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ON THE SOLUTIONS OF A CLASS OF LINEAR SELFADJOINT DIFFERENTIAL EQUATIONS

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Abstract. Let L be a linear selfadjoint ordinary differential operator with coefficients which are real and sufficiently regular on $(-\infty, \infty)$. Let A^+ (A^-) denote the subspace of the solution space of $Ly=0$ such that $y \in A^+$ ($y \in A^-$) iff $D^k y \in L^2[0, \infty)$ ($D^k y \in L^2(-\infty, 0]$) for $k=0, 1, \ldots, m$ where $2m$ is the order of L. A sufficient condition is given for the solution space of $Ly=0$ to be the direct sum of A^+ and A^-. This condition which concerns the coefficients of L reduces to a necessary and sufficient condition when these coefficients are constant. In the case of periodic coefficients this condition implies the existence of an exponential dichotomy of the solution space of $Ly=0$.

1. **Introduction.** The object of study of this paper is the general linear homogeneous selfadjoint differential equation which for convenience we shall write in the form

$$\sum_{k=0}^{m} (-1)^k D^k a_k D^k y = 0, \tag{1}$$

where $D^k y \equiv d^k y/dt^k$.

Except when otherwise stated we will assume throughout that for each $k=0, 1, \ldots, m$, $a_k(t)$ is real valued, $a_k \in C^k(-\infty, \infty)$ and $a_m(t) \neq 0$ for all $t \in (-\infty, \infty)$.

The motivation for this paper comes from the case when $a_k(t) = c_k = \text{constant}$, $k=0, 1, \ldots, m$. In this case the solutions of (1) are determined entirely by the zeros of the polynomial

$$p(\lambda) = \sum_{k=0}^{m} (-1)^k c_k \lambda^{2k}. \tag{2}$$

Since only even powers of λ appear in p it follows that if $\mu \neq 0$ is a zero of p of multiplicity r then $-\mu$ is also a zero of p of multiplicity r and the functions $t^j e^{\mu t}$, $t^j e^{-\mu t}$, $j=0, 1, \ldots, r-1$, form a set of $2r$ linearly independent solutions of (1). Consequently if $p(\lambda)$ has no zero or purely imaginary roots and S denotes the set of solutions of (1) considered as a complex vector space of dimension $2m$, then S has a simple geometrical description. Namely, if E^+ denotes the subspace of S consisting

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of solutions of (1) which together with their derivatives tend to zero exponentially as $t \to \infty$ and E^- denotes the subspace of S consisting of solutions of (1) which together with their derivatives tend to zero as $t \to -\infty$ exponentially then dimension $E^+ =$dimension $E^- = m$, dimension $E^+ \cap E^- = 0$. Therefore S will split into the direct sum of E^+ and E^-.

The objective of this paper is to give a partial extension of this simple observation to a class of equations of the form (1) with variable coefficients. For simplicity we will only consider real solutions. Henceforth S will denote the set of real solutions of (1) considered as a real vector space of dimension $2m$.

THEOREM 1. *Assume that for each $k=0, 1, \ldots, m$, $a_k(t)$ is bounded below on $(-\infty, \infty)$ and define*

$$c_k = \inf a_k(t). \tag{3}$$

Let A^+ and A^- denote the subspaces of S defined by

$$A^+ = \left\{ v \in S \,\middle|\, \begin{matrix} D^k v \in L^2[0, \infty) \\ 0 \leqq k \leqq m \end{matrix} \right\}, \tag{4}$$

$$A^- = \left\{ v \in S \,\middle|\, \begin{matrix} D^k v \in L^2(-\infty, 0] \\ 0 \leqq k \leqq m \end{matrix} \right\}. \tag{5}$$

If

$$c_m > 0 \tag{6}$$

and the polynomial p defined by (2) *has no zero or purely imaginary roots then*

$$\dim A^+ \geqq m, \qquad \dim A^- \geqq m.$$

If, in addition, each $a_k(t)$ is bounded above as well as below then

$$\textit{dimension } A^+ = \textit{dimension } A^- = m$$

and

$$\textit{dimension } A^+ \cap A^- = 0$$

so that S is the direct sum of A^+ and A^-.

