EXISTENCE AND UNIQUENESS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR TWO DIMENSIONAL SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS([1])

BY

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Abstract. The paper considers the nonlinear system $x'=f(t, x, y)$, $y'=g(t, x, y)$ with linear and nonlinear two point boundary conditions. With a Lipschitz condition, an interval of uniqueness for linear boundary conditions is determined using a comparison theorem. A corresponding existence theorem is established. Under the assumption of uniqueness, a general existence theorem is established for quite general nonlinearities in the functions and in the boundary conditions. Examples are provided. The results extend previous work on second order scalar differential equations.

Introduction. Questions of the existence and uniqueness of solutions of the two point boundary value problem

$$(1.1) \qquad y'' + f(t, y, y') = 0,$$

$$(1.2) \qquad y(a) = A,$$

$$(1.3) \qquad y(b) = B,$$

where $f(t, y, y')$ satisfies a Lipschitz condition, have a long history, going back to Picard [9], 1893. The problem is to determine, in terms of the Lipschitz constants, the best possible interval $[a, b]$ on which there exists a unique solution of (1.1), (1.2), (1.3). Recent results and references on this problem can be found in [2], [3]. A second question of interest is, given that solutions of (1.1), (1.2), (1.3) are unique, i.e. given that there exists at most one solution, when does this imply that there exists at least one solution. The first results along this line are due to Lasota and Opial [6] and Jackson [3].

This paper investigates these two types of questions for the second order system

$$(1.4) \qquad x' = f_1(t, x, y), \qquad y' = f_2(t, x, y)$$

subject to more general boundary conditions

$$(1.5) \qquad g_1(x(a), y(a)) = c_1,$$

$$(1.6) \qquad g_2(x(b), y(b)) = c_2.$$

In §2 the Lipschitzian problem is treated and in §3 the question of proving existence given uniqueness is explored. The theorems are illustrated in §4. When a Lipschitz condition is assumed, as in [2], we take it to be of the form

$$(1.7) \quad \begin{aligned} L_1(x_1 - x_2) &\leqq f_1(t, x_1, y) - f_1(t, x_2, y) \leqq L_2(x_1 - x_2) && \text{if } x_1 \geqq x_2, \\ K_1(y_1 - y_2) &\leqq f_1(t, x, y_1) - f_1(t, x, y_2) \leqq K_2(y_1 - y_2) && \text{if } y_1 \geqq y_2, \\ M_1(x_1 - x_2) &\leqq f_2(t, x_1, y) - f_2(t, x_2, y) \leqq M_2(x_1 - x_2) && \text{if } x_1 \geqq x_2, \\ N_1(y_1 - y_2) &\leqq f_2(t, x, y_1) - f_2(t, x, y_2) \leqq N_2(y_1 - y_2) && \text{if } y_1 \geqq y_2. \end{aligned}$$

If $f_i(t, x, y)$, $i = 1, 2$, are continuous on $[a, b] \times R \times R$ and satisfy (1.7), solutions of the initial value problem are unique and can be continued over $[a, b]$. The advantage of using (1.7) instead of the usual Lipschitz condition with absolute value is that the added information in (1.7) can lead to sharper estimates on the allowable size of $b - a$.

Many of the arguments used in this paper take place in the $x - y$ plane. By making the substitution $x = \rho \cos \theta$, $y = \rho \sin \theta$, one obtains a new set of differential equations for ρ and θ. The functions $\theta(t)$, the *polar angle*, and $\rho(t)$, the *polar radius*, play an important role.

2. **Existence and uniqueness of solutions for Lipschitzian problems.**   Consider the boundary value problem

$$(2.1) \qquad\qquad x' = f_1(t, x, y), \qquad y' = f_2(t, x, y),$$

$$(2.2) \qquad\qquad Ax(a) + By(a) = c_1,$$

$$(2.3) \qquad\qquad Cx(b) + Dy(b) = c_2, \qquad b > a,$$

where the $f_i(t, x, y)$ satisfy a Lipschitz condition of the form (1.7). Theorem 2.1 deals with the problem of finding the best possible bound on the size of $b - a$, given the Lipschitz constants, for there to exist at most one solution. An existence theorem for this case is given as Theorem 2.2. The conditions (2.2) and (2.3) can be replaced by a class of nonlinear boundary conditions and it is shown that Theorem 2.1 can be extended to this case at the expense of increased computational complexity.

A principal tool in the approach here will be a comparison theorem of Perov [5, p. 177] (see also [8]). Solutions of (2.1) will be compared with those of an auxiliary system

$$(2.4) \qquad\qquad x' = \tilde{f}_1(t, x, y), \qquad y' = \tilde{f}_2(t, x, y),$$

where the $\tilde{f}_i(t, x, y)$, $i = 1, 2$, will be constructed from the Lipschitz constants. First of all, define $\{f_1, f_2\} \geqq \{\tilde{f}_1, \tilde{f}_2\}$ in a region D if the following inequality is satisfied there.

$$(2.5) \quad x f_2(t, x, y) - y f_1(t, x, y) \geqq x \tilde{f}_2(t, x, y) - y \tilde{f}_1(t, x, y), \qquad (t, x, y) \in D.$$

Let $\varphi(t)$ be the polar angle (see Introduction) for $(x(t), y(t))$ a solution of (2.1); $\tilde{\varphi}(t)$, the polar angle for $(\tilde{x}(t), \tilde{y}(t))$, a solution of (2.4).

