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LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS ON ARBITRARY POINT SETS

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Abstract. Boundary-value problems for differential operators Λ of order $2m$ which are the Euler derivatives of quadratic functionals are considered. The boundary conditions require the solution F to coincide with a given function $f \in \mathscr{H}_L(\boldsymbol{R})$ at the points of an arbitrary closed set B, to satisfy at the isolated points of B the knot conditions of $2m$-spline interpolations, and to lie in $\mathscr{H}_L(\boldsymbol{R})$. Existence of solutions (called "Λ-splines knotted on B") is proved by consideration of the associated variational problem. The question of uniqueness is treated by decomposing the problem into an equivalent set of problems on the disjoint intervals of the complement of B', where B' denotes the set of limit points of B. It is also shown that Λ, considered as an operator from $\mathscr{L}_2(\boldsymbol{R})$ to $\mathscr{L}_2(\boldsymbol{R})$, with appropriately restricted domain, has a unique selfadjoint extention Λ_B if one postulates that the domain of Λ_B contains only functions of $\mathscr{H}_L(\boldsymbol{R})$ which vanish on B. $I+\Lambda_B$ has a bounded inverse which serves to solve the inhomogeneous equation $\Lambda F = G$ with homogeneous boundary conditions. Approximations to the Λ-splines knotted on B are constructed, consisting of Λ-splines knotted on finite subsets B_n of B, with $\cup B_n$ dense in B. These approximations F_n converge to F in the sense of $\mathscr{H}_L(\boldsymbol{R})$.

1. **The boundary-value problem.** Suppose B is a closed set of real numbers and f a (real-valued) function defined on B. In a previous article Golomb and Schoenberg [1] considered the problem of extending the function f to the real line in such way that the extended function F has a square-integrable mth derivative (more precisely, $F \in \mathscr{H}_2^m(\boldsymbol{R})$). Of special interest is the extension that minimizes $\int_{\boldsymbol{R}} (D^m F)^2$. It is readily seen that the minimizing extension satisfies the differential equation $D^{2m}F(x) = 0$ in intervals that are free of points of B, and that F has a continuous derivative of order $2m-2$ at isolated points of B. In the same article it was shown that, conversely, the solutions F of this differential equation problem are extensions of f that minimize $\int_{\boldsymbol{R}} (D^m F)^2$. In this way, the original extension problem (which is also an interpolation problem) becomes a boundary-value problem in differential equations, but one of an unusual kind, since the boundary involved is not that of a finite or infinite interval, but that of an arbitrary open set in $\boldsymbol{R}$.

In this paper such boundary-value problems are considered, not for the operator D^{2m}, but for general linear differential operators Λ with variable coefficients that

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are the Euler derivatives of quadratic functionals. The existence of solutions is proved even for cases where there is no uniqueness. With some restrictions on the operator Λ and/or the set B uniqueness of the solution is then proved. The usual proofs for existence and uniqueness are not applicable here since the conditions $F(x)=f(x)$ for $x \in B$ are equivalent to an infinite system of linear equations for the unknown coefficients in the linear combinations of a fundamental set of solutions (if B is an infinite set). It is then shown that the operator Λ, with domain essentially restricted by the boundary conditions $F(x)=0$ for $x \in B$ and some condition on the behavior at infinity, when considered as an operator from $\mathscr{L}_2(\boldsymbol{R})$ to $\mathscr{L}_2(\boldsymbol{R})$ has a unique selfadjoint extension (the closure of Λ), which is explicitly described. In connection with this, it is proved that the problem $\Lambda F=G$, $F(x)=0$ for $x \in B$, has a unique solution for every $G \in \mathscr{L}_2(\boldsymbol{R})$.

2. **Existence of solutions.** The differential operator to be considered in this section is of the form $\Lambda = L^*L$, where

$$L = a_m D^m + a_{m-1} D^{m-1} + \cdots + a_1 D + a_0 \tag{2.1}$$

with (real-valued) coefficient functions $a_k \in \mathscr{C}^m(\boldsymbol{R})$ $(k=0, 1, \ldots, m)$, L^* denoting the formal adjoint of L. Throughout it will be assumed that $a_m(x) \geqq \alpha > 0$ for all $x \in \boldsymbol{R}$, so that every finite point is a regular point for Λ, but the boundary-value problem to be considered is singular since the boundaries are at infinity. There is one global condition which the solutions F are expected to satisfy in all cases:

$$\int_{\boldsymbol{R}} (LF)^2 < \infty. \tag{2.2}$$

We say that F is in $\mathscr{H}_L(\boldsymbol{R})$, which is a Hilbert space with norm to be defined below, the main term of which is given by (2.2).

Let $f \in \mathscr{H}_L(\boldsymbol{R})$ be a given function, B a closed set on $\boldsymbol{R}$ (bounded or not), B' the set of limit points of B. The complete boundary-value problem for the unknown function F is posed by the following set of conditions:

($\boldsymbol{R}$i) $\Lambda F(x)=0$, $x \in \boldsymbol{R}-B$,

($\boldsymbol{R}$ii) $F(x)=f(x)$, $x \in B$,

($\boldsymbol{R}$iii) $F \in \mathscr{H}_L(\boldsymbol{R}) \cap \mathscr{C}^{2m}(\boldsymbol{R}-B) \cap \mathscr{C}^{2m-2}(\boldsymbol{R}-B')$.

Condition ($\boldsymbol{R}$i) requires that F satisfy the differential equation $\Lambda F=0$ of order $2m$ on the open set $\boldsymbol{R}-B$. ($\boldsymbol{R}$ii) demands that, at the boundary points of this set, F coincide with the given function f. ($\boldsymbol{R}$iii) is the boundary condition (2.2) at infinity ($F \in \mathscr{H}_L(\boldsymbol{R})$), and specifies that the solution possess continuous derivatives of order $\leqq 2m$ at interior points ($F \in \mathscr{C}^{2m}(\boldsymbol{R}-B)$) together with continuous derivatives of order $\leqq 2m-2$ everywhere except at accumulations of boundary points ($F \in \mathscr{C}^{2m-2}(\boldsymbol{R}-B')$). If f is defined only on B then any $\mathscr{H}_L$-extension of f to $\boldsymbol{R}$ will result in the same boundary-value problem. Necessary and sufficient conditions for the existence of such an extension are discussed in [1].

The main existence theorem is the following:

THEOREM 1. *There always exists a solution of the boundary-value problem* ($\boldsymbol{R}$i, ii, iii).

Proof. For F, G in $\mathscr{H}_L(\boldsymbol{R})$, let the inner product be defined as

$$(F, G)_L = \sum_{k=0}^{m-1} D^kF(0)D^kG(0) + \int_{\boldsymbol{R}} LF\cdot LG. \tag{2.3}$$

It is readily seen that $\mathscr{H}_L(\boldsymbol{R})$ is a Hilbert space if its elements are taken to be functions F which have absolutely continuous derivatives of order $\leqq m-1$ and for which LF is square-integrable. We now quote a lemma from [2], which is useful here.

LEMMA. *Suppose $\mathscr{H}_1$, $\mathscr{H}_2$ are Hilbert spaces; R is a bounded linear transformation with nullspace $\mathscr{N}$ that maps $\mathscr{H}_1$ onto $\mathscr{H}_2$; and $\mathscr{U}_0$ is a subspace of $\mathscr{H}_1$. If $\mathscr{N}+\mathscr{U}_0$ is closed then $R\mathscr{U}_0$ is closed.*

For the proof of the above theorem we use this lemma with $\mathscr{H}_1=\mathscr{H}_L(\boldsymbol{R})$, $\mathscr{H}_2=\mathscr{L}_2(\boldsymbol{R})$, and $R=L$. R is onto since $Lf=h$ has (locally) a solution f for every (locally) integrable h, and clearly $f\in\mathscr{H}_L(\boldsymbol{R})$ if $h\in\mathscr{L}_2(\boldsymbol{R})$. For $\mathscr{U}_0$ we use the subspace of $\mathscr{H}_L(\boldsymbol{R})$, consisting of the functions that vanish at the points of B; it is clearly closed, and since the nullspace of L is m-dimensional, we conclude that $L\mathscr{U}_0$ is closed. Then also $L\mathscr{U}$ is closed, where $\mathscr{U}=\mathscr{U}_0+f$, i.e. $\mathscr{U}$ is the (nonempty) flat

$$\mathscr{U} = \{F\in\mathscr{H}_L(\boldsymbol{R}) : F(x) = f(x),\ x\in B\}. \tag{2.4}$$

Therefore, there exists $F_*\in\mathscr{U}$ such that

$$\int_{\boldsymbol{R}} (LF_*)^2 = \inf_{F\in\mathscr{U}} \int_{\boldsymbol{R}} (LF)^2. \tag{2.5}$$

We now show that F_* is a solution of the boundary-value problem. Clearly, $F_*\in\mathscr{H}_L(\boldsymbol{R})$, and F_* satisfies ($\boldsymbol{R}$ii). From (2.5) it follows immediately that

$$\int_{\boldsymbol{R}} LF_*\cdot LG = 0 \tag{2.6}$$

for every $G\in\mathscr{U}_0$. Suppose I is an open interval in $\boldsymbol{R}-B$. The orthogonality condition (2.6) holds for every $G\in\mathscr{C}^\infty(\boldsymbol{R})$ with compact support in I. Integration by parts in (2.6) gives $\int_I F_*\cdot L^*LG=0$. Thus, F_* is a weak solution of $L^*LF=0$, and considering the assumptions made on the coefficients of L, one infers by familiar arguments (see, e.g. [3, §8]) that F_* is indeed a classical solution of $L^*LF(x)=0$, for $x\in I$. Thus, F_* satisfies ($\boldsymbol{R}$i) and $F_*\in\mathscr{C}^{2m}(\boldsymbol{R}-B)$.

Next, suppose J is an open interval in $\boldsymbol{R}-B'$ containing exactly one point x_* of B. Then (2.6) holds for every function $G\in\mathscr{C}^\infty(\boldsymbol{R})$ with compact support in J and vanishing at x_*. Since the functions $F_*^{(k)}$ $(k=0, 1, \ldots, m-1)$ are continuous at x_*, integration by parts in (2.6) gives

$$\begin{aligned} 0 = \int_J (LF_*)(LG) = {} & (DG)\{[a_2LF_*]-[Da_3LF_*]+\cdots+(-1)^{m-2}[D^{m-2}a_mLF_*]\} \\ & +(D^2G)\{[a_3LF_*]-\cdots+(-1)^{m-3}[D^{m-3}a_mLF_*]\} \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & +(D^{m-2}G)\{[a_{m-1}LF_*]-[Da_mLF_*]\} \\ & +(D^{m-1}G)[a_mLF_*]. \end{aligned} \tag{2.7}$$

Here, (F) stands for $F(x_*)$, and $[F]$ stands for $F(x_*-0)-F(x_*+0)$. By choosing G so that $(DG)=(D^2G)=\cdots=(D^{m-2}G)=0$, while $(D^{m-1}G)=1$, we conclude $(a_m^2)[D^mF_*]=0$, hence $[D^mF_*]=0$. Then we choose $(DG)=\cdots=(D^{m-3}G)=(D^{m-1}G)=0$, $(D^{m-2}G)=1$, and conclude $(a_m^2)[D^{m+1}F_*]=0$, hence $[D^{m+1}F_*]=0$. Continuing this way, we obtain

$$[D^mF_*] = [D^{m+1}F_*] = \cdots = [D^{2m-2}F_*] = 0, \tag{2.8}$$

hence $F_* \in \mathscr{C}^{(2m-2)}(J)$.

Altogether we have proved that $F=F_*$ satisfies conditions ($\boldsymbol{R}$i, ii, iii).

Suppose the set B is bounded from one or both sides. Then the solution F_* given in the proof of Theorem 1 is of "lower degree at infinity." This is the content of

COROLLARY 1.1. *Suppose $F=F_*$ is a solution of problem* ($\boldsymbol{R}$i, ii, iii), *as determined in the proof of Theorem* 1. *If* $\inf B=a>-\infty$ *then $LF_*(x)=0$ for $x<a$. If* $\sup B=b<\infty$ *then $LF_*(x)=0$ for $x>b$.*

Proof. Suppose $\inf B=a>-\infty$ and $LF_*(x)\neq 0$ for some $x<a$. Then $LF_*(x)\neq 0$ in some interval contained in $(-\infty, a)$. If we determine a function F_a from the conditions

$$\begin{aligned} LF_a(x) &= 0, & x &< a,\\ D^kF_a(a) &= D^kF_*(a), & k &= 0, 1, \ldots, m-1, \end{aligned} \tag{2.10}$$

and set

$$\begin{aligned} F(x) &= F_*(x), & x &\geqq a,\\ &= F_a(x), & x &< a, \end{aligned} \tag{2.11}$$

we have a function $F\in\mathscr{U}$ for which $\int_R (LF_a)^2<\int_R (LF_*)^2$. This contradicts the definition of F_*.

For the case where the set B has a finite bound that is not a limit point the preceding corollary leads to a "natural" boundary condition.