If $v \in A^+$ $(v \in A^-)$ then

$$\lim_{t \to \infty} D^k v = 0 \qquad \left(\lim_{t \to -\infty} D^k v = 0\right), \qquad k = 0, 1, \ldots, m-1.$$

To the best of our knowledge the only literature connected with Theorem 1 is a remarkable paper by M. Švec [3] which deals with the fourth order equation $d^4y/dt^4 + p(t)y = 0$ where p is defined and continuous on a half-infinite interval $[c, \infty)$. Švec showed that if p is bounded below by a positive constant then there exist two linearly independent solutions of the differential equation which belong to $L^2[c, \infty)$ and tend to zero as $t \to \infty$. As an application of Theorem 2, which is similar to Theorem 1 but concerns the differential equation (1) when the a_k are only defined on a half-infinite interval $[c, \infty)$, we shall generalize Švec's result.

The proof of Theorem 1 will be deferred until after we have established some auxiliary lemmas.

2. Some preliminary lemmas.

LEMMA 2.1. *Let d_k, $k=0, 1, \dots, m$, be real numbers with the property that*

$$q(\omega) = \sum_{k=0}^{m} d_k \omega^{2k} \geqq 0$$

for all real ω. Let f be a real function of class C^{m-1} on $[-T, T]$, $T>0$, and sectionally of class C^m on this interval, i.e. there exist numbers t_j, $j=1, \dots, N-1$, such that

$$-T = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

and f is of class C^m on each of the intervals $[t_{j-1}, t_j]$, $j=1, \dots, N$. If

$$D^k f(-T) = D^k f(T) = 0, \qquad k = 0, 1, \dots, m-1,$$

then

$$\int_{-T}^{T} \sum_{k=0}^{m} d_k (D^k f(s))^2 \, ds \geqq 0.$$

Proof. If for $t \in [-\pi, \pi]$ we define $F(t)=f(tT/\pi)$ then F is of class C^{m-1} on $[-\pi, \pi]$, F is sectionally of class C^m on this interval,

$$D^k F(-\pi) = D^k F(\pi) = 0, \qquad 0 \leqq k \leqq m-1, \tag{7}$$

and

$$\int_{-T}^{T} \sum_{k=0}^{m} d_k (D^k f(s))^2 \, ds = \frac{1}{r} \int_{-\pi}^{\pi} \sum_{k=0}^{m} d_k r^{2k} (D^k F(u))^2 \, du, \tag{8}$$

where $r=\pi/T$.

For each $j=0, \pm 1, \pm 2, \dots$ let

$$\gamma_j = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} F(u) e^{-iju} \, du,$$

Integration by parts and (7) yield

$$(-ij)^k \gamma_j = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} D^k F(u) e^{-iju} \, du, \tag{9}$$

for $k=1, \dots, m-1$. Since $D^m F$ is sectionally continuous it follows by dividing the interval of integration in (9) into suitable subintervals that (9) is also true for $k=m$.

The orthonormal functions

$$(1/(2\pi)^{1/2}) e^{iju}, \qquad j = 0, \pm 1, \pm 2, \dots,$$

form a complete set in $L^2[-\pi, \pi]$, so by Parseval's formula

$$\int_{-\pi}^{\pi} (D^k F(u))^2 \, du = \sum_{j=-\infty}^{\infty} j^{2k} |\gamma_j|^2$$

for $k=0, 1, \dots, m$ ($0^0 \equiv 1$ in the above and following identity). Hence

$$\int_{-\pi}^{\pi} \sum_{k=0}^{m} d_k r^{2k} (D^k F(u))^2 \, du = \sum_{k=0}^{m} d_k r^{2k} \sum_{j=-\infty}^{\infty} j^{2k} |\gamma_j|^2$$

$$= \sum_{j=-\infty}^{\infty} |\gamma_j|^2 \sum_{k=0}^{m} d_k (jr)^{2k} = \sum_{j=-\infty}^{\infty} |\gamma_j|^2 q(rj) \geqq 0.$$

By (8), this proves the lemma.