Perov Comparison Theorem. *Suppose $(x(t), y(t))$ and $(\tilde{x}(t), \tilde{y}(t))$ exist over $[a, b]$, neither vector being zero there. Let $D = [a, b] \times R \times R$. Suppose $\tilde{f}_i(t, cx, cy) = c\tilde{f}_i(t, x, y)$, $c > 0$, $i = 1, 2$, and suppose that solutions of the initial value problem for (2.4) are unique. If $\{f_1, f_2\} \geq \{\tilde{f}_1, \tilde{f}_2\}$ and $\varphi(a) \geq \tilde{\varphi}(a)$, then $\varphi(t) \geq \tilde{\varphi}(t)$, $t \in [a, b]$. The inequalities may be reversed throughout.*

First, define

$$
\begin{aligned}
P_1(u, v) &= L_1 u + K_1 v && \text{if } uv \geq 0, \\
&= L_2 u + K_1 v && \text{if } uv < 0; \\
P_2(u, v) &= M_2 u + N_2 v && \text{if } uv \geq 0, \\
&= M_2 u + N_1 v && \text{if } uv < 0; \\
Q_1(w, z) &= L_2 w + K_2 z && \text{if } wz \geq 0, \\
&= L_1 w + K_2 z && \text{if } wz < 0; \\
Q_2(w, z) &= M_1 w + N_1 z && \text{if } wz \geq 0, \\
&= M_1 w + N_2 z && \text{if } wz < 0.
\end{aligned}
$$

The comparison system will be

$$
(2.6) \qquad u' = P_1(u, v), \qquad v' = P_2(u, v)
$$

and

$$
(2.7) \qquad w' = Q_1(w, z), \qquad z' = Q_2(w, z).
$$

The systems (2.6) and (2.7) can have a discontinuity in the right-hand side at a change of quadrant. If differentiability of solutions is not required at these points (left and right derivatives exist but are unequal), solutions exist and are uniquely defined by their initial conditions. Within a quadrant, the right-hand sides are linear and solutions may be computed explicitly.

Let $y = \alpha x$ and $y = \beta x$ be two lines in the $x - y$ plane. Define $p(\alpha, \beta)$ to be the time it takes for a solution of (2.6) which begins at $t = 0$ at a point on $y = \alpha x$ to first reach a point on the line $y = \beta x$ (if a solution does reach it) or to be $+\infty$ if no solution beginning on $y = \alpha x$ at $t = 0$ reaches $y = \beta x$ for any $t \geq 0$. Since (2.6) is piecewise linear and homogeneous, $p(\alpha, \beta)$ is independent of the choice of the initial condition on $y = \alpha x$ and of the initial "time" $t = 0$. This definition can be interpreted in terms of the polar angle $\omega(t)$ of (2.6), i.e. if $\omega(0) = \alpha$, $\pi/2 \geq \alpha \geq -\pi/2$, and $\omega(b) = \beta \pmod{\pi}$, $\pi/2 \geq \beta \geq -\pi/2$, where b is the first nonnegative time that $\omega(t) = \beta \pmod{\pi}$, then $p(\alpha, \beta) = b$. Since the equation has piecewise constant coefficients, $p(\alpha, \beta)$ can be determined explicitly if needed. Define $q(\alpha, \beta)$ in the same way for the system (2.7). Note that $p(\alpha, \alpha) = q(\alpha, \alpha) = 0$.

THEOREM 2.1. *Let $f_i(t, x, y)$, $i=1, 2$, be continuous on $[a, b] \times R \times R$ and satisfy
(1.7) there. If $b-a < M$, where*

$$M = \min [p(\tan^{-1} -A/B, \tan^{-1} -C/D), q(\tan^{-1} -A/B, \tan^{-1} -C/D)],$$

there is at most one solution of the boundary value problem (2.1), (2.2), (2.3). *This
result is best possible.*

Proof. Suppose there were two solutions $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ of (2.1), (2.2),
(2.3). Let $\bar{x}(t) = x_2(t) - x_1(t)$ and $\bar{y}(t) = y_2(t) - y_1(t)$ and suppose labeling is chosen
so that $\bar{x}(a) > 0$ or if $\bar{x}(a) = 0$, $\bar{y}(a) > 0$. Such a choice is possible since $(\bar{x}(a), \bar{y}(a)) \neq
(0, 0)$ by uniqueness of solutions of initial value problems. Then $(\bar{x}(t), \bar{y}(t))$ satisfies
the following:

(2.8) $x' = F_1(t, x, y), \qquad y' = F_2(t, x, y),$

(2.9) $Ax(a) + By(a) = 0,$

(2.10) $Cx(b) + Dy(b) = 0,$

where $F_i(t, x, y) = f_i(t, x + x_1(t), y + y_1(t)) - f_i(t, x_1(t), y_1(t))$, $i=1, 2$. Note that the
$F_i(t, x, y)$ satisfy the same Lipschitz condition (1.7) as the $f_i(t, x, y)$.