COROLLARY 1.2. *If B is a point set in $\boldsymbol{R}_a=[a,\infty)$ and $a\notin B'$ then there exists a solution of the boundary-value problem*

($\boldsymbol{R}_a$i) $\Lambda F(x)=0$, $x\in \boldsymbol{R}_a-B$,
($\boldsymbol{R}_a$ii) $F(x)=f(x)$, $x\in B$,
($\boldsymbol{R}_a$iii) $F\in\mathscr{H}_L(\boldsymbol{R}_a)\cap\mathscr{C}^{2m}(\boldsymbol{R}_a-B)\cap\mathscr{C}^{2m-2}(\boldsymbol{R}_a-B')$,
($\boldsymbol{R}_a$iv) $D^kLF(a)=0$, $k=0, 1, \ldots, m-2$.

Proof. By Corollary 1.1 we have a function F defined on $\boldsymbol{R}$ which satisfies conditions ($\boldsymbol{R}$i, ii, iii) and for which $LF(x)=0$, $x<a$. Clearly, its restriction to $\boldsymbol{R}_a$ satisfies the above conditions.

If B belongs to $\boldsymbol{R}_b=(-\infty, b]$ and $b\notin B'$, then Corollary 2.2 holds with $\boldsymbol{R}_a$ replaced by $\boldsymbol{R}_b$, and ($\boldsymbol{R}_a$iv) replaced by $D^kLF(b)=0$ $(k=0, 1, \ldots, m-2)$. If B belongs to $I_{ab}=[a, b]$ and $a\notin B'$, $b\notin B'$ then there exists a solution of the problem

(I_{ab}i) $\Lambda F(x)=0$, $x\in I_{ab}-B$,
(I_{ab}ii) $F(x)=f(x)$, $x\in B$,
(I_{ab}iii) $F\in\mathscr{C}^{2m}(I_{ab}-B)\cap\mathscr{C}^{2m-2}(I_{ab}-B')$,
(I_{ab}iv) $D^kLF(a)=D^kLF(b)=0$, $k=0, 1, \ldots, m-2$.

3. **Decomposition of boundary-value problem.** Let the point sets B, B' be defined as in the preceding section. The open set $\boldsymbol{R}-B'$ is the union of disjoint intervals J_ν, which we refer to as the *discrete components* of B. We show now that the boundary-value problem ($\boldsymbol{R}$i, ii, iii) breaks up into a number of such problems, one for each interval J_ν.

The intervals J_ν are of three different types. We say the discrete component J of B is of *type* I if J is finite, $J=(a, b)$. In this case, $a \in B'$ and $b \in B'$, and since the solution F of problem ($\boldsymbol{R}$i, ii, iii) coincides with $f \in \mathscr{C}^{m-1}$ at the points of B, we have $F^{(k)}(a)=f^{(k)}(a)$, $F^{(k)}(b)=f^{(k)}(b)$ $(k=0, 1, \ldots, m-1)$. Thus, the restriction of F to $J=(a, b)$ satisfies the following conditions:

(Ii) $\Lambda F(x)=0$, $x \in J-B$,

(Iii) $F(x)=f(x)$, $x \in J \cap B$,

(Iiii) $F \in \mathscr{C}^{2m}(J-B) \cap \mathscr{C}^{2m-2}(J)$,

(Iiv) $D^kF(a)=D^kf(a)$, $D^kF(b)=D^kf(b)$, $k=0, 1, \ldots, m-1$.

The discrete component J of B is of *type* II if J is semi-infinite, either $J_a=(a, \infty)$ or $J_b=(-\infty, b)$. In the first case $a \in B'$, in the second case $b \in B'$. Considering the first case only, we conclude as above that $D^kF(a)=D^kf(a)$ $(k=0, 1, \ldots, m-1)$ for any solution F of ($\boldsymbol{R}$i, ii, iii). Hence the restriction of F to J_a satisfies

(IIi) $\Lambda F(x)=0$, $x \in J_a-B$,

(IIii) $F(x)=f(x)$, $x \in J_a \cap B$,

(IIiii) $F \in \mathscr{H}_L(J_a) \cap \mathscr{C}^{2m}(J_a-B) \cap \mathscr{C}^{2m-2}(J_a)$,

(IIiv) $D^kF(a)=D^kf(a)$, $k=0, 1, \ldots, m-1$.

The discrete component J of B is of *type* III if J is infinite, $J=\boldsymbol{R}=(-\infty, \infty)$. This is the case if and only if the set B is discrete (finite or infinite, bounded or unbounded). The restriction of F to J is F itself, which satisfies the conditions

(IIIi) $\Lambda F(x)=0$, $x \in \boldsymbol{R}-B$,

(IIIii) $F(x)=f(x)$, $x \in B$,

(IIIiii) $F \in \mathscr{H}_L(\boldsymbol{R}) \cap \mathscr{C}^{2m}(\boldsymbol{R}-B) \cap \mathscr{C}^{2m-2}(\boldsymbol{R})$.

Now if, for each discrete component J_ν of B, we have a solution F_{J_ν} of (Ii, ii, iii, iv), (IIi, ii, iii, iv), (IIIi, ii, iii) depending on whether J_ν is of type I, II, or III, then the function F on $\boldsymbol{R}$, defined by

$$\begin{aligned} F(x) &= f(x), && x \in B', \\ F(x) &= F_{J_\nu}(x), && x \in J_\nu,\ \nu = 1, 2, \ldots, \end{aligned} \tag{3.1}$$

is clearly a solution of boundary-value problem ($\boldsymbol{R}$i, ii, iii). Thus, the problem is indeed decomposed into independent problems, one corresponding to each discrete component J_ν of B.

It is also clear that the restriction of the function F_* of Theorem 1, which minimizes $\int_{\boldsymbol{R}} (LF)^2$, minimizes the integral $\int_{J_\nu} (LF)^2$ among all functions in $\mathscr{H}_L(J_\nu)$ that interpolate f on $B \cap J_\nu$. Conversely any function F whose restriction to J_ν $(\nu=1, 2, \ldots)$ minimizes $\int_{J_\nu} (LF)^2$ minimizes $\int_{\boldsymbol{R}} (LF)^2$.

4. **Unicity conditions.** By the results of the preceding section, the solution of problem ($\boldsymbol{R}$i, ii, iii) is unique if and only if the restriction of the problem to each of the discrete components J_ν of B has a unique solution. In the following we refer to these restricted problems, as formulated in §3 in equations (Ii–iv), (IIi–iv) and (IIIi–iii), as problems (I), (II) and (III), respectively.

For a bounded component J (of type I, see §3) the solution of problem (I) is always unique, as will be shown below. This is so even if J contains no point of B, the reason being that we always have the $2m$ terminal conditions (Iiv). If J is of type II, say $J=(a, \infty)$, then we have only the m terminal conditions (IIiv), and if $J=(-\infty, \infty)$ there are no terminal conditions. The global condition $F \in \mathscr{H}_L(J)$ is not strong enough to replace the missing terminal conditions, even in the presence of (IIiv), as the following example shows.

EXAMPLE 1. For $J=(a, \infty)$ consider the operator L defined by $LF(x)=e^x D^m F(x)$. Then $L^*F(x)=(-1)^m D^m e^x F(x)$, and $\Lambda F(x)=(-1)^m D^m e^{2x} D^m F(x)$. Suppose F_0 is so chosen that $D^m F_0(x)=e^{-2x}$ and $D^k F_0(a)=0$ ($k=0, 1, \ldots, m-1$). Then $\Lambda F_0=0$ and $F_0 \in \mathscr{H}_L(J)$. Hence, problem (II) with $J \cap B=\varnothing$ has more than one solution.

To assure uniqueness in the case of isolated bounds of $J \cap B$ we impose the condition

(4.1) If $+\infty$ $(-\infty)$ is an isolated point of $J \cap B$ then $L^*G=0$ has no nontrivial solution that is square-integrable near $+\infty$ $(-\infty)$.

If B' (the set of limit points of B) is empty and B contains no Tchebychev set of the operator L, then ($\boldsymbol{R}$i, ii, iii) has more than one solution since any nullfunction of L that vanishes at the points of B may be added to a particular solution to produce a new solution. For example, if $L=D^m$ $(m \geqq 1)$ and B has fewer than m points, then ($\boldsymbol{R}$i, ii, iii) clearly has more than one solution, although condition (4.1) is satisfied in this case. We are led to require additionally

(4.2) If $B'=\varnothing$ then B contains a Tchebychev set of L.

By a Tchebychev set here is meant a finite subset B_0 of B such that no nontrivial nullfunction of L can vanish for each point of B_0.

That (4.2) cannot replace (4.1) is clear from Example 1, where $B' \neq \varnothing$. Another example where we have nonuniqueness, although B contains a Tchebychev set of L, is the following:

EXAMPLE 2. For $J=\boldsymbol{R}$ consider the operator L defined by $LF(x)=e^{x^2}D^m F(x)$. Let B be an arbitrary set of m points and let F_0 be the solution of $D^m F_0(x)=e^{-2x^2}$ that vanishes at the points of B. Then $\Lambda F_0=0$ and $F_0 \in \mathscr{H}_L(J)$. Hence, problem ($\boldsymbol{R}$i, ii, iii) has more than one solution although B is a Tchebychev set of L.

In the remainder of this and in the following section it is assumed, without further mention, that conditions (4.1) and (4.2) are satisfied. Under this assumption, uniqueness of the solution of problems (I), (II), (III) of §3 will be proved. However, problems (II) and (III) must be modified somewhat when $+\infty$ (and/or $-\infty$) is a limit point of $J \cap B$. Then we have a more complex situation in regard to uniqueness since there are knots (discontinuities of the $(2m-1)$th derivative) arbitrarily close to $+\infty$ or $-\infty$. It is not impossible that there is more than one solution in such cases (although conditions (4.1) and (4.2) are satisfied), but we have not been able to produce such an example. Be this as it may, we impose for these cases a terminal condition that singles out the unique solution which minimizes $\int_J (LF)^2$. We set

$$B[F, G] = \sum_{k=0}^{m-1} \sum_{l=k+1}^{m} (-D)^{l-k-1} a_l LF \cdot D^k G \tag{4.3}$$

and we require

$$\text{(IV)}\qquad \begin{aligned} &\lim_{x\to+\infty\,(-\infty)} B[F, G](x) = 0 \quad \text{if } +\infty\ (-\infty) \text{ is a limit point of } J \cap B,\\ &\lim_{x\to+\infty} B[F, G](x) - \lim_{x\to-\infty} B[F, G](x) = 0\\ &\qquad\qquad\qquad\qquad \text{if } +\infty \text{ and } -\infty \text{ are limit points of } J \cap B \end{aligned}$$

for every function $G \in \mathscr{H}_L(\boldsymbol{R})$ that vanishes at the points of B. This condition is clearly a linear condition on the unknown function F, and it restricts the behavior of F only at the points $+\infty$ and/or $-\infty$, if these are limit points of $J \cap B$.

We show that condition (IV) does not restrict the class of problems for which solutions exist. Indeed, we have

THEOREM 2. *The restriction to J of the solution F_* of problem* ($\boldsymbol{R}$i, ii, iii) *which minimizes $\int_{\boldsymbol{R}} (LF)^2$ satisfies condition* (IV).

Proof. Because of the minimizing property of F_*, we have

$$\int_J LF_* \cdot LG = 0 \tag{4.4}$$

for every function $G \in \mathscr{H}_L(J)$ which vanishes at the points of $J \cap B$. Suppose first $J=(a, \infty)$ and $J \cap B=\{x_n\}$, $\lim x_n=\infty$. Then (4.4) becomes

$$\lim_{x\to\infty} \int_a^x LF_* \cdot LG = 0 \tag{4.5}$$

and integration by parts, as in (2.7), gives

$$\lim_{x\to\infty} B[F_*, G](x) = B[F_*, G](a). \tag{4.6}$$

But since $a \in B'$, we have $D^k G(a)=0$ ($k=0, 1, \ldots, m-1$), hence the term on the right-hand side of (4.6) vanishes.

Next, assume $J=(a, \infty)$ and $J \cap B$ is a bisequence $\cdots x_{-2}<x_{-1}<x_1<x_2<\cdots$ with $\lim x_{-n}=a$, $\lim x_n=\infty$. By Lemma 2a, §5, below, we can find, for $\varepsilon>0$ and $x \in J$ given, a function $G_\varepsilon \in \mathscr{H}_L(a, x)$ which vanishes near a and at the points of B in (a, x), which equals G near x, and which is such that

$$\left| \int_a^x LF_* \cdot LG - \int_a^x LF_* \cdot LG_\varepsilon \right| < \varepsilon. \tag{4.7}$$

But $\int_a^x LF_* \cdot LG_\varepsilon = B[F_*, G_\varepsilon](x) = B[F_*, G](x)$, hence (4.5) and (4.7) give (IV) again. The case $J=(-\infty, b)$ is disposed of in the same way.

Suppose now $J=(-\infty, \infty)$ and B is the sequence $x_1 < x_2 < \cdots$, $\lim x_n = \infty$. Then by ($\boldsymbol{R}$i) $L^*LF_*(x)=0$ for $x<x_1$, LF_* square-integrable near $-\infty$, and since we assume (4.1), $LF_*(x)=0$ for $x \leqq x_1$. Hence, also $D^kLF_*(x_1)=0$ $(k=0, 1, \ldots, m-2)$, and (4.4) gives

$$0 = \lim_{x\to\infty} \int_{x_1}^{x} LF_* \cdot LG = \lim_{x\to\infty} B[F_*, G](x). \tag{4.8}$$

Finally, if $J=(-\infty, \infty)$ and B is the bisequence $\cdots x_{-2} < x_{-1} < x_1 < x_2 < \cdots$, $\lim x_{-n} = -\infty$, $\lim x_n = \infty$, then (4.4) gives

$$0 = \lim_{x\to-\infty;\, y\to+\infty} \int_{x}^{y} LF_* \cdot LG = \lim_{y\to\infty} B[F, G](y) - \lim_{x\to-\infty} B[F, G](x), \tag{4.9}$$

hence (IV) again. Thus Theorem 2 is proved.