LEMMA 2.2. *Let the real numbers $d_0, d_1, \dots, d_m$ satisfy the same hypothesis as in Lemma* 2.1. *Let f be a real valued function defined and of class C^m on the interval* $[0, T]$, $T>0$. *If*

$$D^k f(T) = 0, \qquad 0 \leqq k \leqq m-1, \tag{10}$$

and for some fixed integer j with $0 \leqq j \leqq m-1$,

$$D^k f(0) = 0, \qquad k \neq j, \; 0 \leqq k \leqq m-1, \tag{11}$$

then

$$\sum_{k=0}^{m} \int_0^T d_k (D^k f(s))^2 \, ds \geqq 0.$$

Proof. We define a function g on $[-T, T]$ as follows:
If j is an even integer

$$\begin{aligned} g(t) &= f(t), & 0 \leqq t \leqq T, \\ &= f(-t), & -T \leqq t < 0. \end{aligned}$$

If j is an odd integer

$$\begin{aligned} g(t) &= f(t), & 0 \leqq t \leqq T, \\ &= -f(-t), & -T \leqq t < 0. \end{aligned}$$

Using (11) it is easy to verify that g is of class C^{m-1} on $[-T, T]$ and sectionally of class C^m on this interval since $D^m g$ has both left-hand and right-hand limits at $t=0$. From (10) $D^k g(-T) = D^k g(T) = 0$, $0 \leqq k \leqq m-1$. Thus Lemma 1.1 is applicable and

$$\int_{-T}^{T} \sum_{k=0}^{m} d_k (D^k g(s))^2 \, ds \geqq 0.$$

But

$$\int_0^T \sum_{k=0}^{m} d_k (D^k f(s))^2 \, ds = \frac{1}{2} \int_{-T}^{T} \sum_{k=0}^{m} d_k (D^k g(s))^2 \, ds$$

and the assertion of the lemma follows.

LEMMA 2.3. *Let the real numbers $d_0, d_1, \ldots, d_m$ satisfy the same hypothesis as in Lemma* 2.1. *If $f \in C^m(-\infty, \infty)$ and $D^k f \in L^2(-\infty, \infty)$, $k=0, 1, \ldots, m$, then*

$$\int_{-\infty}^{\infty} \sum_{k=0}^{m} d_k (D^k f(s))^2 \, ds \geqq 0.$$

Proof. This result is almost an immediate consequence of Lemma 2.1. Let $\varphi(t)$ be a real valued function defined and of class C^∞ on the real line such that $\varphi(t)=1$ for $t \leqq \frac{1}{2}$ and $\varphi(t)=0$ for $t \geqq 1$. For each positive integer n let θ_n be the C^∞ function defined by

$$\begin{aligned} \theta_n(t) &= 1, & 0 &\leqq t \leqq n, \\ &= \varphi(t-n), & n &< t, \\ &= \theta_n(-t), & t &< 0. \end{aligned}$$

Let $f_n = \theta_n f$ for $n=1, 2, \ldots$. Since $D^k \theta_n$ is bounded independently of n for $0 \leqq k \leqq m$ there exists a fixed constant L such that

$$(D^k f_n)^2 \leqq L \sum_{j=0}^{m} (D^j f)^2$$

for k and n in the same range. For each fixed $t \in (-\infty, \infty)$, $\lim_{n \to \infty} D^k f_n(t) = D^k f(t)$ so by the dominated convergence theorem

$$\int_{-\infty}^{\infty} \sum_{k=0}^{m} d_k (D^k f(s))^2 \, ds = \lim_{n \to \infty} \int_{-\infty}^{\infty} \sum_{k=0}^{n} d_k (D^k f_n(s))^2 \, ds.$$

Since for each n, f_n has compact support, it follows from Lemma 2.1 that

$$\int_{-\infty}^{\infty} \sum_{k=0}^{n} d_k (D^k f_n(s))^2 \, ds \geqq 0.$$

This proves the lemma.

3. **Proof of Theorem 1.** In addition to the preliminary lemmas the proof of Theorem 1 will depend on a certain identity which we first establish.