Let $(u(t), v(t))$ and $(w(t), z(t))$ be solutions of (2.6) and (2.7) with initial con-
ditions $(u(a), v(a)) = (w(a), z(a)) = (\bar{x}(a), \bar{y}(a))$. Let $\varphi(t)$, $\omega(t)$, $\psi(t)$ denote the polar
angles of $(\bar{x}(t), \bar{y}(t))$, $(u(t), v(t))$ and $(w(t), z(t))$, respectively. P_1, P_2, Q_1, Q_2 are
positive homogeneous and one verifies that $\{P_1, P_2\} \geq \{F_1, F_2\} \geq \{Q_1, Q_2\}$. Since
$\varphi(a) = \omega(a) = \psi(a)$ it follows from the Perov Comparison Theorem that $\omega(t) \geq
\varphi(t) \geq \psi(t)$ and in particular $\omega(b) \geq \varphi(b) \geq \psi(b)$.

Now, $b-a < M$ says that no solution of (2.6) or (2.7) which begins on a line
through the origin with slope $\tan^{-1} -A/B$ at $t=a$ can reach a line through the
origin with slope $\tan^{-1} -C/D$ at or before $t=b$. The comparison theorem says
that if neither $\omega(t)$ or $\psi(t)$ crosses a line with slope $\tan^{-1} C/D$, then neither does a
solution of (2.8), (2.9). Hence $(\bar{x}(b), \bar{y}(b))$ cannot satisfy (2.10) unless $(\bar{x}(t), \bar{y}(t)) =
(0, 0)$. Such is not the case and this contradiction establishes the theorem.

The result is "best possible" since if $b-a = M$, either (2.6) or (2.7) has a non-
trivial solution satisfying (2.9) and (2.10) as well as the trivial solution.

If the lines given by (2.2) and (2.3) are parallel, then $M=0$ and Theorem 2.1 does
not apply. That the theorem is not true in this case may be seen from the problem
$y'' + y' = 0$, $y'(a) = y'(b) = 0$ which has any constant as a solution. In the important
special case $x(a) = c_1$, $x(b) = c_2$ this can be remedied by assuming $K_1 > 0$ (which is
always the case for second order equations). Then it can be shown that at $\theta = \pm \pi/2$,
$\theta'(t) < 0$. That is, solutions cross $\theta = \pm \pi/2$ moving counterclockwise in the plane
so that nontrivial solutions of (2.8) can move from $x(a) = 0$ to $x(b) = 0$ only by
passing through an elapsed angle of π. (For $y(a) = c_1$, $y(b) = c_2$ this can be remedied
by assuming $M_2 < 0$.) Then the comparison equations can be used to determine
an interval so small that $\theta(t)$ cannot pass through such an angle.

The existence theorem corresponding to Theorem 2.1 is

THEOREM 2.2. *If the hypotheses of Theorem 2.1 hold, then there exists a solution of* (2.1), (2.2), (2.3).

Proof. Let $(x_1(t), y_1(t))$ be an arbitrary solution of (2.1), (2.2). If (2.3) is not satisfied, then either $Cx_1(b) + Dy_1(b) = \delta < c_2$ or $> c_2$. We suppose the former; a similar argument applies in the second case. The idea of the proof is to find initial conditions for a solution which lies on the opposite side of the line $Cx + Dy = c_2$. This will be done using (2.8) and (2.9). In particular, we seek a solution $(r(t), s(t))$ of (2.8), (2.9) with $Cr(b) + Ds(b) < c_2 - \delta$, so that $(x_2(t), y_2(t)) = (r(t) + x_1(t), s(t) + y_1(t))$ satisfies (2.1), (2.2) and $Cx_2(b) + Dx_2(b) > c_2$.

Let $(u(t), v(t))$, $(w(t), z(t))$ be solutions of (2.6), (2.7) with polar angles $\omega(t)$, $\psi(t)$, with $\omega(a) = \psi(a) = \tan^{-1} - A/B$. The rays $\omega(b)$ and $\psi(b)$ are not parallel to $Cx + Dy = 0$ so $\omega(b)$ and $\psi(b)$ intersect the line $Cx + Dy = c_2$ at points whose radii in polar coordinates we label R_ω and R_ψ. Let $R = \max [R_\omega, R_\psi]$. We need the following very elementary lemma.

LEMMA. *Let $\rho(t)$ be the polar radius of* (2.8). *Then $\rho(t)$ satisfies*

$$\rho(a)e^{-\sigma(t-a)} \leqq \rho(t) \leqq \rho(a)e^{\sigma(t-a)},$$

$$a \leqq t \leqq b, \qquad \sigma = \sum_{i=1}^{2} (|K_i| + |L_i| + |M_i| + |N_i|).$$

Proof of lemma. The equation for $r(t)$ is

$$r' = F_1(t, r \cos\theta, r \sin\theta) \cos\theta + F_2(t, r \cos\theta, r \sin\theta) \sin\theta.$$

Using the Lipschitz condition, (1.7), it follows that $\sigma r \geqq r' \geqq -\sigma r$ and an integration establishes the lemma.