5. **Uniqueness of solutions.** As stated in §4, only problems (i.e., operators L and sets B) will be considered for which conditions (4.1) and (4.2) are satisfied. J will denote any one of the discrete components of B (see §3). To make possible integration by parts in the presence of infinitely many discontinuities in the interval of integration, we establish first some approximation lemmas. The first one is based on a result concerning the behavior of a function $F \in \mathscr{H}_L(J)$ at a boundary point of J if this boundary point is a limit point of zeros of F (for a special case of this result, see [1, Lemma 1]).

LEMMA 1. *Suppose* $x_1 > x_2 > \ldots$, $\lim x_n = 0$, $I=(0, x_1)$, $m \geqq 1$, L *a differential operator of order* m, $F \in \mathscr{H}_L(I)$, $F(x_n)=0$ $(n=1, 2, \ldots)$. *Then, as* $x \to 0$,

$$D^{m-k}F(x) = o(x^{k-1/2}), \qquad k = 1, 2, \ldots, m. \tag{5.1}$$

Proof. We prove the lemma first for $m=1$. In this case the operator $L=L_1$ is of the form a_1D+a_0, and $a_1(x) \geqq \alpha > 0$ for $x \in I$. Assume (5.1) does not hold for $m=k=1$. Then there is a sequence $\xi_1 > \xi_2 > \cdots > 0$, $\lim \xi_\nu = 0$, and a constant $C>0$ such that

$$|F(\xi_\nu)| \geqq C\xi_\nu^{1/2}, \qquad \nu = 1, 2, \ldots. \tag{5.2}$$

We may assume that each interval (x_{n+1}, x_n) contains no more than one ξ_ν, say

$$x_{n(\nu)+1} < \xi_\nu < x_{n(\nu)}, \qquad \nu = 1, 2, \ldots. \tag{5.3}$$

Let H be the function which is defined by 0 outside of the intervals $[x_{n(\nu)+1}, x_{n(\nu)}]$

($\nu=1, 2, \ldots$) and by

$$
\begin{aligned}
H(x) &= \left[Q(x)\int_{x_*}^{x} P\right]\Big/\left[Q(\xi_\nu-)\int_{x_*}^{\xi_\nu} P\right], && x_{n(\nu)+1} \leqq x < \xi_\nu, \\
&= 1, && x = \xi_\nu, \\
&= \left[Q(x)\int_{x_*}^{x} P\right]\Big/\left[Q(\xi_\nu+)\int_{x_*}^{\xi_\nu} P\right], && \xi_\nu < x \leqq x_n(\nu); \qquad (5.4)\\
Q(x) &= \exp\left[-\int_{x_*}^{x} (a_0/a_1)\right], && \\
P &= (a_1 Q)^{-2}, && x \neq \xi_\nu,
\end{aligned}
$$

inside $[x_{n(\nu)+1}, x_{n(\nu)}]$, where $x_* = x_{n(\nu)+1}$ if $x_{n(\nu)+1} \leqq x < \xi_\nu$ and $x_* = x_{n(\nu)}$ if $\xi_\nu < x \leqq x_{n(\nu)}$. H is a solution of the problem

$$
\begin{aligned}
L_1^* L_1 H(x) &= 0, && x \in I - X, \\
H(x_n) = 0, \quad H(\xi_\nu) &= 1, && n, \nu = 1, 2, \ldots, \\
H \in \mathscr{C}(I) \cap \mathscr{C}^2(I-X),
\end{aligned}
$$

where X is the union of $x_1, x_2, \ldots$ and $\xi_1, \xi_2, \ldots$. Therefore, the function F_* defined on I by

$$
\begin{aligned}
F_*(x) &= F(\xi_\nu) H(x), && x_{n(\nu)+1} \leqq x \leqq x_{n(\nu)} \ (\nu = 1, 2, \ldots), \\
&= 0, && \text{otherwise},
\end{aligned} \tag{5.5}
$$

is a solution of the boundary-value problem

$$
\begin{aligned}
L_1^* L_1 F_*(x) &= 0, && x \in I - X, \\
F_*(x) &= F(x), && x \in X, \\
F_* \in \mathscr{C}(I) &\cap \mathscr{C}^2(I-X),
\end{aligned} \tag{5.6}
$$

and it is easily seen that the solution of (5.6) is unique. It then follows from the proof of Theorem 1 that

$$
\int_I (L_1 F_*)^2 \leqq \int_I (L_1 F)^2. \tag{5.7}
$$

But

$$
\begin{aligned}
\int_{x_{n(\nu)+1}}^{x_{n(\nu)}} (L_1 F_*)^2 &= \left(\int_{x_{n(\nu)+1}}^{\xi_\nu} + \int_{\xi_\nu}^{x_{n(\nu)}}\right)(L_1 F_*)^2 \\
&= F_*(\xi_\nu) a_1(\xi_\nu)\{L_1 F_*(\xi_\nu - 0) - L_1 F_*(\xi_\nu + 0)\} \\
&= a_1^2(\xi_\nu) F^2(\xi_\nu)\left[P(\xi_\nu-)\left(\int_{x_{n(\nu)+1}}^{\xi_\nu} P\right)^{-1} - P(\xi_\nu+)\left(\int_{x_{n(\nu)}}^{\xi_\nu} P\right)^{-1}\right].
\end{aligned} \tag{5.8}
$$

Clearly, there is a positive constant γ such that

$$
a_1^2(x) P(x) \geqq \gamma, \quad P(x) \leqq \gamma^{-1}, \qquad x_{n(\nu)+1} \leqq x \leqq x_{n(\nu)}, \quad x \neq \xi_\nu.
$$

Thus (5.8) together with (5.2) gives

$$\begin{aligned}\int_{x_{n(\nu)+1}}^{x_{n(\nu)}} (L_1F_*)^2 &\geqq C^2\xi_\nu\gamma\left\{\left(\int_{x_{n(\nu)+1}}^{\xi_\nu} P\right)^{-1}+\left(\int_{\xi_\nu}^{x_{n(\nu)}} P\right)^{-1}\right\} \\ &\geqq C^2\gamma\xi_\nu\left(\int_{x_{n(\nu)+1}}^{\xi_\nu} P\right)^{-1} \\ &\geqq C^2\gamma^2, \qquad \nu = 1, 2, \ldots,\end{aligned} \tag{5.9}$$

hence $\int_I (L_1F_*)^2=\infty$, in contradiction to (5.7).

We now proceed by induction. We assume the lemma is proved for differential operators of order $m-1$ ($m\geqq 2$). Let Y_0 be the solution of $LY=0$ for which $Y(0)=1$, $Y'(0)=\cdots=Y^{(m-1)}(0)=0$. Then $LF=MD(Y_0^{-1}F)$, where $M=b_{m-1}D^{m-1}+\cdots+b_0$, with certain coefficients $b_k\in\mathscr{C}^m(I)$, and $b_{m-1}=a_mY_0$. Clearly, $b_{m-1}(x)\geqq\beta$ for some positive β, at least in some interval $I_N=[0, x_N]$. Since $F(x_n)=0$ ($n=N, N+1,\ldots$) we conclude that there is a sequence $x_N>y_1>y_2>\ldots$, $\lim y_n=0$, and that $D(Y_0^{-1}F)(y_n)=0$. As $F\in\mathscr{H}_L(I)$, $D(Y_0^{-1}F)$ is in $\mathscr{H}_M(I_N)$. The induction assumption implies that, for $x\to 0$,

$$D^{m-1-k}D(Y_0^{-1}F)(x) = o(x^{k-1/2}), \qquad k = 1, 2, \ldots, m-1,$$

and this gives immediately

$$D^{m-k}F(x) = o(x^{k-1/2}), \qquad k = 1, 2, \ldots, m-1. \tag{5.10}$$

Moreover, $F(x)=\int_0^x DF=o(x^{m-1/2})$, hence (5.10) also holds for $k=m$, and the lemma is proved.

LEMMA 2a. *Suppose the hypotheses of Lemma* 1 *are satisfied. Then there exists, for each* $\varepsilon>0$, *a function* $\Phi_\varepsilon\in\mathscr{H}_L(I)$ *which vanishes at* $x_1, x_2, \ldots$ *and near* 0, *and for which*

$$\int_I (LF-L\Phi_\varepsilon)^2 < \varepsilon^2. \tag{5.11}$$

Proof. Let $0\leqq E\leqq 1$ be an infinitely differentiable function for which

$$\begin{aligned} E(x) &= 0, \qquad x\leqq 0, \\ &= 1, \qquad x\geqq 1,\end{aligned} \tag{5.12}$$

and set, for $n=1, 2, \ldots$,

$$F_n(x) = F(x)E(nx-1). \tag{5.13}$$

Then $F_n(x)=F(x)$ for $x\geqq 2n^{-1}$, and $F_n(x)=0$ for $x\leqq n^{-1}$. Thus, $F_n\in\mathscr{H}_L(I)$ and F_n vanishes at $x_1, x_2, \ldots$, and near 0. We have

$$\begin{aligned} &LF(x)-LF_n(x) \\ &\quad = LF(x)[1-E(nx-1)] - \sum_{k=1}^{m}\sum_{j=1}^{k} a_k(x)\binom{k}{j}n^jD^{k-j}F(x)D^jE(nx-1).\end{aligned} \tag{5.14}$$

The first term in (5.14) is 0 for $x \geqq 2n^{-1}$, and as $n \to \infty$

$$\int_I \{LF(x)[1-E(nx-1)]\}^2\,dx \leqq \int_0^{2n^{-1}} \{LF(x)\}^2\,dx = o(1). \tag{5.15}$$

The sum term in (5.14) is 0 for $x \leqq n^{-1}$ and for $x \geqq 2n^{-1}$. Thus, by Lemma 1, as $n \to \infty$

$$\begin{aligned}\int_I \{D^{k-j}F(x)D^jE(nx-1)\}^2\,dx &= n^{-1}\int_0^1 \{D^{k-j}F(n^{-1}(x+1))D^jE(x)\}^2\,dx \\ &= o(n^{2k-2m-2j}) = o(n^{-2j}), \\ &\qquad j = 1,\dots,k;\ k = 1,\dots,m.\end{aligned} \tag{5.16}$$

Using (5.15) and (5.16) in (5.14) gives

$$\int_I (LF-LF_n)^2 = o(1) \quad \text{as } n \to \infty,$$

and this proves (5.11), for $\Phi_\varepsilon = F_n$, with n sufficiently large.

If $D^kF(x_1)=0$ ($k=0,1,\dots,m-1$) in the preceding lemma, then also $D^k\Phi_\varepsilon(x_1)=0$. This follows immediately from the construction of Φ_ε. An obvious extension of Lemma 2a is

LEMMA 2b. *Suppose* $\cdots x_{-2}<x_{-1}<x_1<x_2<\cdots$, $\lim x_{-n}=a>-\infty$, $\lim x_n=b<\infty$, $I=(a,b)$, $F\in\mathscr{H}_L(I)$, $F(x_{-n})=F(x_n)=0$ $(n=1,2,\dots)$, *and* $\varepsilon>0$. *There exists a function* $\Phi_\varepsilon\in\mathscr{H}_L(I)$ *which vanishes at* $\cdots x_{-2}, x_{-1}, x_1, x_2, \cdots$ *and near a and b, such that*

$$\int_I (LF-L\Phi_\varepsilon)^2 < \varepsilon^2. \tag{5.17}$$

We can now prove the uniqueness theorem for bounded discrete components of B.

THEOREM 3a. *The solution of boundary-value problem* (I) *on the bounded interval J is unique.*

Proof. Let G be the difference of two solutions. Then by (I):

$$\begin{array}{llll} & \text{(i)} & \Lambda G(x) = 0, & x\in J-B, \\ (5.18) & \text{(ii)} & G(x) = 0, & x\in J\cap B, \\ & \text{(iii)} & G\in\mathscr{C}^{2m}(J-B)\cap\mathscr{C}^{2m-2}(J), & \\ & \text{(iv)} & D^kG(a) = D^kG(b) = 0, & k = 0,1,\dots,m-1. \end{array}$$

If J contains only the finitely many (or no) points $x_1<x_2<\cdots<x_n$ of B we write

$$\int_J LG\cdot LG = \int_a^{x_1} + \sum_{k=1}^{n-1}\int_{x_k}^{x_{k+1}} + \int_{x_n}^b \tag{5.19}$$

and carry out $m-1$ integration by parts in each of the integrals on the right-hand side of (5.19). By using the fact that G satisfies (5.18), one verifies that $\int_J (LG)^2=0$, and since LG is continuous, $LG=0$. Using (5.18iv) once more, it follows that $G=0$.