For each solution y of (1) we define a function $F[y]$ on $(-\infty, \infty)$ by the formula

$$F[y](t) = \sum_{k=1}^{m} \sum_{j=0}^{k-1} (-1)^{j+k} (D^j y)(t)(D^{k-j-1} a_k D^k y)(t). \tag{12}$$

According to (1)

$$\int_0^t y(s) \sum_{k=0}^{m} (-1)^k (D^k a_k D^k y)(s) \, ds = 0$$

so by the integration by parts formula

$$\int_0^t y D^k z \, ds = \sum_{j=0}^{k-1} (-1)^j (D^j y)(D^{k-j-1} z) \Big]_0^t + (-1)^k \int_0^t z D^k y \, ds,$$

we obtain the important identity

$$F[y](t) = F[y](0) - \sum_{k=0}^{m} \int_0^t a_k(s)(D_k y(s))^2\, ds. \tag{13}$$

The proof of Theorem 1 will be broken up into several lemmas.

LEMMA 3.1. *Let the coefficients $a_k(t)$ be bounded below on $(-\infty, \infty)$ and assume that the numbers c_k satisfy the hypothesis of Theorem 1. Let v be a solution of (1) such that for some number $T>0$,*

$$D^k v(T) = 0, \qquad 0 \leqq k \leqq m-1, \tag{14}$$

and for some fixed integer j with $0 \leqq j \leqq m-1$,

$$D^k v(0) = 0, \qquad k \neq j, 0 \leqq k \leqq m-1. \tag{15}$$

There exists a number $M>0$ independent of both v and T, such that

$$\sum_{k=0}^{m} \int_0^T (D^k v(s))^2\, ds \leqq MF[v](0). \tag{16}$$

Proof. Since the polynomial $p(\lambda) = \sum_{k=0}^{m} (-1)^k c_k \lambda^{2k}$ has no zero or purely imaginary roots it follows that if $Q(\omega) \equiv p(i\omega) = \sum_{k=0}^{m} c_k \omega^{2k}$ then $Q(\omega) \neq 0$ for all $\omega \in (-\infty, \infty)$. According to assumption (6) $c_m > 0$ and hence

$$\lim_{\omega \to \pm\infty} Q(\omega) = +\infty. \tag{17}$$

Thus $Q(\omega) > 0$ for all real ω and in particular $Q(0) = c_0 > 0$. This together with (17) implies the existence of a number $\delta > 0$ such that if

$$d_k \equiv c_k - \delta, \qquad 0 \leqq k \leqq m, \tag{18}$$

then

$$q(\omega) \equiv \sum_{k=0}^{m} d_k \omega^{2k} \geqq 0, \qquad \omega \in (-\infty, \infty). \tag{19}$$

Now by (14) and (12) it follows that $F[v](T) = 0$ and so by (13)

$$F[v](0) = \sum_{k=0}^{m} \int_0^T a_k(s)(D^k v(s))^2\, ds.$$

From (2) $a_k(t) \geqq c_k$, $0 \leqq k \leqq m$, so by using (18) we have

$$F[v](0) \geqq \sum_{k=0}^{m} \int_0^T c_k(D^k v(s))^2\, ds = \sum_{k=0}^{m} \int_0^T d_k(D^k v(s))^2\, ds + \delta \sum_{k=0}^{m} \int_0^T (D^k v(s))^2\, ds.$$

From (14), (15) and (19) we observe that the function v and the numbers d_k satisfy the hypothesis of Lemma 2.2 and hence

$$\sum_{k=0}^{m} \int_0^T d_k(D^k v(s))^2\, ds \geqq 0.$$

The assertion of the lemma follows by setting $M=1/\delta$.