Proof of theorem (continued). Choose an initial radius $\rho(a)$ such that $\rho(a) > Re^{\sigma|b-a|}$. Let $(r(t), s(t))$ be a solution of (2.8), (2.9) such that $r^2(a) + s^2(a) \geqq \rho^2(a)$. We know then that $\theta(t)$, the polar angle of this solution, satisfies

(2.11) $$\psi(b) \leqq \theta(b) \leqq \omega(b)$$

and the polar radius satisfies $\rho(b) \geqq R$. Any point (x, y) which satisfies

(2.12) $$Cx + Dy < c_2 - \delta$$

lies on the origin side of $Cx + Dy = c_2 - \delta$. (Note that $c_2 - \delta > 0$ in the case we are considering.) Hence any points of the sector (2.11) which satisfy (2.12) have $\rho(b) < R$. Thus

(2.13) $$Cr(b) + Ds(b) \geqq c_2 - \delta.$$

Let $x_2(t), y_2(t)$ be a solution of (2.1) with initial conditions $x_2(a) = r(a) + x_1(a)$ and $y_2(a) = s(a) + y_1(a)$. This solution satisfies (2.2) and from (2.13)

$$c_2 - \delta \leqq C(x_2(b) - x_1(b)) + D(y_2(b) - y_1(b)) = Cx_2(b) + D_2(b) - \delta$$

or

$$Cx_2(b) + Dy_2(b) \geqq c_2.$$

Let l denote the segment of $Ax+By=c_1$ between $(x_1(a), y_1(a))$ and $(x_2(b), y_2(b))$. The set of solutions of (2.1) evaluated at b given by

$$[x(b), y(b) \mid (x(a), y(a)) \in l]$$

is a connected set (by a simple strengthening of Kneser's theorem, see [12]). Thus there exists an initial condition on l, i.e., a point on l such that the solution of (2.1) with these initial values satisfies (2.3).

We indicate now an extension of Theorem 2.1 to the boundary value problem

(2.1) $$x' = f_1(t, x, y), \qquad y' = f_2(t, x, y),$$

(2.14) $$g_1(x(a), y(a)) = c_1,$$

(2.15) $$g_2(x(b), y(b)) = c_2.$$

The $g_i(x, y)$, $i = 1, 2$, are assumed to satisfy a Lipschitz condition like that of the $f_i(t, x, y)$, i.e., to satisfy

(2.16)
$$
\begin{aligned}
A_1(x-y) &\leq g_1(x, z) - g_1(y, z) \leq A_2(x-y) & \text{if } x \geq y, \\
B_1(x-y) &\leq g_1(z, x) - g_1(z, y) \leq B_2(x-y) & \text{if } x \geq y, \\
C_1(x-y) &\leq g_2(x, z) - g_2(y, z) \leq C_2(x-y) & \text{if } x \geq y, \\
D_1(x-y) &\leq g_2(z, x) - g_2(z, y) \leq D_2(x-y) & \text{if } x \geq y.
\end{aligned}
$$

The situation is considerably more complicated than that of Theorem 2.1. Let G_1 denote the region (possibly null) in the two sectors

$$
\begin{aligned}
A_2x+B_2y &\geq 0, & A_1x+B_1y &\leq 0, & x &\geq 0, y \geq 0; \\
A_2x+B_1y &\geq 0, & A_1x+B_2y &\leq 0, & x &\geq 0, y \leq 0.
\end{aligned}
$$

Let G_2 denote the union of the following four sectors:

$$
\begin{aligned}
C_2x+D_2y &\geq 0, & C_1x+D_1y &\leq 0, & x &\geq 0, y \geq 0; \\
C_2x+D_1y &\geq 0, & C_1x+D_2y &\leq 0, & x &\geq 0, y \leq 0; \\
C_1x+D_1y &\geq 0, & C_2x+D_2y &\leq 0, & x &\leq 0, y \leq 0; \\
C_1x+D_2y &\geq 0, & C_2x+D_1y &\leq 0, & x &\leq 0, y \geq 0.
\end{aligned}
$$

It will be required that $G_1 \cap G_2 - \{(0, 0)\} = \varnothing$. In the case the $g_i(x, y)$ are linear this is taken care of in the nonparallel condition on the lines inherent in the definition of M. Define

$$M = \min_{i=1,2;j=1,2} \left[p\left(\tan^{-1} -\frac{A_i}{B_j}, \tan^{-1} -\frac{C_i}{D_j}\right), q\left(\tan^{-1} -\frac{A_i}{B_j}, \tan^{-1} -\frac{C_i}{D_j}\right)\right].$$

THEOREM 2.3. *Let $f_i(t, x, y)$, $i = 1, 2$, be continuous and satisfy (1.7) in $[a, b] \times R \times R$. Let $q_i(x, y)$, $i = 1, 2$, be continuous in $R \times R$, and satisfy (2.16) there. Suppose further that $G_1 \cap G_2 - \{(0, 0)\} = \varnothing$. If $b - a < M$, there exists at most one solution of (2.1), (2.14), (2.15).*