If $J \cap B$ consists of the infinitely many points $x_1>x_2>\cdots$, then $\lim x_n=a$, and by Lemma 2a one can find, for $\varepsilon>0$ given, a function $\Phi_\varepsilon \in \mathscr{H}_L(J)$ which vanishes at the points $x_1, x_2, \ldots$, and near a, such that

$$\left| \int_J (LG)^2 - \int_J LG \cdot L\Phi_\varepsilon \right| < \varepsilon^2. \tag{5.20}$$

Moreover, by the remark following Lemma 2a, $D^k\Phi_\varepsilon(b)=0$ $(k=0, 1, \ldots, m-1)$. Since the support of Φ_ε contains only a finite number of the sequence $\{x_n\}$, we may proceed with the integral $\int_J LG \cdot L\Phi_\varepsilon$ as above with the integral $\int_J LG \cdot LG$, and we find $\int_J LG \cdot L\Phi_\varepsilon=0$. Since (5.20) holds for every $\varepsilon>0$, we conclude $\int_J (LG)^2=0$, and $G=0$, as before.

The case where $J \cap B=\{x_n\}$, $x_1<x_2<\cdots$, is disposed of in the same way. If $J \cap B$ contains the bisequence $\cdots x_{-2}<x_{-1}<x_1<x_2<\cdots$, then $\lim x_{-n}=a$, $\lim x_n=b$, and we use Lemma 2b in place of Lemma 2a, to establish (5.20). Again we conclude $G=0$.

We turn to the case where the discrete component J of B is unbounded, either of type II or III (see §3). We first assume $J \cap B$ is bounded and prove

Theorem 3b. *The solution of problems* (II) *and* (III) *is unique if $J \cap B$ is bounded.*

Proof. Let G be the difference of two solutions. If $J=(a, \infty)$ and $x_1<x_2<\cdots<x_n$ are the points of B in J, then $L^*LG(x)=0$ for $x \geqq x_n$, and since LG is square-integrable near ∞, it follows by (4.1) that $LG(x)=0$ for $x \geqq x_n$. In particular,

$$D^kLG(x_n) = 0, \qquad k = 0, 1, \ldots, m-2. \tag{5.21}$$

We write

$$\int_J LG \cdot LG = \int_a^{x_1} + \sum_{k=1}^{n-2} \int_{x_k}^{x_{k+1}} + \int_{x_{n-1}}^{x_n}$$

and carry out m integration by parts in the integrals on the right-hand side. Using $D^kG(a)=0$ $(k=0, 1, \ldots, m-1)$ and (5.21), we find $LG=0$, hence $G=0$.

If $x_1>x_2>\cdots$ are the points of B in J then $\lim x_n=a$, and by Lemma 2a there is, for every $\varepsilon>0$, a function $\Phi_\varepsilon \in \mathscr{H}_L(J)$ which vanishes at $x_1, x_2, \ldots$ and near a, such that $\left|\int_J (LG)^2-\int_J LG \cdot L\Phi_\varepsilon\right|<\varepsilon^2$. The support of Φ_ε contains only a finite number of the $x_1, x_2, \ldots$, and integration by parts can be carried out for the integral $\int_J LG \cdot L\Phi_\varepsilon$ in the same way as above. Using the fact that $D^kLG(x_1)=0$ for $k=0, 1, \ldots, m-2$ (compare (5.21)), one obtains $\int_J LG \cdot L\Phi_\varepsilon=0$, hence $\int_J (LG)^2<\varepsilon^2$, $LG=0$, thus again $G=0$.

The case $J=(-\infty, b)$ is disposed of in the same way. If $J=(-\infty, \infty)$, then $J \cap B=B$ is necessarily a finite set, which by (4.2) includes at least m points. If

$x_1 < x_2 < \cdots < x_n$ are the points of B, we conclude as above that $D^k LG(x_1) = D^k LG(x_n) = 0$ for $k = 0, 1, \ldots, m-2$. We use integration by parts again to find $\int_R (LG)^2 = 0$, hence $LG = 0$, and because of (4.2), $G = 0$.

There remains the case of an unbounded discrete component J of B containing points of B that converge to $+\infty$ and/or $-\infty$. In this case condition (IV) (§4) is made use of.

THEOREM 3c. *Problems* (II), (IV) *and* (III), (IV) *have unique solutions if* $J \cap B$ *contains a sequence that converges to* $+\infty$ *and/or* $-\infty$.

Proof. Let G be the difference of two solutions. If $J = (a, \infty)$ and $x_1 < x_2 < \cdots$ are the points of $J \cap B$, then $\lim x_n = \infty$ and G satisfies (II), (IV) with $f = 0$. Therefore,

$$\int_J (LG)^2 = \lim_{x \to \infty} \int_a^x LG \cdot LG = \lim_{x \to \infty} B[G, G](x) = 0. \tag{5.22}$$

From (IIiv) it follows that $G = 0$. If $J = (a, \infty)$ and $\cdots x_{-2} < x_{-1} < x_1 < x_2 \cdots$ are the points of $J \cap B$, then $\lim x_{-n} = a$ and $\lim x_n = \infty$. Given $\varepsilon > 0$ and $x > a$, there is by Lemma 2a a function $G_\varepsilon \in \mathscr{H}_L(a, x)$ which vanishes near a and at the points of B in (a, x), and which equals G near x, such that

$$\left| \int_a^x (LG)^2 - \int_a^x LG \cdot LG_\varepsilon \right| < \varepsilon^2. \tag{5.23}$$

But $\int_a^x LG \cdot LG_\varepsilon = B[G, G_\varepsilon](x) = B[G, G](x)$, hence $\int_a^x (LG)^2 = B[G, G](x)$. As before, we find by use of (IV) and (IIiv) that $G = 0$. The case $J = (-\infty, b)$ is disposed of in the same way. If $J = (-\infty, \infty)$ and $x_1 < x_2 < \cdots$ are the points of $J \cap B$, then $\lim x_n = \infty$ and G satisfies (III), (IV) with $f = 0$. Therefore, $L^*LG(x) = 0$ for $x \leqq x_1$, and since LG is square-integrable near $-\infty$, it follows by (4.1) that $LG(x) = 0$ for $x \leqq x_1$, hence $D^k LG(x_1) = 0$ $(k = 0, 1, \ldots, m-2)$. Integration by parts gives

$$\int_{-\infty}^{\infty} (LG)^2 = \lim_{x \to \infty} \int_{x_1}^x LG \cdot LG = \lim_{x \to \infty} B[G, G](x) = 0,$$

hence $LG = 0$, and because of (4.2), $G = 0$. Finally, if $J = (-\infty, \infty)$ and $\cdots x_{-2} < x_{-1} < x_1 < x_2 \cdots$ are the points of B, then $\lim x_{-n} = -\infty$ and $\lim x_n = +\infty$. By (IV),

$$\begin{aligned} \int_{-\infty}^{\infty} (LG)^2 &= \lim_{y \to \infty;\, x \to -\infty} \int_x^y LG \cdot LG \\ &= \lim_{y \to \infty} B[G, G](y) - \lim_{x \to \infty} B[G, G](x) = 0, \end{aligned} \tag{5.24}$$

hence $LG = 0$, and because of (4.2), $G = 0$.

We add another uniqueness theorem for the case where $J \cap B$ contains points that converge to $+\infty$ and/or $-\infty$. In this theorem condition (IV) is replaced by a different one:

(V) $\lim_{x \to +\infty (-\infty)} D^k F(x)$ exists if $+\infty$ $(-\infty)$ is a limit point of $J \cap B$

($k=0, 1, \ldots, m-1$) and we make the additional hypothesis:

(5.25) The coefficients a_k ($k = 0, 1, \ldots, m$) of L are bounded.

To prove uniqueness for this case, we establish another approximation lemma.

LEMMA 3a. *Suppose* (5.25) *holds*, $x_1 < x_2 < \cdots$, $\lim x_n = \infty$, $I = (x_1, \infty)$, $F \in \mathscr{H}_L(I)$, $F(x_n) = 0$ $(n = 1, 2, \ldots)$, $\lim_{x\to\infty} D^k F(x) = 0$ $(k = 0, 1, \ldots, m-1)$, *and* $\varepsilon > 0$. *There exists a function* $\Phi_\varepsilon \in \mathscr{H}_L(I)$ *which vanishes at* $x_1, x_2, \ldots$, *and near* $+\infty$, *and for which*

$$\int_I (LF - L\Phi_\varepsilon)^2 < \varepsilon^2. \tag{5.26}$$

Proof. We use the function E defined in (5.12) and set $\bar{E}(x) = 1 - E(x)$,

$$F_n(x) = F(x)E(n+1-x), \qquad n = 1, 2, \ldots. \tag{5.27}$$

Then $F_n(x) = 0$ for $x \geqq n+1$, $F_n(x_\nu) = 0$ $(\nu = 1, 2, \ldots)$, and $F_n \in \mathscr{H}_L(I)$. Clearly,

$$\begin{aligned} \left\{\int_I (LF - LF_n)^2\right\}^{1/2} \leqq \left\{\int_{n+1}^{\infty} (LF)^2\right\}^{1/2} + \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} \\ \times \left\{\int_n^{n+1} a_l^2(x)[D^{l-k}F(x) D^k \bar{E}(n+1-x)]^2\right\}^{1/2} \end{aligned} \tag{5.28}$$

and

$$\begin{aligned} &\int_n^{n+1} a_l^2(x)[D^{l-k}F(x)D^k\bar{E}(n+1-x)]^2 \\ &\leqq \sup_{x\in I} a_l^2(x) \int_n^{n+1} [D^{l-k}F(x)D^k\bar{E}(n+1-x)]^2\, dx \\ &\leqq \sup_{x\in I} a_l^2(x) \cdot \sup_{x\in[0,1]} [D^k\bar{E}(x)]^2 \cdot \int_0^1 [D^{l-k}F(x+n)]^2\, dx, \\ &\qquad l = 0, \ldots, m;\ k = 0, \ldots, l. \end{aligned} \tag{5.29}$$

But for $l-k<m$ the sequence of functions $\{D^{l-k}F(x+n)\}$ is bounded and converges to 0 on $0 \leqq x \leqq 1$, hence $\int_0^1 [D^{l-k}F(x+n)]^2\, dx = o(1)$ for $l-k \leqq m$. Also, of course, $\int_{n+1}^{\infty} (LF)^2 = o(1)$. (5.28) and (5.29) together prove (5.26) for $\Phi_\varepsilon = F_n$ with n sufficiently large.

If $D^kF(x_1) = 0$ $(k = 0, 1, \ldots, m-1)$ in the preceding lemma then also $D^k\Phi_\varepsilon(x_1) = 0$. An obvious extension of Lemmas 2a, 2b and 3a is

LEMMA 3b. *Suppose* (5.25) *holds*, $\cdots x_{-2} < x_{-1} < x_1 < x_2 < \cdots$, $\lim x_{-n} = x_{-\infty} \geqq -\infty$, $\lim x_n = x_\infty \leqq \infty$, $I = (x_{-\infty}, x_\infty)$, $F \in \mathscr{H}_L(I)$, $F(x_{-n}) = F(x_n) = 0$ $(n = 1, 2, \ldots)$, $\lim_{x\to+\infty\,(-\infty)} D^kF(x) = 0$ $(k = 0, 1, \ldots, m-1)$ *if* $x_{\infty\,(-\infty)} = +(-)\infty$, and $\varepsilon > 0$. *Then there exists a function* $\Phi_\varepsilon \in \mathscr{H}_L(\boldsymbol{R})$ *which vanishes at* $\ldots x_{-2}, x_{-1}, x_1, x_2, \ldots$ *and near* $x_{-\infty}$ *and* x_∞, *such that*

$$\int_I (LF - L\Phi_\varepsilon)^2 < \varepsilon^2. \tag{5.30}$$

THEOREM 3d. *Problems* (II), (V) *and* (III), (V) *for an operator L satisfying* (5.25) *have unique solutions if $J \cap B$ contains a sequence that converges to $+\infty$ and/or $-\infty$.*

Proof. We present the proof only for one of the possible cases: $J=(a, \infty)$ and $J \cap B$ is the bisequence $\cdots x_{-2}<x_{-1}<x_1<x_2\cdots$, with $\lim x_{-n}=a$, $\lim x_n=+\infty$. Let G be the difference of two solutions. Because of (V), the limits $\lim_{x\to+\infty} D^kG(x)$ ($k=0, 1, \ldots, m-1$) exist. Since $G(x_n)=0$ ($n=1, 2, \ldots$), we conclude $\lim G(x)=0$. By Rolle's theorem there is a sequence $y_1<y_2<\cdots$, $\lim y_n=+\infty$, such that $DG(y_n)=0$ ($n=1, 2, \ldots$). It follows that $\lim DG(x)=0$. More generally, we conclude

$$\lim_{x\to+\infty} D^kG(x) = 0, \qquad k = 0, 1, \ldots, m-1. \tag{5.31}$$

By Lemma 3b it follows now that there exists a function $\Phi_\varepsilon \in \mathscr{H}_L(J)$ which vanishes at the points of $J \cap B$ and near a and $+\infty$, such that

$$\left| \int_J (LG)^2 - \int_J LG \cdot L\Phi_\varepsilon \right| < \varepsilon^2. \tag{5.32}$$

The support of Φ_ε includes only finitely many of the points of $J \cap B$. Hence, integration by parts gives $\int_J LG \cdot L\Phi_\varepsilon=0$, and (5.32) leads to $LG=0$. Since $D^kG(a)=0$ ($k=0, 1, \ldots, m-1$) by (IIiv), we conclude $G=0$.

In certain cases equation (5.31) is implied by the hypotheses $G(x_n)=0$ ($n=1, 2, \ldots$) and $G \in \mathscr{H}_L(J)$. In these cases, condition (V) may be omitted in Theorem 3d.

