}^K - r_{t,n}|(x, dy) + \varepsilon. \tag{18}$$

Now $r_{t,1}^K(x, dy) = M_t^K(x, dy)$, so $r_{t,1}^K(x, dy) \to r_{t,1}(x, dy)$ in variation by Proposition 4. If we assume that $r_{t,k}^K(x, dy) \to r_{t,k}(x, dy)$ in variation for all $k \leqq n-1$, $x \in B$ and $t>0$, then

$$\begin{aligned}
&\int_B |r_{t,n}^K - r_{t,n}|(x, dy) \\
&\quad \leqq \int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_{u,n-1}^K(y, dz) \cdot M_{t-u}^K(x, dy) - r_{u,n-1}(y, dz) \cdot M_{t-u}(x, dy)|\, du \\
&\quad \leqq \int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_{u,n-1}^K - r_{u,n-1}|(y, dz) \cdot |M_{t-u}|(x, dy) \cdot du \\
&\qquad\qquad + \int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_{u,n-1}^K|(y, dz) \cdot |M_{t-u}^K - M_{t-u}|(x, dy) \cdot du \\
&\quad \equiv \text{(i)} + \text{(ii)}, \quad \text{say.}
\end{aligned}$$

We consider (i). $\int_B |r_{u,n-1}^K - r_{u,n-1}|(y, dz) \to 0$ as $P \to I$ for each $u>0$ and $y \in B$. Moreover, by (17)

$$\int_B |r_{u,n-1}^K - r_{u,n-1}|(y, dz) \leqq 2\, \frac{Q^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\, u^{n/2-3/2}$$

for all P. Since

$$\int_0^t \int_B u^{n/2-3/2} \cdot |M_{t-u}|(x, dy) \cdot du \leqq Q \int_0^t u^{n/2-3/2} \cdot (t-u)^{-1/2} \cdot du$$

and the integral on the right-hand side exists, we conclude by Dominated Convergence that (i) $\to 0$ as $P \to I$.

We consider (ii).

$$\text{(ii)} \leqq \frac{Q^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^t \int_B |M_{t-u}^K - M_{t-u}|(x, dy) \cdot u^{n/2-3/2} \cdot du.$$

Now $\int_B |M_{t-u}^K - M_{t-u}|(x, dy) \to 0$ for each $u > 0$ and moreover

$$\int_B |M_{t-u}^K - M_{t-u}|(x, dy)$$

is dominated (for all P and for all $x \in B$) by $2Q(t-u)^{-1/2}$, which is integrable from 0 to t with respect to $u^{(n-3)/2}\,du$. Thus (ii) $\to 0$ as $P \to I$, and so $r_{t,k}^K(x, dy) \to r_{t,k}(x, dy)$ for each $k = 1, 2, \ldots$. The proposition now follows from (18).

Proposition 6. *As $P \to I$, $q_t^K(x, dy) \to q_t(x, dy)$ in variation norm, for each $x \in B$, $t > 0$.*

Proof.

$$\begin{aligned}
&\int_B |q_t^K - q_t|(x, dy) \\
&\quad \leqq \int_B |\hat{m}_t^K - \hat{m}_t|(x, dy) \\
&\qquad + \int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_u^K(y, dz) \cdot \hat{m}_{t-u}^K(x, dy) - r_u(y, dz) \cdot \hat{m}_{t-u}(x, dy)| \cdot du \\
&\quad \equiv \text{(i)} + \text{(ii)}, \quad \text{say.}
\end{aligned}$$

Now (i) $\to 0$ as $P \to I$ by Proposition 3, and

$$\begin{aligned}
\text{(ii)} \leqq &\int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_u^K - r_u|(y, dz) \cdot |\hat{m}_{t-u}|(x, dy) \cdot du \\
&+ \int_{0;(u)}^t \int_{B;(y)} \int_{B;(z)} |r_u^K|(y, dz) \cdot |\hat{m}_{t-u}^K - \hat{m}_{t-u}|(x, dy) \cdot du.
\end{aligned}$$

From [12, Proposition 3], $\int_B |r_t^K|(x, dy) \leqq c_{t_0} t^{-1/2}$, where c_{t_0} is a constant independent of t and of x. c_{t_0} can easily be seen to be independent of P. Also $\int_B |\hat{m}_t^K|(x, dy)$ is dominated by a constant, independent of P, x and t. By an argument similar to that of Proposition 5 we may now establish that (ii) $\to 0$ as $P \to I$.

Corollary 6.1. *For each $t > 0$, $x \in B$, $q_t(x, dy)$ is a probability measure on B.*

This is an immediate consequence of Proposition 6.

VII. The semigroup property.

THEOREM 1. *For* $f \in \mathscr{B}(B)$, *s and* $t>0$, $q_t q_s f(x) = q_{t+s} f(x)$.

Proof. Let $f \in \mathscr{B}(B)$, and for a finite signed Borel measure μ on B let $\|\mu\|$ denote the total variation of μ. Denoting the measure $q_t(x, dy)$ by $q_{t,x}(dy)$ we have

$$|(q_t q_s - q_{t+s})f(x)| \leqq |q_t(q_s - q_s^K)f(x)| + |(q_t - q_t^K)q_s^K f(x)| + |(q_{t+s}^K - q_{t+s})f(x)|,$$

where

$$|q_t(q_s - q_s^K)f(x)| \leqq \|f\|_\infty \int_{B;(y)} \int_{B;(z)} |q_s - q_s^K|(y, dz) \cdot |q_t|(x, dy).$$

Now $\int_B |q_s - q_s^K|(y, dz)$ converges to 0 as $P \to I$ pointwise for $y \in B$. Moreover this term is dominated by 2. It now follows by Dominated Convergence that $|q_t(q_s - q_s^K)f(x)| \to 0$ as $P \to I$. Considering the two remaining terms, we have

$$|(q_t - q_t^K)q_s^K f(x)| \leqq \|q_{t,x} - q_{t,x}^K\| \cdot \|q_s^K f\|_\infty \leqq \|q_{t,x} - q_{t,x}^K\| \cdot \|f\|_\infty$$

and

$$|(q_{t+s}^K - q_{t+s})f(x)| \leqq \|q_{t+s,x}^K - q_{t+s,x}\| \cdot \|f\|_\infty,$$

both of which converge to zero as $P \to I$ by Proposition 6.

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