Proof. We sketch the proof which is essentially that of Theorem 2.1. Suppose there were two solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ and as before let $(\bar{x}(t), \bar{y}(t))$ be the difference of them where $\bar{x}(a) > 0$ or $\bar{x}(a) = 0$ and $\bar{y}(a) > 0$. Then $(\bar{x}(t), \bar{y}(t))$ satisfies

$$(2.17) \qquad x' = F_1(t, x, y), \qquad y' = F_2(t, x, y),$$

where the $F_i(t, x, y)$ are as defined in Theorem 2.1. From the Lipschitz conditions (2.16)

$$A_2 \bar{x}(a) + \left\{ \begin{array}{l} B_2 \bar{y}(a) \text{ if } \bar{y}(a) \geqq 0 \\ B_1 \bar{y}(a) \text{ if } \bar{y}(a) < 0 \end{array} \right\} \geqq 0,$$

$$A_1 \bar{x}(a) + \left\{ \begin{array}{l} B_1 \bar{y}(a) \text{ if } \bar{y}(a) \geqq 0 \\ B_2 \bar{y}(a) \text{ if } \bar{y}(a) < 0 \end{array} \right\} \leqq 0,$$

$$\left\{ \begin{array}{l} C_2 \bar{x}(b) \text{ if } \bar{x}(b) \geqq 0 \\ C_1 \bar{x}(b) \text{ if } \bar{x}(b) < 0 \end{array} \right\} + \left\{ \begin{array}{l} D_2 \bar{y}(b) \text{ if } \bar{y}(b) \geqq 0 \\ D_1 \bar{y}(b) \text{ if } \bar{y}(b) < 0 \end{array} \right\} \geqq 0,$$

$$\left\{ \begin{array}{l} C_1 \bar{x}(b) \text{ if } \bar{x}(b) \geqq 0 \\ C_2 \bar{x}(b) \text{ if } \bar{x}(b) < 0 \end{array} \right\} + \left\{ \begin{array}{l} D_1 \bar{y}(b) \text{ if } \bar{y}(b) \geqq 0 \\ D_2 \bar{y}(b) \text{ if } \bar{y}(b) < 0 \end{array} \right\} \leqq 0.$$

The first two inequalities say that $(\bar{x}(a), \bar{y}(a)) \in G_1$ and the last two that $(\bar{x}(b), \bar{y}(b)) \in G_2$.

To reach G_2, $(\bar{x}(t), \bar{y}(t))$ must cross one of the lines bounding G_1 a last time $t_1 \geqq a$. Since $(\bar{x}(b), \bar{y}(b)) \in G_2$, so $(\bar{x}(t), \bar{y}(t))$ must cross one of the lines bounding G_2 for a first time $t_2 \leqq b$. M is defined so that neither angle $\omega(t)$ or $\psi(t)$ of the comparison equations (2.6) and (2.7) can pass through any angle between a boundary line of G_1 and one of G_2. Hence by the Perov Comparison Theorem neither can $\varphi(t)$, the polar angle of the system (2.17).

3. Existence from uniqueness.

In this section the boundary value problem

$$(3.1) \qquad x' = f_1(t, x, y), \qquad y' = f_2(t, x, y),$$

$$(3.2) \qquad g_1(x(a), y(a)) = c_1,$$

$$(3.3) \qquad g_2(x(b), y(b)) = c_2,$$

will be considered under the hypothesis that there is at most one solution of (3.1), (3.2), (3.3). The object is to establish an existence theorem for suitable curves (3.2) and (3.3).

The first theorems of this sort for two point boundary value problems are due to Lasota and Opial [6] and Jackson [3]. Other results of this kind may be found in [1], [10], [11], [13], [14]. The conditions on $g_i(x, y)$, $i = 1, 2$, will be strengthened as to smoothness over those imposed in §2, but there will be no condition of the form (2.16). The major additional assumption is one which relates the $g_i(x, y)$ and the $f_i(t, x, y)$. We state this as

(H) If $\{(x_n, y_n)\}$ is a sequence of points in the plane such that $g_2(x_n, y_n)$ and

$$\frac{\partial}{\partial x} g_2(x_n, y_n) f_1(t_n, x_n, y_n) + \frac{\partial g_2}{\partial y} (x_n, y_n) f_2(t_n, x_n, y_n)$$

are bounded for bounded $\{t_n\}$, then $\{(x_n, y_n)\}$ lies in a bounded region.

First it needs to be shown that (H) is satisfied in the usually studied case

$$z'' + f(t, z, z') = 0, \qquad z(a) = A, \qquad z(b) = B.$$

In this case, $g_1(x, y) = g_2(x, y) = x$, $f_1(t, x, y) = y$, and $f_2(t, x, y) = -f(t, x, y)$. $g_2(x_n, y_n)$ bounded says that x_n is bounded. Since $\partial g_2/\partial x = 1$, $\partial g_2/\partial y = 0$, the second condition becomes

$$1 \cdot y_n + 0 \cdot f_2(b, x_n, y_n) = y_n.$$

Thus the (x_n, y_n) lie in a bounded region and (H) is satisfied. A less trivial example will be given in the next section.