LEMMA 3.2. *Let the hypothesis of Lemma* 3.1 *hold. For each integer j with* $0\leqq j\leqq m-1$ *there exists a solution* v_j *of* (1) *such that*

$$D^k v_j \in L^2[0,\infty), \qquad 0 \leqq k \leqq m, \tag{20}$$
$$D^k v_j(0) = 0, \qquad k \neq j,\, 0 \leqq k \leqq m-1,$$

and

$$D^j v_j(0) \neq 0. \tag{21}$$

Proof. Let z_i, $0\leqq i\leqq 2m-1$, denote the solution of (1) defined by the initial conditions

$$D^k z_i(0) = \delta_{ik} = 0, \quad i \neq k, \qquad = 1, \quad i = k. \tag{22}$$

The solutions $z_0, z_1, \ldots, z_{2m-1}$ obviously form a basis for the vector space S.

Let $0\leqq j\leqq m-1$. By a well-known result of algebra, for each positive integer n there exist $m+1$ numbers, which we denote by $b_n^j, b_n^m, b_n^{m+1}, \ldots, b_n^{2m-1}$, *not all zero* such that

$$b_n^j D^k z_j(n) + \sum_{i=m}^{2m-1} b_n^i D^k z_i(n) = 0 \quad \text{for } k = 0, 1, \ldots, m-1. \tag{23}$$

By a suitable normalization we may further assume that for all $n=0, 1, 2, \ldots,$

$$(b_n^j)^2 + \sum_{i=m}^{2m-1} (b_n^i)^2 = 1. \tag{24}$$

For each positive integer n consider the solution

$$v_{jn} = b_n^j z_j + \sum_{i=m}^{2m-1} b_n^i z_i. \tag{25}$$

From (22) and (23) $D^k v_{jn}(0)=0$, $k\neq j$, $0\leqq k\leqq m-1$, $D^k v_{jn}(n)=0$, $0\leqq k\leqq m-1$. Thus if M is defined as in Lemma 3.1, it follows that for all $n=0, 1, 2, \ldots,$

$$\sum_{k=0}^{m} \int_0^n (D^k v_{jn}(s))^2\, ds \leqq MF[v_{jn}](0). \tag{26}$$

Condition (24) implies the existence of a sequence of integers $\{n_h\}$ and $m+1$ numbers $b^j, b^m, b^{m+1}, \ldots, b^{2m-1}$ such that $\lim_{h\to\infty} b_{n_h}^i = b^i$, $i=j$, $m\leqq i\leqq 2m-1$, and

$$(b^j)^2 + \sum_{i=m}^{2m-1} (b^i)^2 = 1. \tag{27}$$

We will show that the solution

$$v_j = b^j z_j + \sum_{i=m}^{2m-1} b^i z_i \tag{28}$$

fulfills the assertion of the lemma.

Fix $t>0$. Since by (25) the sequences $\{D^k v_{jn_h}\}$ converges uniformly to $D^k v_j$, $0 \leqq k \leqq m$, on bounded intervals

$$\sum_{k=0}^{m} \int_0^t (D^k v_j(s))^2 \, ds = \lim_{h\to\infty} \sum_{k=0}^{m} \int_0^t (D^k v_{jn_h}(s))^2 \, ds.$$

For $n_h \geqq t$ it follows by (26) that

$$\sum_{k=0}^{m} \int_0^t (D^k v_{jn_h}(s))^2 \, ds \leqq \sum_{k=0}^{m} \int_0^{n_h} (D^k v_{jn_h}(s))^2 \, ds \leqq MF[v_{jn_h}](0).$$

From (12), (25), and (28) we see that

$$\lim_{h\to\infty} F[v_{jn_h}](0) = F[v_j](0).$$

Hence

$$\sum_{k=0}^{m} \int_0^t (D^k v_j(s))^2 \, ds \leqq MF[v_j](0).$$

Since $t>0$ was arbitrary this implies that $D^k v_j \in L^2[0, \infty)$ for $0 \leqq k \leqq m$ and

$$\sum_{k=0}^{m} \int_0^\infty (D^k v_j(s))^2 \, ds \leqq MF[v_j](0).$$

Finally, suppose contrary to the lemma $D^j v_j(0)=0$. By (22) and (28) $D^k v_j(0)=0$, $0 \leqq k \leqq m-1$, so by (12) $F[v_j(0)]=0$. Hence

$$\sum_{k=0}^{m} \int_0^\infty (D^k v_j(s))^2 \, ds = 0$$

and $v_j(t)=0$ for all t. This, however, contradicts (27), (28) and the linear independence of the solutions $z_j, z_m, z_{m+1}, \ldots, z_{2m-1}$. Hence $D^j v_j(0) \neq 0$ and the lemma is proved.