We add a uniqueness theorem for problem (I_{ab}) of §2, where I_{ab} is a finite interval which is not a discrete component of B.

THEOREM 3e. *The boundary-value problem (I_{ab}) has a unique solution if B contains a Tchebychev set of the operator L.*

Proof. Let G be the difference of two solutions. G is in $\mathscr{C}^{2m}(I_{ab}-B)$, and in particular has continuous derivatives of order $\leqq 2m$ in one-sided neighborhoods of the points a and b. We define the functions G_a, G_b by the conditions

$$\begin{aligned} LG_a(x) &= 0, \quad x < a, \quad D^kG_a(a) = D^kG(a), \\ LG_b(x) &= 0, \quad x > b, \quad D^kG_b(b) = D^kG(b), \end{aligned} \qquad k = 0, 1, \ldots, m-1, \tag{5.33}$$

and the function $\hat{G}$ on $\boldsymbol{R}$ by

$$\begin{aligned} \hat{G}(x) &= G_a(x), & x &< a, \\ &= G(x), & a &\leqq x \leqq b, \\ &= G_b(x), & x &> b. \end{aligned} \tag{5.34}$$

By (I_{ab}iv), $\hat{G}$ has continuous derivatives of order $\leqq 2m-2$ near a and b. If $a \notin B$ then $\inf B=a'>a$ and Corollary 1.1 shows that $D^kLG(a)=0$ for $k=0, 1, \ldots, m$.

Therefore, in this case, $\hat{G}$ has continuous derivatives of order $\leqq 2m$ near a. The same result holds near b if $b \notin B$. Altogether, we have

$$\begin{aligned} \Lambda\hat{G}(x) &= 0, \qquad x \in \boldsymbol{R}-B, \\ \hat{G}(x) &= 0, \qquad x \in B, \\ \hat{G} &\in \mathscr{H}_L(\boldsymbol{R}) \cap \mathscr{C}^{2m}(\boldsymbol{R}-B) \cap \mathscr{C}^{2m-2}(\boldsymbol{R}-B'). \end{aligned}$$

Since B contains a Tchebychev set for L, we conclude $\hat{G}=0$, by Theorem 3a. In particular, $G=0$.

6. **The selfadjoint boundary-value operator.** In this section we determine the selfadjoint operator defined by the homogeneous boundary-value problem ($\boldsymbol{R}$i, ii, iii), with $f=0$. As the underlying space we take $\mathscr{L}_2(\boldsymbol{R})$ over C, with inner product

$$(F, G) = \int_{\boldsymbol{R}} F\bar{G}. \tag{6.1}$$

The operator L is the same as in (2.1), except that we now assume a_m^{-1} and $a_0, \ldots, a_m$ bounded (besides $a_k \in \mathscr{C}^m(\boldsymbol{R})$). For convenience, we consider the operator

$$\Lambda = L^*L+I \tag{6.2}$$

rather than L^*L. It is clear that the selfadjoint extension and the spectral decomposition of L^*L are easily derived from those of Λ. As the initial domain of the operator we take

$$\mathscr{D}_B^0 = \{F \in \mathscr{C}^{2m}(\boldsymbol{R}-B) \cap \mathscr{C}^{2m-2}(\boldsymbol{R}-B') : F, LF, L^*LF \in \mathscr{L}_2(\boldsymbol{R}) \text{ and } F(x) = 0,\ x \in B\}. \tag{6.3}$$

The conditions "F and L^*LF in $\mathscr{L}_2(\boldsymbol{R})$" are added to the previous ones because Λ is to act as an operator from $\mathscr{L}_2(\boldsymbol{R})$ to $\mathscr{L}_2(\boldsymbol{R})$. In (6.3) it is understood that $L^*LF(x)$ remains undefined at the points $x \in B-B'$. We denote by $\mathscr{L}_2^B(\boldsymbol{R})$ the subspace of functions in $\mathscr{L}_2(\boldsymbol{R})$ which vanish at the points of B (this is different from $\mathscr{L}_2(\boldsymbol{R})$ only if B has positive measure). Clearly, $\mathscr{D}_B^0$ is dense in $\mathscr{L}_2^B(\boldsymbol{R})$.

In connection with this problem, we introduce the space $\mathscr{H}_L^B(\boldsymbol{R})$ of functions F with absolutely continuous derivatives of order $\leqq m-1$, with F and LF in $\mathscr{L}_2(\boldsymbol{R})$, with $F(x)=0$ for $x \in B$, and with inner product

$$(F, G)_L = \int_{\boldsymbol{R}} (F\bar{G}+LF\cdot\overline{LG}). \tag{6.4}$$

It is well known that this is a Hilbert space. Clearly, $\|F\|_L \geqq \|F\|$, and $\mathscr{H}_L^B(\boldsymbol{R})$ is densely imbedded in $\mathscr{L}_2^B(\boldsymbol{R})$.

If $\{J_\nu\}$ is the family of discrete components of B (see §3) then

$$\int_{\boldsymbol{R}} (F\bar{G}+LF\cdot\overline{LG}) = \sum_\nu \int_{J_\nu} (F\bar{G}+LF\cdot\overline{LG}) \tag{6.5}$$

for all $F, G \in \mathscr{H}_L^B(\boldsymbol{R})$. Thus, $\mathscr{H}_L^B(\boldsymbol{R})$ appears as the direct sum of the spaces $\mathscr{H}_L^B(J_\nu)$.

The space $\mathscr{H}_L^B(J_\nu)$ is defined like $\mathscr{H}_L^B(\boldsymbol{R})$ above, except that the condition

$$D^kF(c) = 0, \qquad k = 0, 1, \dots, m-1, \tag{6.6}$$

for each finite endpoint c of J_ν is added.

Suppose J is one of the discrete components of B. If J is bounded, $J=(a, b)$, and $J \cap B$ is finite, then $a \in B'$, $b \in B'$, hence $D^kG(a)=D^kG(b)=0$ $(k=0, 1, \dots, m-1)$ for any $G \in \mathscr{H}_L^B(\boldsymbol{R})$, and integration by parts gives

$$\int_J L^*LF\cdot \bar{G} = \int_J LF\cdot\overline{LG}, \qquad F \in \mathscr{D}_B^0,\ G \in \mathscr{H}_L^B(\boldsymbol{R}). \tag{6.7}$$

To establish this formula for a general component J we need another approximation lemma.

Lemma 4. *Given $G \in \mathscr{H}_L^B(\boldsymbol{R})$ and $\varepsilon>0$, there exists a function $\Phi_\varepsilon \in \mathscr{H}_L^B(J)$ whose support includes only finitely many points of $B \cap J$ and such that*

$$\int_J [|G-\Phi_\varepsilon|^2+|LG-L\Phi_\varepsilon|^2] < \varepsilon^2. \tag{6.8}$$

Proof. Assume first $J=(a, b)$, then $a \in B'$, $b \in B'$, and since $G \in \mathscr{H}_L^B(\boldsymbol{R})$, $D^kG(a) = D^kG(b)=0$ $(k=0, 1, \dots, m-1)$. Then the lemma is proved by arguments exactly like those used in the proof of Lemma 2a of the preceding section. We next assume $J=(a, \infty)$ and a is not a limit point of $J \cap B$. Under our hypotheses on the coefficients of L, $G \in \mathscr{H}_L^B(\boldsymbol{R})$ implies $D^kG \in \mathscr{L}_2(\boldsymbol{R})$ $(k=0, 1, \dots, m)$ (see, for example, [4, VI. 6.2]), hence as $n \to \infty$, assuming, without loss of generality, G is real

$$\int_n^\infty (D^kG)^2 = o(1), \qquad k = 0, 1, \dots, m. \tag{6.9}$$

We now set

$$G_n(x) = G(x)E(n+1-x), \qquad x > a, \tag{6.10}$$

where E is the function defined in (5.12). Then $G_n(x)=0$ for $x \geqq n+1$, $G_n \in \mathscr{H}_L^B(J)$, and the support of G_n includes no more than a finite number of points of $J \cap B$. Clearly,

$$\begin{aligned}\left\{\int_J (LG-LG_n)^2\right\}^{1/2} &\leqq \sum_{l=0}^m \left\{\int_J a_l^2(x)[D^lG(x)(1-E(n+1-x))]^2\,dx\right\}^{1/2} \\ &\leqq \sum_{l=0}^m \sup_{x\in J} |a_l(x)|\left\{\int_J [D^lG(x)(1-E(n+1-x))]^2\,dx\right\}^{1/2}.\end{aligned} \tag{6.11}$$

On the other hand,

$$\int_J [D^lG(x)D^0(1-E(n+1-x))]^2\,dx \leqq \int_n^\infty [D^lG(x)]^2\,dx,$$

$$\begin{aligned}\int_J [D^{l-k}G(x)\cdot D^k(1-E(n+1-x))]^2\,dx &= \int_0^1 [D^{l-k}G(x+n)D^kE(1-x)]^2\,dx \\ &\leqq \sup_{x\in[0,1]} [D^kE(1-x)]^2 \int_n^\infty [D^{l-k}G(x)]^2\,dx, \\ &\qquad l = 0, 1, \dots, m;\ k = 1, \dots, l.\end{aligned} \tag{6.12}$$

By (6.9), the right-hand terms of (6.12) go to 0 as $n \to \infty$; hence the same is true of the right-hand term of (6.11). This shows that (6.8) is valid for $\Phi_\varepsilon = G_n$ with n sufficiently large.

If $J=(a, \infty)$ and a is also a limit point of $J \cap B$, then a combination of the arguments in the proof of Lemma 2a and those used above yield the same result. This is also true of the various possible cases with $J=(-\infty, \infty)$.

We can now establish formula (6.7) for a general component J. If $J=(a, \infty)$ and a is not a limit point of $J \cap B$, then we make use of the fact that $a \in B'$, hence $D^kG(a)=0$ $(k=0, 1, \ldots, m-1)$ if $G \in \mathscr{H}^B_L(\boldsymbol{R})$. Using Lemma 4, we find $\Phi_\varepsilon \in \mathscr{H}^B_L(J)$ which is equal to G near a, equal to 0 near ∞, whose support includes only finitely many points of $J \cap B$ and for which (6.8) holds. Then integration by parts can be carried out and gives $\int_J L^*LF\cdot \Phi_\varepsilon = \int_J LF\cdot L\Phi_\varepsilon$, and as $\varepsilon \to 0$, (6.7). If $J=(a, \infty)$ and a is a limit point of $J \cap B$, then we find, by Lemma 4 again, $\Phi_\varepsilon \in \mathscr{H}^B_L(J)$ with compact support in J, such that (6.8) holds. Integration by parts can be carried out and gives $\int_J L^*LF\cdot \Phi_\varepsilon = \int_J LF\cdot L\Phi_\varepsilon$, hence (6.7). By similar arguments one disposes of the remaining cases.

Since (6.7) holds for every discrete component $J=J_\nu$ of B and since it trivially holds for $J=B'$, it follows that (6.7) is valid for $J=\boldsymbol{R}$. Therefore, we have proved

$$(\Lambda F, G) = (F, G)_L, \qquad F \in \mathscr{D}^0_B,\ G \in \mathscr{H}^B_L(\boldsymbol{R}). \tag{6.11}$$

In particular, $(\Lambda F, F) = \|F\|^2_L \geqq 0$ for each $F \in \mathscr{D}^0_B$, and this shows that Λ is a symmetric operator.

We consider selfadjoint extensions Λ_B of Λ, but only such whose domain is contained in $\mathscr{H}^B_L(\boldsymbol{R})$:

$$\operatorname{dom}(\Lambda_B) \subset \mathscr{H}^B_L(\boldsymbol{R}). \tag{6.12}$$

This restriction symbolizes the boundary condition for our problem. We prove

THEOREM 4. *The operator* $\Lambda = L^*L + I$ *from* $\mathscr{L}_2(\boldsymbol{R})$ *to* $\mathscr{L}_2(\boldsymbol{R})$ *with domain* (6.3) *has a unique selfadjoint extension* Λ_B *with domain a subset of* $\mathscr{H}^B_L(\boldsymbol{R})$.

Proof. The construction of Λ_B is essentially that of the so-called Friedrichs extension (see [5; §124]). If $G \in \mathscr{H}^B_L(\boldsymbol{R})$ and $\Phi \in \mathscr{L}_2(\boldsymbol{R})$ then

$$|(G, \Phi)| \leqq \|G\|\,\|\Phi\| \leqq \|G\|_L\|\Phi\| \tag{6.13}$$

and, therefore, there exists a unique $F \in \mathscr{H}^B_L(\boldsymbol{R})$, such that

$$(G, \Phi) = (G, F)_L, \qquad G \in \mathscr{H}^B_L(\boldsymbol{R}). \tag{6.14}$$

We write $F = \Gamma\Phi$, and since by (6.13), $\|F\|_L \leqq \|\Phi\|$, we see that Γ is a bounded linear transformation from $\mathscr{L}_2(\boldsymbol{R})$ to $\mathscr{H}^B_L(\boldsymbol{R})$. If Ψ is arbitrary element of $\mathscr{L}_2(\boldsymbol{R})$, and we use $G = \Gamma\Psi$ in (6.14), we obtain

$$\begin{aligned}(\Gamma\Psi, \Phi) &= (\Gamma\Psi, F)_L = \overline{(F, \Gamma\Psi)_L} \\ &= \overline{(F, \Psi)} = (\Psi, \Gamma\Phi), \qquad \Phi, \Psi \in \mathscr{L}_2(\boldsymbol{R}).\end{aligned}$$

This shows that Γ is selfadjoint. Let Γ_B denote the restriction of Γ to $\mathscr{L}_2^B(\boldsymbol{R})$. Of course, Γ_B is a bounded selfadjoint operator on $\mathscr{L}_2^B(\boldsymbol{R})$.