Condition (H) can be equivalently formulated as follows: For any bounded interval I,

$$g_2^2(x, y) + [\text{grad } g_2(x, y) \cdot f(x, y)]^2 \to \infty \quad \text{as } x^2 + y^2 \to \infty$$

uniformly with respect to $t \in I$, where $f = (f_1, f_2)$. (The author is indebted to the referee for this comment and for other suggestions to improve the clarity of presentation.)

THEOREM 3.1. *For $i = 1, 2$, let $f_i(t, x, y)$ be continuous on $[a, b+\gamma] \times R \times R$, $\gamma > 0$, and let $g_i(x, y)$ be continuously differentiable. Let $g_1(x, y) = c_1$, when parametrized by arc length, leave every bounded region for sufficiently large arc length in either direction, and let $g_2(x, y) = c$ have a real locus for all real c. If (H) holds, if all solutions with initial conditions at $t = a$ exist over $[a, b+\gamma]$, and if there exists at most one solution of (3.1), (3.2) and $g_2(x(s), y(s)) = c$ for $s \in (b - \varepsilon, b + \varepsilon)$ some $\varepsilon > 0$ and all c, then there exists at least one (and hence exactly one) solution of (3.1), (3.2), (3.3).*

REMARKS. The condition that $g_2(x, y) = c$ have a real locus for every real c may be replaced by a real locus for c in some open interval provided the conclusion is restricted to c in that interval. A similar theorem is true if the interval $[a, b+\gamma]$ is replaced by $(a-\gamma, b]$ and the uniqueness holds in $(a-\varepsilon, a+\varepsilon)$. That the theorem is not true if uniqueness is not required in a neighborhood has been shown by a counterexample in [6]. Also in [6] an example is given of a second order scalar equation with linear boundary conditions which satisfies all of the hypotheses of Theorem 3.1 except (H) and for which solutions fail to exist.

Proof. Let P_0 be an arbitrary point of $g_1(x, y) = c_1$ and suppose that $g_1(x, y) = c_1$ is parameterized by arc length with P_0 as initial point. Hereafter we refer to points of this curve by a single real number denoting the parameter. For a fixed s, $a \leq s \leq$

$b+\varepsilon$, define $T_s(P)$, $P \in g_1(x, y) = c_1$, to be the point set $\{(x(s, P), y(s, P))\}$ where $(x(t, P), y(t, P))$ denotes a solution through P at $t = a$, i.e. $T_s(P)$ is the funnel section at s of all solutions through P. Since all solutions are continuable, the mapping takes the points of $g_1(x, y) = c_1$ into nonempty subsets of the plane. On the range of $T_s(P)$ define

$$S(T_s(P)) = \{g_2(x(s, P), y(s, P)) \mid (x(s, P), y(s, P)) \in T_s(P)\}.$$

Let $\Gamma_s(P) = S(T_s(P))$. If the range of Γ_b is the whole real line, there exists a solution of (3.1), (3.2), (3.3) for every choice of c_2.

LEMMA 3.1. *$\Gamma_s(R)$ is connected for $s \in [a, b+\varepsilon)$.*

Proof. Since $g_2(x, y)$ is continuous $\Gamma_s(P)$ will be a compact connected set if $T_s(P)$ is. That $T_s(P)$ is compact and connected is a consequence of Kneser's theorem and the assumed continuability of solutions. In fact, if A is a compact connected set, $T(A)$ is also. (This is stated in [12] for the autonomous case and can be proved in the nonautonomous case as well, following [4].)

Suppose $\Gamma_s(R) = A \cup B$ where A and B are separated. Let $Q_1 \in A$ and $Q_2 \in B$. For $i = 1, 2$ there exists a solution $(x_i(t, P_i), y(t, P_i))$ such that $P_i \in g_1(x, y) = c_1$, and $g_2(x_i(s, P_i), y_i(s, P_i)) = Q_i$. Let E be the arc of $g_1(x, y) = c_1$ joining P_1 and P_2. In view of the remarks in the preceding paragraph, $\Gamma_s(E)$ is connected. Yet $\Gamma_s(P_1) \in A$, $\Gamma_s(P_2) \in B$, and A and B are separated. This establishes the lemma.

By uniqueness of solutions of boundary value problems, if $s \in (b-\varepsilon, b+\varepsilon)$, $\Gamma_s(P_1) \cap \Gamma_s(P_2) = \varnothing$. Hence $\Gamma_s(P)$ is an interval or a point, and the range of $\Gamma_s(R)$ may be ordered in the obvious way.

LEMMA 3.2. *Γ_s is monotone for $s \in (b-\varepsilon, b+\varepsilon)$.*

Proof. Let $P_1 < P_2 < P_3$ (ordered by arc length) and suppose $\Gamma(P_2) < \Gamma(P_1) < \Gamma(P_3)$ (the other cases follow similarly). Let $x_i \in \Gamma_s(P_i)$, $i = 1, 2, 3$. $\Gamma_s([P_2, P_3])$ is a connected set so there exists a point P^*, $P_2 \leqq P^* \leqq P_3$, such that $x_1 \in T(P^*)$. Since $P_1 < P_2 \leqq P^*$ this contradicts uniqueness of solutions of boundary value problems.