From this lemma the first assertion of Theorem 1 follows immediately. For each j with $0 \leqq j \leqq m-1$, let v_j be the solution whose existence was established above. If v_0, v_1, v_{m-1} were not linearly independent, there would exist numbers $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$, not all zero such that

$$\sum_{j=0}^{m-1} \gamma_j v_j(t) = 0$$

for all t. But $D^k v_j(0)=0$, $k \neq j$, $0 \leqq k \leqq m-1$, $D^j v_j(0) \neq 0$, so $\gamma_j = 0, j=0, 1, \ldots, m-1$. This contradiction proves that the set $\{v_j\}_{j=0}^{m-1}$ is linearly independent and hence $\dim A^+ \geqq m$.

The proof that, under the hypothesis of Lemma 3.1, $\dim A^- \geqq m$ follows easily from the inequality $\dim A^+ \geqq m$ by means of a convenient artifice. For $k=0, 1, \ldots, m$, define functions $\tilde{a}_k(t) = a_k(-t)$, $t \in (-\infty, \infty)$. Clearly $\tilde{a}_k \in C^k(-\infty, \infty)$ and $\inf \tilde{a}_k = \inf a_k = c_k$. Therefore, by what we have just shown, there exist m linearly independent solutions $\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_{m-1}$ of the differential equation

$$(1') \qquad \sum_{k=0}^{m} (-1)^k D^k(\tilde{a}_k D^k y) = 0$$

such that $D^k\tilde{v}_j \in L^2[0, \infty)$ for $0 \leqq k \leqq m$, $0 \leqq j \leqq m-1$. If for $j=0, 1, \ldots, m-1$, $\omega_j(t) = \tilde{v}_j(-t)$, it is easy to verify that ω_j is a solution of

$$\sum_{k=0}^{m} (-1)^k D^k(a_k D^k y) = 0. \tag{1}$$

Therefore, since $D^k\omega_j \in L^2(-\infty, 0]$, $0 \leqq k \leqq m$, and the set $\{\omega_j\}_{j=0}^{m-1}$ is linearly independent, dim $A^- \geqq m$.

The second assertion of Theorem 1 is a consequence of the following:

LEMMA 3.3. *Suppose in addition to the hypothesis of Lemma* 3.1, a_k *is bounded above as well as below for* $0 \leqq k \leqq m$. *If* u *is a solution of* (1) *such that* $D^k u \in L^2(-\infty, \infty)$ *for* $0 \leqq k \leqq m$, *then* $u(t) = 0$ *for all* $t \in (-\infty, \infty)$.

Proof. Referring to the proof of Lemma 2.3 we see that there exists a sequence of function $\{u_n\}_{n=1}^{\infty}$ such that

$$u_n(t) = 0 \quad \text{if } |t| \geqq n+1, \quad u_n \in C^{2m}(-\infty, \infty), \tag{29}$$

and

$$\lim_{n \to \infty} D^k u_n = D^k u \quad \text{in } L^2(-\infty, \infty) \quad \text{for } 0 \leqq k \leqq m. \tag{30}$$

Since for $n = 1, 2, \ldots$

$$\int_{-\infty}^{\infty} u_n(s) \sum_{k=0}^{m} (-1)^k D^k(a_k D^k u)(s)\, ds = 0,$$

it follows from (29) and integration by parts that

$$\int_{-\infty}^{\infty} \sum_{k=0}^{m} a_k(s)(D^k u_n(s))(D^k u(s))\, ds = 0.$$