If $\Gamma_B\Phi=0$ for some $\Phi\in\mathscr{L}_2^B(\boldsymbol{R})$, then by (6.14), $(G,\Phi)=0$ for all $G\in\mathscr{H}_L^B(\boldsymbol{R})$, and since $\mathscr{H}_L^B(\boldsymbol{R})$ is dense in $\mathscr{L}_2^B(\boldsymbol{R})$, it follows that $\Phi=0$. Thus, Γ_B is injective, and its left inverse Λ_B, for which

$$\Lambda_B\Gamma_B\Phi = \Phi, \qquad \Phi\in\mathscr{L}_2^B(\boldsymbol{R}), \tag{6.15}$$

is defined. If $\mathscr{D}_B$ denotes the domain of Λ_B (=the range of Γ_B) then (6.14) becomes

$$(\Lambda_B F, G) = (F, G)_L, \qquad F\in\mathscr{D}_B,\ G\in\mathscr{H}_L^B(\boldsymbol{R}). \tag{6.16}$$

Conversely, if $F\in\mathscr{H}_L^B(\boldsymbol{R})$ is such that $(G,F)_L=(G,\Phi)$ for some $\Phi\in\mathscr{L}_2^B(\boldsymbol{R})$, then by (6.14) $F=\Gamma_B\Phi$, hence $\Phi=\Lambda_B F$. Thus, (6.16) defines $\mathscr{D}_B$ and Λ_B completely. Λ_B, as the left-inverse of the selfadjoint Γ_B, is selfadjoint. Its domain $\mathscr{D}_B$ is a dense subspace of $\mathscr{H}_L^B(\boldsymbol{R})$. In fact, if $G_0\in\mathscr{D}_B^\perp$, then $0=(G_0,F)_L=(G_0,\Lambda_B F)$ for all $F\in\mathscr{D}_B$. Since the range of Λ_B is $\mathscr{L}_2^B(\boldsymbol{R})$, this proves $G_0=0$. Comparing (6.16) with (6.11), we conclude that Λ_B is an extension of Λ. If Λ' is any selfadjoint extension of Λ with domain $\mathscr{D}'\subset\mathscr{H}_L^B(\boldsymbol{R})$ then by (6.14)

$$(F, \Gamma_B\Lambda' G)_L = (F, \Lambda' G) = (\Lambda' F, G) = (\Lambda F, G) = (F, G)_L \tag{6.17}$$

for any $F\in\mathscr{D}_B^0$, $G\in\mathscr{D}'$. By (6.16), $(\Lambda F, G)=(F,G)_L$. Thus, (6.17) gives

$$(F, \Gamma_B\Lambda' G - G)_L = 0$$

for $F\in\mathscr{D}_B^0$, hence also for $F\in\mathscr{H}_L^B(\boldsymbol{R})$. Therefore, $G=\Gamma_B\Lambda' G$, $G\in\mathscr{D}_B$ and $\Lambda_B G=\Lambda' G$. We have shown that Λ_B is an extension of Λ', and since Λ' is assumed to be selfadjoint (hence maximal symmetric), it follows that $\Lambda'=\Lambda_B$. The proof of Theorem 4 is complete.

We now give an explicit characterization of the selfadjoint extension Λ_B. Let $\mathscr{H}_{L^*L}(\Omega)$ denote the class of functions F having absolutely continuous derivatives of order $\leqq 2m-1$ on the open set $\Omega\subset\boldsymbol{R}$ and such that $L^*LF\in\mathscr{L}_2(\Omega)$. Then we have

COROLLARY 4.1. *The selfadjoint extension Λ_B is characterized by*

$$\begin{aligned} \operatorname{dom}(\Lambda_B) &= \mathscr{D}_B = \mathscr{H}_L^B(\boldsymbol{R})\cap\mathscr{H}_{L^*L}(\boldsymbol{R}-B)\cap\mathscr{C}^{2m-2}(\boldsymbol{R}-B'), \\ \Lambda_B F(x) &= L^*LF(x)+F(x), \qquad x\in\boldsymbol{R}-B,\ F\in\mathscr{D}_B. \end{aligned} \tag{6.18}$$

Proof. Assume $F\in\mathscr{D}_B$. We first show that $F\in\mathscr{H}_{L^*L}(\boldsymbol{R}-B)$. Let J be one of the discrete components of the set B. By (6.16) we must have

$$\int_J \Lambda_B F\cdot\bar G = \int_J F\cdot\bar G + \int_J LF\cdot L\bar G \tag{6.19}$$

for every function $G\in\mathscr{H}_L^B(\boldsymbol{R})$. If we denote $(\Lambda_B-I)F$ by Φ, LF by Ψ, (6.19) becomes $(\Phi, G)=(\Psi, LG)$, and by well-known arguments (see, e.g. [4, VI.1.9]), one concludes that Ψ has absolutely continuous derivatives of order $\leqq m-1$ in

$J-B$, and that $\Phi=L^*\Psi$. For F this means, F has absolutely continuous derivatives of order $\leqq 2m-1$ in $J-B$, and $\Lambda_B F(x)=L^*LF(x)+F(x)$ for $x\in J-B$. Since F and $\Lambda_B F$ are in $\mathscr{L}_2(\boldsymbol{R})$ we conclude $F\in\mathscr{H}_{L^*L}(\boldsymbol{R}-B)$.

We show next that $F\in\mathscr{C}^{2m-2}(\boldsymbol{R}-B')$. Let J be as above and assume $D^{2m-k_*}F$ is discontinuous, for some maximal k_*, $2\leqq k_*\leqq m$, at some point $x_*\in J\cap B$. Then one can show that $D^{m-k_*}LF$ is discontinuous at x_*. We choose $G\in\mathscr{H}_L^B(\boldsymbol{R})$ so that x_* is the only point of B in its support and that $D^{k_*-1}G(x_*)\neq 0$, whereas $D^{k-1}G(x_*)=0$ for $k\neq k_*$, $k=1,2,\ldots,m$. Then, proceeding as in (2.7), one obtains

$$\int_J LF\cdot\overline{LG}-\int_J L^*LF\cdot\overline{G}\neq 0, \tag{6.20}$$

and this contradicts (6.16).

So far we have shown that $\mathscr{D}_B$ is included in $\mathscr{H}_L^B(\boldsymbol{R})\cap\mathscr{H}_{L^*L}(\boldsymbol{R}-B)\cap\mathscr{C}^{2m-2}(\boldsymbol{R}-B')$. Assume now F belongs to the latter set and J is as above. Then integration by parts gives immediately

$$\int_J LF\cdot\overline{LG} = \int_J L^*LF\cdot\overline{G} \tag{6.21}$$

for every $G\in\mathscr{H}_L^B(J)$ with compact support in J. By Lemma 4, equation (6.21) is valid for every $G\in\mathscr{H}_L^B(\boldsymbol{R})$. Since this is true for every discrete component $J=J_\nu$ of B, and trivial for $J=B'$, we have proved

$$(F,G)_L = ((L^*L+I)F,G), \qquad G\in\mathscr{H}_L^B(\boldsymbol{R}). \tag{6.22}$$

Comparison with (6.16) shows that $F\in\mathscr{D}_B$, and this completes the proof of the corollary.

In the proof of Theorem 4 it was shown that the range of the operator Λ_B is $\mathscr{L}_2^B(\boldsymbol{R})$. Therefore, we have

COROLLARY 4.2. *For every $G\in\mathscr{L}_2^B(\boldsymbol{R})$ there exists a unique solution $F=\Gamma_B G$ of the equation $(L^*L+I)F=G$ which belongs to*

$$\mathscr{H}_L^B(\boldsymbol{R})\cap\mathscr{H}_{L^*L}(\boldsymbol{R}-B)\cap\mathscr{C}^{2m-2}(\boldsymbol{R}-B').$$

The solution operator Γ_B is continuous.

For the boundary-value problem (I_{ab}i–iv) on the finite interval I_{ab} essentially the same analysis can be carried out. The main difference is expressed in the "natural boundary conditions" (I_{ab}iv) for the endpoints of the interval. Thus, we start with the operator $\Lambda=L^*L+I$ with domain.

$$\begin{aligned}\mathscr{D}_B^0 = \{F\in\mathscr{C}^{2m}(I_{ab}-B)\cap\mathscr{C}^{2m-2}(I_{ab}-B') : F(x) &= 0 \text{ for } x\in B\\ \text{and } D^kLF(a) = D^kLF(b) &= 0 \text{ for } k=0,1,\ldots,m-2\}.\end{aligned} \tag{6.22}$$

The results are comprised in

COROLLARY 4.3. *Suppose the set B is contained in the interval $I_{ab}=[a,b]$, whose endpoints are not limit points of B. Then the operator $\Lambda=L^*L+I$ from $\mathscr{L}_2(I_{ab})$ to*

$\mathscr{L}_2(I_{ab})$ *with domain* (6.22) *has a unique selfadjoint extension* Λ_B *with domain* $\mathscr{D}_B$ *a subset of* $\mathscr{H}_L^B(I_{ab})$. *Explicitly*:

$$\begin{aligned}\mathscr{D}_B &= \{F \in \mathscr{H}_L^B(I_{ab}) \cap \mathscr{H}_{L^*L}(I_{ab}-B) \cap \mathscr{C}^{2m-2}(I_{ab}-B') : \\ &\qquad D^kLF(a) = D^kLF(b) = 0 \text{ for } k = 0, 1, \ldots, m-2\},\end{aligned} \tag{6.23}$$

$$\Lambda_B F(x) = L^*LF(x)+F(x), \qquad x \in I_{ab}-B,\ F \in \mathscr{D}_B.$$

For every $G \in \mathscr{L}_2^B(I_{ab})$ *there exists a unique solution* $F=\Gamma_B G$ *of the equation* $(L^*L+I)F=G$ *which belongs to* $\mathscr{D}_B$.

In the special case where B *is a finite set, the above descriptions of* $\mathscr{D}_B$ *and* $\mathscr{D}_B^0$ *are simplified*:

$$\begin{aligned}\mathscr{D}_B^0 &= \{F \in \mathscr{C}^{2m}(I_{ab}-B) \cap \mathscr{C}^{2m-2}(I_{ab}) : F(x) = 0 \text{ for } x \in B \\ &\qquad \text{and } D^kLF(a) = D^kLF(b) = 0 \text{ for } k = 0, 1, \ldots, m-2\}, \\ \mathscr{D}_B &= \{F \in \mathscr{H}_2^{2m}(I_{ab}-B) \cap \mathscr{C}^{2m-2}(I_{ab}) : F(x) = 0 \text{ for } x \in B \\ &\qquad \text{and } D^kLF(a) = D^kLF(b) = 0 \text{ for } k = 0, 1, \ldots, m-2\}.\end{aligned} \tag{6.24}$$

It is clear in this case that the selfadjoint extension Λ_B is simply the closure of Λ. The same is true for each of the selfadjoint extensions in this section.

7. **Approximations of the solution.** Let J be a discrete component of B. We will construct a sequence of approximations converging to the unique solution F of (Ii, ii, iii, iv) if J is an interval of type I while if J is an interval of type II or III the approximations will converge, respectively, to that solution F of boundary-value problem II or III which is singled out as the unique solution of the associated minimization problem. For the case $J=\boldsymbol{R}$ we will require that B contain a Tchebychev set $\{x_1 < \cdots < x_m\}$ for L. Notice that we do not require that condition (4.1) be satisfied and, in general, the solutions of the boundary-value problems II and III will not be unique. The convergence of the approximations takes place in $\mathscr{H}_L(J)$, which implies, in particular, uniform convergence of the derivatives through order $m-1$ on compact subsets of J.

Suppose then that J is an interval of type I of the form $J=(a, b)$ and let $B_n =\{x_1, x_2, \ldots, x_n\}\subset J \cap B$ if $J \cap B$ is not empty. The boundary-value problem

$$\begin{aligned}&(\mathrm{I}^n\mathrm{i}) && \Lambda F_n(x) = 0, && x \in J-B_n, \\ &(\mathrm{I}^n\mathrm{ii}) && F_n(x) = f(x), && x \in B_n, \\ &(\mathrm{I}^n\mathrm{iii}) && F_n \in \mathscr{H}_L(J) \cap \mathscr{C}^{2m}(J-B_n) \cap \mathscr{C}^{2m-2}(J) \\ &(\mathrm{I}^n\mathrm{iv}) && D^kF_n(a) = D^kf(a),\quad D^kF_n(b) = D^kf(b), && k = 0, 1, \ldots, m-1,\end{aligned} \tag{7.1}$$

has a unique solution, and

$$\inf_{G\in\mathscr{U}_n} \int_J (LG)^2 = \int_J (LF_n)^2 \tag{7.2}$$

where $\mathscr{U}_n=\{G \in \mathscr{H}_L(J) : G(x)=f(x),\ x \in B_n,\ D^kG(a)= D^kf(a),\ D^kG(b)= D^kf(b), k=0, 1, \ldots, m-1\}$. The solution F_n is characterized by the orthogonality property

$$\int_J LF_n(LG-Lf) = 0 \quad \text{for all } G \in \mathscr{U}_n. \tag{7.3}$$

Now if $B \cap J$ is an infinite point set, say $B \cap J$ contains a sequence converging to a or b, and if we write

$$B \cap J = \bigcup_{n=1}^{\infty} B_n$$

where $B_n \subset B_{n+1}$, then we have the following:

THEOREM 5. *Let F_n be the solution of* (I^ni, ii, iii, iv) *for $n=1, 2, \ldots$. Then $F_n \to F$ in $\mathscr{H}_L(J)$ where F is the solution of the boundary-value problem* (Ii, ii, iii, iv) *on J* (*which is unique by Theorem* 3a).