Proof of theorem (continued). If the range of $\Gamma_b(R)$ is not all of R then $\Gamma_b(P)$ is bounded above or below. Suppose $\Gamma_b(R)$ is bounded above and let $\eta = \sup \Gamma_b(R)$. First of all, it will be established that $\eta \notin \Gamma_b(R)$. Suppose that $\eta \in \Gamma_b(R)$. Then there exists a solution of (3.1), $(x(t), y(t))$, which satisfies (3.2) and $g_2(x(b), y(b)) = \eta$. Let $P_1 = (x(a), y(a))$ (recall P is a parameter denoting arc length on $g_1(x, y) = c_1$). Choose P_2 and P_3 such that $P_2 < P_1 < P_3$. Then, in view of the monotonicity, either $\Gamma_b(P_2) > \Gamma_b(P_1)$ or $\Gamma_b(P_3) > \Gamma_b(P_1)$. Since $\eta \in \Gamma_b(P_1)$, this contradicts the maximality of η and hence $\eta \in \Gamma_b(R)$ is impossible.

Since η is the sup $\Gamma_b(R)$ there exist $z_n \in \Gamma_b(R)$ such that $z_n \uparrow \eta$. To each z_n there corresponds a unique point in the $x-y$ plane P_n, $P_n \in g_1(x, y) = c$, and a unique solution of (3.1), $(x_n(t), y_n(t))$ through P_n at $t = a$, such that $g_2(x_n(b), y_n(b)) = z_n$.

Note that Lemma 3.2 guarantees that P_n depends monotonely on n. Let $\psi_n(t) = g_2(x_n(t), y_n(t))$, $t \in [a, b+\gamma)$. Then $\psi_n(t)$ is differentiable and

$$\psi_n'(t) = (\partial g_2/\partial x)(x_n(t), y_n(t))f_1(t, x_n(t), y_n(t))$$
$$+ (\partial g_2/\partial y)(x_n(t), y_n(t))f_2(t, x_n(t), y_n(t)).$$

The object of the following argument is to select a convergent subsequence of initial conditions, i.e. points in the plane (x_n, y_n). It is an adaptation of an argument originally given by Lasota and Opial [6].

In terms of the parametrization discussed in the beginning of the proof, for fixed t in $(b-\varepsilon, b+\varepsilon)$, $\psi_n(t)$ is monotone in n (either increasing or decreasing) since $\{P_n\}$ is monotone in n. Further, $\psi_n'(t) \geq 0$ or $\psi_n'(t) \leq 0$ for infinitely many n—for definiteness assume $\psi_n(t)$ is monotone increasing in n for $t \in (b-\varepsilon, b+\varepsilon)$ and $\psi_n'(b) \leq 0$ for infinitely many n. We restrict the argument to these n. Let $t^* \in [b, b+\varepsilon)$. Then

$$(3.4) \qquad \frac{\psi_n(t^*)-\psi_n(b)}{t^*-b} \geq \frac{\psi_1(t^*)-\psi_n(b)}{t^*-b} \geq K= \min\left[0, \frac{\psi_1(t^*)-\eta}{t^*-b}\right].$$

Let $S_n = [t \mid t \geq b; 0 \geq \psi_n'(t) \geq K]$. The sign of $\psi_n'(b)$, (3.4), and the mean value theorem imply that S_n is not empty. Clearly, S_n is closed. Let $s_n = \min S_n$. Then $b \leq s_n \leq b+\varepsilon$, so there exists a convergent subsequence (which we still label s_n), i.e. $s_n \to s_0$. Now $0 \geq \psi_n'(s_n) \geq K$. Note that if $\psi_n'(b)=0$, $b=s_n$. If $\psi_n'(b)<0$, $\psi_n(t)<\eta$ until $\psi_n'(t)=0$ for some $t>b$. This t is in S_n, hence $\psi_1(s_n)<\psi_n(s_n)<\eta$. Rewriting the $\psi_n(s_n)$ and $\psi_n'(s_n)$ in terms of $(x_n(s_n), y_n(s_n))$ we are in the situation described by hypothesis (H). Thus $\{(x_n(s_n), y_n(s_n))\}$ lies in a bounded region of the plane.

Choose a convergent subsequence, i.e. let $s_n \to s_0$, $x_n = x_n(s_n) \to x_0$, $y_n = y_n(s_n) \to y_0$. Let S_δ denote a sphere of radius δ about (s_0, x_0, y_0) where $\delta < b+\varepsilon-s_0$. For large n, $(s_n, x_n, y_n) \in S_\delta$. Since all solutions exist over $[a, s_0+\delta]$, there exists a k such that $x_n^2(a)+y_n^2(a)<k^2$ [15, p. 16]. Let $P_n=(x_n(a), y_n(a))$. Let $Q_1>P>Q_2$ be points of $g_1(x, y)=c_1$ such that $\|Q_1\|>k$ and $\|Q_2\|>k$ ($\|\cdot\|$ is Euclidean norm). Such points exist by the assumption that $g_1(x, y)=c$ does not lie in a bounded region. From the monotonicity of Γ_b it follows that $\Gamma_b(Q_1)>\Gamma_b(P_n)$ or $\Gamma_b(Q_2)>\Gamma_b(P_n)$ for every n. Hence either $\Gamma_b(Q_1) \geq \eta$ or $\Gamma_b(Q_2) \geq \eta$. This is a contradiction of the maximality of η. The other cases are established similarly.