By the boundedness of a_k, $0 \leqq k \leqq m$, (30) implies that

$$\int_{-\infty}^{\infty} \sum_{k=0}^{m} a_k(s)(D^k u(s))^2\, ds = 0.$$

Let the numbers $d_0, d_1, \ldots, d_m$ and $\delta > 0$ be defined as in the proof of Lemma 3.1. Since $\sum_{k=0}^{m} d_k \omega^{2k} \geqq 0$, Lemma 2.3 implies that

$$\int_{-\infty}^{\infty} \sum_{k=0}^{m} d_k(D^k u(s))^2\, ds \geqq 0.$$

Therefore

$$\begin{aligned} \delta \sum_{k=0}^{m} \int_{-\infty}^{\infty} (D_k u(s))^2\, ds &\leqq \delta \sum_{k=0}^{m} \int_{-\infty}^{\infty} (D_k u(s))^2\, ds + \sum_{k=0}^{m} \int_{-\infty}^{\infty} d_k(D^k u(s))^2\, ds \\ &= \sum_{k=0}^{m} \int_{-\infty}^{\infty} c_k(D^k u(s))^2\, ds \leqq \sum_{k=0}^{m} \int_{-\infty}^{\infty} a_k(s)(D^k u(s))^2\, ds = 0 \end{aligned}$$

and so $u(t) = 0$ for all $t \in (-\infty, \infty)$.

The second assertion of Theorem 1 now follows by a well known result in algebra. Assuming the hypothesis of Lemma 3.3 we have as an equivalent statement

$$\text{dimension } A^+ \cap A^- = 0.$$

Therefore

$$\text{dimension } A^+ + \text{dimension } A^- \leqq \text{dimension } S = 2m$$

(see for example [2, §12, problem 7(b)]). But we have shown that $\dim A^+ \geqq m$, $\dim A^- \geqq m$; hence $\dim A^+ = \dim A^- = m$.

The final statement of Theorem 1 is a consequence of the following elementary fact:

LEMMA 3.4. *If $f \in C^1[0, \infty)$ and $f \in L^2[0, \infty)$, $f' \in L^2[0, \infty)$, then* $\lim_{t\to\infty} f(t) = 0$.

Proof. The hypothesis implies that $2ff' \in L^1[0, \infty)$. Therefore the identity $f(t)^2 = f(0)^2 + 2\int_0^t f(s)f'(s)\,ds$ implies that $\lim_{t\to\infty} f(t)$ exists. But $f \in L^2[0, \infty)$ so $\lim_{t\to\infty} f(t) = 0$.

This concludes the proof of Theorem 1.

4. **Equations defined on a half-infinite interval—Examples.** The following statement is actually a corollary of Theorem 1:

THEOREM 2. *Let a_k, $0 \leqq k \leqq m$, be real functions defined on the half-infinite interval $[b, \infty)$ with $a_k \in C^k$. Assume each a_k is bounded below and if $c_k = \inf a_k$, $0 \leqq k \leqq m$, then $c_m > 0$ and the polynomial* (2) *has no zero or purely imaginary roots. If A denotes the vector space of real solutions of*

$$\sum_{k=0}^{m} (-1)^k D^k(a_k D^k y) = 0 \tag{1}$$

which together with their first m derivatives belong to $L^2[b, \infty)$, then $\dim A \geqq m$. *If each a_k is bounded above as well as below on $[b, \infty)$, then* $\dim A = m$.

Proof. Let φ be a real C^∞ function defined on $(-\infty, \infty)$ such that

$$\begin{aligned} 0 \leqq \varphi(t) &\leqq 1, & t &\in (-\infty, \infty), \\ \varphi(t) &= 0, & t &\leqq b+1, \\ \varphi(t) &= 1, & t &\geqq b+2. \end{aligned} \tag{31}$$

For $k = 0, 1, \ldots, m$, define $a_k^* \in C^k(-\infty, \infty)$ by the formula

$$a_k^*(t) = [1 - \varphi(t)]c_k + \varphi(t)a_k(t).$$

Since for $k = 0, 1, \ldots, m$

$$\inf_{(-\infty,\infty)} a_k^* = \inf_{[b,\infty)} a_k = c_k,$$

Theorem 1 implies that the differential equation

$$\sum_{k=0}^{m} (-1)^k D^k(a_k^* D^k y) = 0 \tag{1''}$$

has m linearly independent solutions which together with their first m derivatives belong to $L^2[0, \infty)$. For $t \geqq b+2$ these solutions are also solutions of (1). Continuing these solutions back from $b+2$ to b we obtain m linearly independent solutions of (1) which are in A. This proves the first assertion of Theorem 2.