Proof. By (7.2) and (7.3) we have

$$\int_J (LF_n)^2 \leqq \int_J (LF)^2, \tag{7.4}$$

$$\int_J (LF_n)^2 \leqq \int_J (LF_N)^2, \qquad n \leqq N, \tag{7.5}$$

$$\int_J (LF_N - LF_n)^2 = \int_J (LF_N)^2 - \int_J (LF_n)^2, \qquad n \leqq N. \tag{7.6}$$

Using the norm

$$\|u\|_L^2 = \sum_{k=0}^{m-1} (u^{(k)}(a))^2 + \int_J (Lu)^2, \tag{7.7}$$

it follows from (7.4), (7.5) and (7.6) that F_n converges in $\mathscr{H}_L(J)$, say $F_n \to G$. Since convergence in the norm (7.7) implies pointwise convergence it follows that $G \in \bigcap_{n=1}^{\infty} \mathscr{U}_n$. It follows from (7.4) that

$$\int_J (LG)^2 = \lim_{n\to\infty} \int (LF_n)^2 \leqq \int_J (LF)^2.$$

Since F is the unique solution of the minimization problem

$$\inf_{u \in \bigcap_n \mathscr{U}_n} \int_J (Lu)^2 = \int_J (LF)^2$$

it follows that $F(x) = G(x)$ for all $x \in J$ and hence $F_n \to F$ as asserted.

If J is an interval of type II, say $J = (a, \infty)$ for concreteness, and $B_n \subset J \cap B$ is chosen as before, then the boundary-value problem

$$\begin{array}{lll} (\mathrm{II}^n\mathrm{i}) & \Lambda F_n(x) = 0, & x \in J - B_n, \\ (\mathrm{II}^n\mathrm{ii}) & F_n(x) = f(x), & x \in B_n, \\ (\mathrm{II}^n\mathrm{iii}) & F_n \in \mathscr{H}_L(J) \cap \mathscr{C}^{2m}(J - B_n) \cap \mathscr{C}^{2m-2}(J), & \\ (\mathrm{II}^n\mathrm{iv}) & D^k F_n(a) = D^k f(a), & k = 0, 1, \ldots, m-1, \end{array} \tag{7.8}$$

has a unique solution, and $\int_J (LF_n)^2 \leqq \int_J (LG)^2$, $G \in \mathscr{U}_n$, where now

$$\mathscr{U}_n = \{G \in \mathscr{H}_L(J) : G(x) = f(x),\ x \in B_n,\ G^{(k)}(a) = f^{(k)}(a),\ k = 0, 1, \ldots, m-1\}.$$

As before, the solution F_n is characterized by the orthogonality property

$$\int_J LF_n(Lf-LG) = 0 \quad \text{for all } G \in \mathscr{U}_n.$$

By using the minimizing and orthogonality properties of F_n one shows as before that $F_n \to F$ in $\mathscr{H}_L(J)$ with respect to the norm defined by (7.7), where F is the unique solution of (IIi, ii, iii, iv) which solves the minimization problem

$$\inf_{G\in\mathscr{U}} \int_J (LG)^2 = \int_J (LF)^2,$$

where

$$\mathscr{U} = \{G \in \mathscr{H}_L(J) : G(x) = f(x),\, x \in B,\, G^{(k)}(a) = f^{(k)}(a),\, k = 0, 1, \ldots, m-1\}. \tag{7.9}$$

Now, if $J=\boldsymbol{R}$, then let $B_m=\{x_1<\cdots<x_m\}$ be a Tchebychev set for B and define a norm by

$$\|G\|_L^2 = \sum_{k=1}^m (G(x_k))^2 + \int_{\boldsymbol{R}} (LG)^2 \tag{7.10}$$

on $\mathscr{H}_L(\boldsymbol{R})$. The norm defined by (7.10) is clearly equivalent to the norm determined by (2.3). Consider sets $B_m \subset B_n \subset B_{n+1} \subset B$ as before.

Setting $\xi_n = \min_{x\in B_n} x$ and $\eta_n = \max_{x\in B_n} x$ then there is a unique solution F_n of the boundary-value problem

$$(7.11)\quad \begin{array}{lll} (\mathrm{III}^n\mathrm{i}) & \Lambda F_n(x) = 0, & x \in \boldsymbol{R} - B_n, \\ (\mathrm{III}^n\mathrm{ii}) & F_n(x) = f(x), & x \in B_n, \\ (\mathrm{III}^n\mathrm{iii}) & F \in \mathscr{H}_L(\boldsymbol{R}) \cap \mathscr{C}^{2m}(\boldsymbol{R}) \cap \mathscr{C}^{2m-2}(\boldsymbol{R}-B_n), & \\ (\mathrm{III}^n\mathrm{iv}) & D^k(LF)(\xi_n) = D^k(LF)(\eta_n) = 0, & k = 0, 1, \ldots, m-2, \end{array}$$

and

$$\int_{\boldsymbol{R}} (LG)^2 > \int_{\boldsymbol{R}} (LF_n)^2, \qquad G \in \mathscr{U}_n,\ G \neq F_n$$

where

$$\mathscr{U}_n = \{G \in \mathscr{H}_L(\boldsymbol{R}) : G(x) = f(x),\, x \in B_n\}.$$

A corresponding orthogonality property holds and one concludes as before that $F_n \to F$ in $\mathscr{H}_L(\boldsymbol{R})$ with respect to the norm defined by (7.10), where F is that solution of (IIIi, ii, iii, iv) for which

$$\int_{\boldsymbol{R}} (LG)^2 > \int_{\boldsymbol{R}} (LF)^2, \qquad G \in \mathscr{U},\ G \neq F$$

and

$$\mathscr{U} = \{G \in \mathscr{H}_L(\boldsymbol{R}) : G(x) = f(x),\ x \in B\}. \tag{7.12}$$

We summarize this discussion in

THEOREM 6. *The solutions F_n of* (II^{n}i, ii, iii, iv) *converge in $\mathscr{H}_L(J)$ to that solution F of boundary-value problem* II *which minimizes $\int_{\boldsymbol{R}} (LG)^2$ in the class of functions*

(7.9) *if J is an interval of type* II. *If $J=\boldsymbol{R}$ and B contains a Tchebychev set then the solutions F_n of* (III^{n}i, ii, iii, iv) *converge to that solution F of boundary-value problem* III *which minimizes $\int_{\boldsymbol{R}}(LG)^2$ in the class of functions* (7.12).

8. **More general differential operators.** In this section we seek a solution F of a differential equation $\Lambda' F=0$ of order $2m$. In contrast with §2 it is no longer assumed that Λ' is of the form L^*L, but more generally we suppose that there are differential operators L_p, $p=1, 2, \ldots, q$, with coefficients in $\mathscr{C}^m(\boldsymbol{R})$ such that

$$L_p = \sum_{l=0}^{m_p} a_{lp} D^l, \tag{8.1}$$

$$\Lambda' = \sum_{p=1}^{q} L_p^* L_p = a_m^2 D^{2m} + \cdots + a_0, \tag{8.2}$$

where L_p^* denotes the formal adjoint of L_p. Moreover, we suppose that $m=m_q>m_p$ $(1\leqq p<q)$ and $a_m(x)=a_{mq}(x)\geqq\alpha>0$ for all $x\in\boldsymbol{R}$.

Let $\mathscr{H}'=\mathscr{H}'(\boldsymbol{R})$ denote the class of real-valued functions G on $\boldsymbol{R}$ with $G^{(m-1}$ absolutely continuous for which $\sum_{p=1}^{q}\int_{\boldsymbol{R}}(L_pG)^2<\infty$, with the quadratic norm

$$\|G\|'^2 = \sum_{k=0}^{m-1} [G^{(k)}(0)]^2 + \sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pG)^2. \tag{8.3}$$

One verifies easily that $\mathscr{H}'$ with the corresponding inner product is a Hilbert space. Let B be a closed subset of $\boldsymbol{R}$. We distinguish the two cases when B' is nonempty and when B' is empty. In the latter case we assume that B contains a finite subset B_0 such that

$$F\in\mathscr{H}',\quad L_pF = 0\ (p = 1,\ldots,q),\quad F(x) = 0\quad (x\in B_0) \Rightarrow F = 0. \tag{8.4}$$

The boundary-value problem to be considered, for a given $f\in\mathscr{H}'$, is of the form,

$$\Lambda' F(x) = 0, \qquad x\in \boldsymbol{R}-B, \tag{8.5i}$$

$$F(x) = f(x), \qquad x\in B, \tag{8.5ii}$$

$$F\in\mathscr{H}'(\boldsymbol{R})\cap\mathscr{C}^{2m}(\boldsymbol{R}-B)\cap\mathscr{C}^{2m-2}(\boldsymbol{R}-B'). \tag{8.5iii}$$

We now prove

THEOREM 7. *Suppose B is a closed set of real numbers, B' the set of limit points of B, and suppose* (8.4) *holds if B' is empty. Let Λ' be the differential operator* (8.2) *and let $f\in\mathscr{H}'$. Then there exists a solution $F=F_*$ of the boundary-value problem* (8.5i, ii, iii), *where F_* uniquely minimizes $\sum_{p=1}^{q}\int_{\boldsymbol{R}}(L_pF)^2$ among all functions $F\in\mathscr{H}'$ for which $F(x)=f(x)$, $x\in B$.*

Proof. In this case we cannot make use of the results of [2] to prove the existence of an element minimizing $\sum_{p=1}^{q}\int_{\boldsymbol{R}}(L_pF)^2$ since we do not have the analogue of the operator L in §2. However, the existence of the minimizing element is easily demonstrated directly by considering the minimization problem in $\mathscr{H}'(\boldsymbol{R})$ rather than in $\mathscr{L}_2(\boldsymbol{R})$. We seek an element F_* in the flat

$$\mathscr{V} = \{G \in \mathscr{H}' : G(x) = f(x),\ x \in B\}. \tag{8.6}$$

The parallel subspace $\mathscr{V}^0 = \{G \in \mathscr{H}' : G(x)=0,\ x \in B\}$ is clearly closed in $\mathscr{H}'$. We now introduce new quadratic norms in $\mathscr{H}'$, equivalent to (8.3):

$$\begin{aligned} \|G\|^{*2} &= \sum_{k=0}^{m-1} [G^{(k)}(a)]^2 + \sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pG)^2 \quad \text{if } a \in B' \neq \varnothing, \\ \|G\|^{*2} &= \sum_{x \in B_0} [G(x)]^2 + \sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pG)^2 \quad \text{if } B' = \varnothing. \end{aligned} \tag{8.7}$$

Thus $\mathscr{V}^0$ is closed in $\mathscr{H}'$ with respect to the norm (8.7). Now, minimizing $\sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pG)^2$ for $G \in \mathscr{V}$ is equivalent to minimizing $\|G\|^*$ and it follows that there exists a unique $F_* \in \mathscr{V}$ such that

$$\sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pF_*)^2 = \min_{G \in \mathscr{V}} \sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pG)^2. \tag{8.8}$$

From this minimizing property one infers

$$\sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pF_*)(L_pG) = 0 \tag{8.9}$$

for every $G \in \mathscr{V}^0$. As in the proof of Theorem 1 one concludes next that $F_* \in \mathscr{C}^{2m}(\boldsymbol{R}-B)$ and

$$\Lambda' F_*(x) = \sum_{p=1}^{q} L_p^* L_p F_*(x) = 0, \qquad x \in \boldsymbol{R}-B. \tag{8.10}$$

Next suppose J is an open interval in $\boldsymbol{R}-B'$ containing exactly one point x_* of B, and G is an infinitely differentiable function with compact support in J and vanishing at x_*. We use the notation

$$(F) = F(x_*), \qquad [F] = F(x_*-0) - F(x_*+0) \tag{8.11}$$

introduced in §2. Then repeated integration by parts in (8.9) gives

$$\begin{aligned} 0 = \sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_pF_*)(L_pG) &= (DG)\{(-1)^m (a_m^2)[D^{2m-2}F_*] + \cdots\} \\ &+ (D^2G)\{(-1)^{m-1}(a_m^2)[D^{2m-3}F_*] + \cdots\} + \cdots + (D^{m-1}G)(a_m^2)[D^mF_*]. \end{aligned} \tag{8.12}$$

The dots in each of the bracketed expressions stand for terms involving jumps at x_* of derivatives of F_* of order lower than those that are written. By choosing G so that $(DG) = \cdots = (D^{m-2}G) = 0$, $(D^{m-1}G) = 1$, we conclude first $[D^mF_*]=0$. Recursively, we find

$$[F_*^{(m)}] = [F_*^{(m-1)}] = \cdots = [F_*^{(2m-2)}] = 0, \tag{8.13}$$

hence $F_* \in \mathscr{C}^{2m-2}(J)$. Altogether, we have proved that $F = F_*$ is a solution of (8.5i, ii, iii).