Theorem 3.1 includes the theorem in [11] as a special case (which in turn includes the theorem of this type in [1], [3], [6], [10], [14]).

4. **Examples.** In this section the theorems of the preceding two sections will be illustrated. Consider the problem

$$x' = f_1(t, x, y), \qquad y' = f_2(t, x, y);$$
$$g_1(x(a), y(a)) = 0, \qquad g_2(x(b), y(b)) = 0.$$

Suppose $g_1(u, v)=u-v$, $g_2(u, v)=u+v+u/1+u^2$, $g_1(u, v)$ is linear, and $g_2(u, v)$ is

linear in v, so in (2.13) $A_1 = A_2 = 1$, $B_1 = B_2 = -1$, $D_1 = D_2 = 1$. Further, $\frac{7}{8}(u-w) \leq g_2(u, v) - g_2(w, v) \leq 2(u-w)$ as may be seen from bounding the derivative. Thus the region G_1 is the line $y = x$ and G_2 is the closed region between $2x + y = 0$ and $7x + 8y = 0$. (The first and third sectors in the definition of G_2 are null since both lines have negative slope.) Thus if $f_i(t, x, y)$, $i = 1, 2$, are Lipschitzian, Theorem 2.3 applies. For example, suppose $f_1(t, x, y) = \sin y$, $f_2(t, x, y) = x - y$. Then in (1.7), $L_1 = L_2 = 0$, $K_1 = -1$, $K_2 = 1$, $M_1 = M_2 = 1$, $N_1 = N_2 = 1$. The comparison systems (2.6) and (2.7) become $u' = -v$, $v' = u - v$, and $w' = z$, $z' = w - z$. These systems are linear and can be solved explicitly. The computation of M, however, is tedious and will be omitted, but if $b - a < M$, the problem has at most one solution.

To show that a solution exists, Theorem 3.1 will be applied. Except for hypothesis (H) all of the conditions of the theorem are satisfied if $b - a < M$. Suppose that (x_n, y_n) is a set of points such that the conditions of (H) are satisfied but that the (x_n, y_n) do lie in a bounded region. If (x_n, y_n) are such that $g_2(x_n, y_n)$ is bounded, then there exist numbers c_1 and c_2 such that (x_n, y_n) are in the region between $g_2(x, y) = c_1$ and $g_2(x, y) = c_2$. From the form of these curves, i.e. $x + y + x/(1 + x^2) = c$, we observe that if c is bounded, then as $x \to \pm\infty$, $y \to \mp\infty$ (and as $y \to \pm\infty$, $x \to \mp\infty$). Since every point (x_n, y_n) is on some such curve with $c_1 < c < c_2$, and $\{(x_n, y_n)\}$ does not lie in a bounded region, then either $x_n \to \infty$, $y_n \to -\infty$, or $x_n \to -\infty$ and $y_n \to \infty$ as $n \to \infty$. However,

$$\frac{\partial g_2}{\partial x}(x_n, y_n) f_1(t, x_n, y_n) + \frac{\partial g_2}{\partial y}(x_n, y_n) f_2(t, x_n, y_n) = \left(1 + \frac{1 - x_n^2}{(1 + x_n)^2}\right) \sin y + x_n - y_n$$

is bounded. Since the first term of this expression is bounded, $(x_n - y_n)$ must be bounded and this is incompatible with x_n and y_n being unbounded and (eventually) of opposite sign. Thus (x_n, y_n) must lie in a bounded region and (H) holds. Theorem 3.1 then yields an existence theorem.

It is possible to apply Theorem 3.1 without using Theorem 2.3. For example, consider the boundary value problem

$$x' = x + 2y \sin^2 x, \qquad y' = y/2 + \sin^2 x;$$
$$y(a) = c, \qquad x(b) - y^2(b) = c_2.$$

The conditions of Theorem 3.1 are obviously satisfied except for the existence of at most one solution (continuability, for example, follows from [7, p. 9]). To prove the existence of at most one solution, note first that a line $y = c_1$ cuts each curve $x - y^2 = c$ exactly once. Hence if $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ were two solutions, then labeling may be chosen so that

$$g_2(x_1(a), y_1(a)) > g_2(x_2(a), y_2(a)).$$

Let $G_i(t) = g_2(x_i(t), y_i(t))$, $i = 1, 2$.

$$G_i'(t) = x_i(t) + 2y_i(t) \sin^2 x_i(t) - 2y_i(t)[y_i(t)/2 + \sin^2 x_i(t)] = G_i(t).$$

Hence $G_i(t) = G_i(a)e^t$. Since $G_1(a) > G_2(a)$, $G_1(b) > G_2(b)$ contradicting the second boundary condition. Thus there can be at most one solution of the boundary value problem. By Theorem 3.1 there is exactly one solution.

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