Suppose that each a_k is bounded above as well as below on $[b, \infty)$ and contrary to the second assertion of Theorem 2, dim $A \geqq m+1$. This clearly implies that (1″) has $m+1$ linearly independent solutions which together with their first m derivatives belong to $L^2[0, \infty)$. But if each a_k is bounded above on $[b, \infty)$ each a_k^* is bounded above on $(-\infty, \infty)$ so we have a contradiction to Theorem 1. This contradiction proves Theorem 2.

We conclude with some simple but noteworthy examples:

1. Assume that both the first and second hypothesis of Theorem 1 and in addition that each a_k is periodic with the same period $T>0$. It is known (see for example [1, Chapter 3]) that every solution of (1) can be expressed as a linear combination of solutions of the form

$$e^{\lambda t} \sum_{j=0}^{r} p_j(t) t^j \tag{32}$$

where $p_j(t+T)=p_j(t)$. The numbers λ are the *characteristic numbers* of (1). If y is a solution of (1) then $y \in A^+$ $(y \in A^-)$ if and only if in the linear combination of the solutions of the form (32) (comprising y) those solutions with Re $(\lambda) \geqq 0$ (Re $(\lambda) \leqq 0$) do not appear. Hence if E^+ (E^-) denotes the subspace of solutions tending to zero exponentially as $t \to +\infty$ $(t \to -\infty)$ it follows that $E^+ = A^+$, $E^- = A^-$. Hence by Theorem 1,

$$\text{dimension } E^+ = \text{dimension } E^- = m, \tag{33}$$

$$\text{dimension } E^+ \cap E^- = 0. \tag{34}$$

From the above discussion it also follows that if $y \in E^+$ $(y \in E^-)$ and y is not identically zero then y is unbounded on $(-\infty, 0]$ (on $[0, \infty)$). Thus since (33) and (34) imply that every solution y of (1) can be represented uniquely in the form $y = y_1 + y_2$, $y_1 \in E^+$, $y_2 \in E^-$ it follows that there exists no nontrivial solution of (1) bounded on $(-\infty, \infty)$. In particular, (1) *has no periodic solution other than the trivial one.*

2. Consider the fourth order selfadjoint differential equation

$$(ry'')'' + (qy')' + py = 0. \tag{35}$$

If $r \in C^2[b, \infty)$, $q \in C^1[b, \infty)$, $p \in C[b, \infty)$, $\inf r = R > 0$, $\sup q = Q < +\infty$, $\inf p = P > 0$, and either $Q < 0$ or $Q^2 - 4RP < 0$, then by Theorem 2, there exist two independent solutions u_k, $k = 1, 2$, of (35) such that $u_k, u_k' \in L^2[b, \infty)$, $k = 1, 2$. For the special case $r(t) = 1$, $q(t) = 0$ for all $t \in [b, \infty)$, this result was discovered by Švec [3].

3. Finally consider the classical second order selfadjoint equation

$$(ry')' + qy = 0 \tag{36}$$

where $r \in C^1[b, \infty)$, $q \in C[b, \infty)$. If

$$\sup r = R < 0, \qquad \inf q = Q > 0,$$

then by Theorem 2, (36) has a nontrivial solution u such that $u, u' \in L^2[b, \infty)$. It is easy to see that any other solution of (36) with this property must be of the form cu. Indeed if v is a solution with $v(b) > 0$, $v'(b) > 0$ then since $drvv'/dt = r(v')^2 - qv^2 < 0$, $v(t) > 0$, $v'(t) > 0$ for all $t \in [b, \infty)$. Since u and v are independent solutions of (36) any other solution y has the form $c_1 u + c_2 v$ and hence $y, y' \in L^2[b, \infty)$ if and only if $c_2 = 0$. Thus dimension $A = 1$ regardless of whether or not r is bounded below and q is bounded above.

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