Problem (8.5) decomposes into separate problems on the disjoint open intervals J of $\boldsymbol{R} - B'$, exactly as in §3. We can then describe uniqueness conditions as in §4. Indeed, the condition which replaces condition (4.1) is

$$(8.14) \quad \sum_{p=1}^{q} L_p^* L_p F = 0, \quad \sum_{p=1}^{q} \int_I (L_p F)^2 < \infty \Rightarrow L_p F = 0, \qquad p = 1, 2, \ldots, q,$$

if $I \cap B = \varnothing$ and I contains $+\infty$ $(-\infty)$.

Condition (IV) on the solution F becomes in this case:

$$\lim_{x \to +\infty(-\infty)} \sum_{p=1}^{q} B_p[F, G](x) = 0 \quad \text{if } +\infty\ (-\infty) \text{ is a limit point of } J \cap B,$$

$$(\mathrm{IV})' \quad \lim_{x \to \infty} \sum_{p=1}^{q} B_p[F, G](x) - \lim_{x \to -\infty} \sum_{p=1}^{q} B_p[F, G](x) = 0$$

$$\text{if } +\infty \text{ and } -\infty \text{ are limit points of } J \cap B$$

for every function $G \in \mathscr{H}'(\boldsymbol{R})$ that vanishes at the points of B. Here

$$(8.15) \qquad B_p[F, G] = \sum_{k=0}^{m_p - 1} \sum_{l=k+1}^{m_p} (-D)^{l-k-1}(a_{lp} L_p F) D^k G.$$

As in §4 the condition (IV)′ does not restrict the class of problems for which solutions exist. Indeed, we have

THEOREM 8. *The restriction to J of the solution F_* of problem* (8.5i, ii, iii) *which minimizes $\sum_{p=1}^{q} \int_{\boldsymbol{R}} (L_p G)^2$ satisfies condition* (IV)′.

Corresponding to Lemmas 2a and 2b of §5 we have the following two lemmas:

LEMMA 5a. *Let $I = (0, x_1)$ and let $x_1 > x_2 > \cdots$ be a sequence converging to* 0. *Let $F \in \mathscr{H}'(I)$, $F(x_n) = 0$ for $n = 1, 2, \ldots$. Then for each $\varepsilon > 0$ there exists a function Φ_ε in $\mathscr{H}'(I)$ which vanishes at $x_1, x_2, \ldots, x_n, \ldots$ and near* 0, *which agrees with F near x_1 and for which*

$$(8.16) \qquad \sum_{p=1}^{q} \int_I (L_p F - L_p \Phi_\varepsilon)^2 < \varepsilon^2.$$

Proof. By Lemma 2a there exists a function $\Phi_\varepsilon \in \mathscr{H}_{L_q}(I)$ vanishing appropriately and agreeing with F near x_1 such that (5.11), with r.h.s. ε^2/q, holds for $L = L_q$ and,

$$\int_I [L_q F - L_q \Phi_\varepsilon]^2 < \left[q \int_0^{x_1} \int_0^{x_1} [L_p \theta(x, \xi)]^2 \, dx \, d\xi \right]^{-1} \varepsilon^2$$

for $1 \leqq p < q$, where L_p operates on the function $\theta(\cdot, \xi)$ and where $\theta(\cdot, \xi)$ is the unique kernel satisfying $L_q \theta(\cdot, \xi) = \delta_\xi$, $D^k \theta(\xi, \xi) = \delta_{k, m-1}$ $(k = 0, 1, \ldots, m-1)$.

Using the representation

$$D^jG(x) = \int_0^{x_1} D_x^j\theta(x, \xi)L_qG(\xi)\,d\xi, \qquad j = 0, 1, 2, \ldots, m-1,$$

valid for every $G \in \mathscr{H}_{L_q}(I)$ such that $D^kG(0)=0$ for $k=0, 1, \ldots, m-1$, we obtain the estimate, for $1 \leqq p < q$,

$$\begin{aligned}[L_pF-L_p\Phi_\varepsilon]^2 &= \left[\int_0^{x_1} L_p\theta(\cdot, \xi)[L_qF(\xi)-L_q\Phi_\varepsilon(\xi)]\,d\xi\right]^2 \\ &\leqq \int_0^{x_1} [L_p\theta(\cdot, \xi)]^2\,d\xi \int_I [L_qF-L_q\Phi_\varepsilon]^2\end{aligned}$$

and

$$\int_I [L_pF-L_p\Phi_\varepsilon]^2 \leqq \int_0^{x_1}\int_0^{x_1} [L_p\theta(x, \xi)]^2\,dx\,d\xi \int_I [L_qF-L_q\Phi]^2 < \varepsilon^2/q.$$

Now $\Phi_\varepsilon \in \mathscr{H}_{L_p}(I)$ $(1 \leqq p \leqq q)$, hence $\Phi_\varepsilon \in \mathscr{H}'(I)$, and (8.16) is proved.

A similar proof yields

LEMMA 5b. *Suppose* $\cdots x_{-2} < x_{-1} < x_1 < x_2 < \cdots$,

$$\lim x_{-n} = a > -\infty, \quad \lim x_n = b < \infty; \qquad I = (a, b), \quad F \in \mathscr{H}'_\Lambda(I)$$

such that F vanishes at $\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots$. *Then there exists, for each* $\varepsilon > 0$, $\Phi_\varepsilon \in \mathscr{H}'(I)$ *such that* Φ_ε *vanishes near a and b and at* $\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots$ *and such that*

$$\sum_{p=1}^{q} \int_I (L_pF-L_p\Phi_\varepsilon)^2 < \varepsilon^2.$$

We now have, from Lemmas 5a, 5b,

THEOREM 9a. *The solution of boundary-value problem* (I) *for the operator* Λ' *on the bounded interval J is unique.*

THEOREM 9b. *The solution of problems* (II) *and* (III) *for the operator* Λ' *is unique if $J \cap B$ is bounded.*

THEOREM 9c. *Problems* (II), (IV)′ *and* (III), (IV)′ *for the operator* Λ' *have unique solutions if $J \cap B$ contains a sequence that converges to* $+\infty$ *and/or* $-\infty$.

Condition (IV)′ can be replaced by (V) if the following additional hypothesis is assumed:

(8.17) The coefficients a_{lp} $(l = 0, 1, \ldots, m_p; p = 1, 2, \ldots, q)$ are bounded.

Lemmas 3a and 3b of §5 are replaced by the following lemmas:

LEMMA 6a. *Suppose* (8.17) *holds,* $x_1 < x_2 < \cdots$, $\lim x_n = \infty$, $I = (x_1, \infty)$, $F \in \mathscr{H}'(I)$, $F(x_n) = 0$ *for* $n = 1, 2, \ldots$, $\lim_{x\to\infty} D^kF(x) = 0$ $(k = 0, 1, \ldots, m-1)$ *and* $\varepsilon > 0$. *There exists a function* $\Phi_\varepsilon \in \mathscr{H}'(I)$ *which vanishes at* $x_1, x_2, \ldots$ *and near* $+\infty$ *and for which*

$$\sum_{p=1}^{q} \int_I (L_pF-L_p\Phi_\varepsilon)^2 < \varepsilon^2.$$

Proof. Apply (5.28) to each L_p.

In a similar way Lemma 3b is extended by

LEMMA 6b. *Suppose* (8.17) *holds,* $\cdots x_{-2} < x_{-1} < x_1 < x_2 < \cdots$, $\lim x_{-n} = x_{-\infty} \geqq -\infty$, $\lim x_n = x_\infty \leqq \infty$, $I = (x_{-\infty}, x_\infty)$, $F \in \mathscr{H}'(I)$, $F(x_{-n}) = F(x_n) = 0$ *for* $n = 1, 2, \ldots$, $\lim_{x \to x_{\infty}(-\infty)} D^k F(x) = 0$ $(k = 0, 1, \ldots, m-1)$ *if* $x_\infty = +\infty$ $(x_{-\infty} = -\infty)$ *and* $\varepsilon > 0$. *Then there exists a function* $\Phi_\varepsilon \in \mathscr{H}'(\boldsymbol{R})$ *which vanishes at* $\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots$ *and near* $x_{-\infty}$ *and* x_∞ *such that*

$$\sum_{p=1}^{q} \int_I (L_p F - L_p \Phi_\varepsilon)^2 < \varepsilon^2.$$

We then have

THEOREM 10. *Under the hypothesis* (8.17), *problems* (II), (V) *and* (III), (V) *for the operator* Λ' *have unique solutions if* $J \cap B$ *contains a sequence that converges to* $+\infty$ *and/or* $-\infty$.

Proof. By the proof of Theorem 3d, it follows that the difference G of any two solutions of the given boundary-value problem must satisfy $\lim_{x \to +\infty(-\infty)} D^k G(x) = 0$ $(k = 0, 1, \ldots, m-1)$. Lemmas 6a, 6b apply to yield Φ_ε such that, for every $\varepsilon > 0$,

$$\left| \sum_{p=1}^{q} \left(\int_J (L_p G)^2 - \int_J L_p G L_p \Phi_\varepsilon \right) \right| < \varepsilon.$$

But $\sum_{p=1}^{q} \int_J L_p G L_p \Phi_\varepsilon = 0$. Therefore, $\sum_{p=1}^{q} \int_J (L_p G)^2 = 0$ and it follows that $L_p G = 0$ for each $p = 1, 2, \ldots, q$. Thus, $G = 0$ since we have assumed condition (8.4). This proves the uniqueness.

If B is bounded from one or both sides then we expect the solution of problem (8.5i, ii, iii) to be of "lower degree at infinity." This special behavior at ∞ is now to be explored. It cannot be characterized as in the case of problem ($\boldsymbol{R}$i, ii, iii) since the analogue of the operator L is not available now. However, we will show now that the characterization of the behavior at ∞ for solutions of ($\boldsymbol{R}$i, ii, iii) has an equivalent formulation, which can be extended to the more general case. In particular, let $\sup_{x \in B} x = b < \infty$ and suppose the solution F of ($\boldsymbol{R}$i, ii, iii) is unique. Then, by Corollary 2.1, $LF(x) = 0$ for $x > b$.

Defining $\mathscr{H}_{L_b}$ as the Hilbert space of those functions in $\mathscr{H}_L(\boldsymbol{R})$ restricted to (b, ∞), with quadratic norm

$$\|g\|^2_{\mathscr{H}_{L_b}} = \sum_{k=0}^{m-1} [g^{(k)}(b)]^2 + \int_b^\infty (Lg)^2 \tag{8.17}$$

we observe that F_b, the restriction of F to (b, ∞), is in $\mathscr{H}_{L_b}$, and satisfies $(F_b, g)_{\mathscr{H}_{L_b}} = 0$ for every $g \in \mathscr{H}_{L_b}$ satisfying $g^{(k)}(b) = 0$ $(k = 0, 1, \ldots, m-1)$. We designate the subspace of all such functions g by $\mathscr{W}_b^0$ and we define $\mathscr{W}_b$ to be the parallel flat obtained by translating $\mathscr{W}_b^0$ by F_b. Then clearly $F_b \in \mathscr{W}_b \cap \mathscr{W}_b^{0\perp}$. Moreover, if

$G \in \mathscr{W}_b \cap \mathscr{W}_b^{0\perp}$, then for every $g \in \mathscr{W}_b^0$,

$$\int_b^\infty LGLg = 0,$$

and it follows that G solves

$$\inf_{h \in \mathscr{W}_b} \int_b^\infty (Lh)^2 = \int_b^\infty (LG)^2 \tag{8.18}$$

and, since (8.18) has a unique solution, it follows that

$$\int_b^\infty (LG)^2 = \int_b^\infty (LF_b)^2 = 0,$$

i.e., $G = F_b$. We have thus shown that the behavior at ∞ of F, characterized by $LF(x) = 0$ for $x > b$, is equivalent to the characterization that the restriction of F to (b, ∞) lies in $\mathscr{W}_b \cap \mathscr{W}_b^{0\perp}$. This result generalizes in the following manner.

Suppose F uniquely solves (8.5i, ii, iii) where $\sup_{x \in B} x = b < \infty$. Let $\mathscr{H}'_b$ denote the Hilbert space of functions which are the restrictions to (b, ∞) of the functions of $\mathscr{H}'$, with quadratic norm

$$\|g\|^2_{\mathscr{H}'_b} = \sum_{k=0}^{m-1} [g^{(k)}(b)]^2 + \sum_{p=1}^{q} \int_b^\infty (L_p g)^2$$

and let

$$\mathscr{W}'_b = \{g \in \mathscr{H}'_b : g^{(k)}(b) = F^{(k)}(b), k = 0, \ldots, m-1\},$$

with parallel subspace, of codimension m,

$$\mathscr{W}'^0_b = \{g \in \mathscr{H}'_b : g^{(k)}(b) = 0, k = 0, 1, \ldots, m-1\}.$$

We have then

THEOREM 11. *If the set B satisfies* $\sup_{x \in B} x = b < \infty$ *and if the solution F of the boundary-value problem* (8.5i, ii, iii) *is unique then the restriction F_b of F to* (b, ∞) *is of lower degree at infinity. More precisely, F_b lies in the m-dimensional subspace $\mathscr{W}'^{0\perp}_b$ of $\mathscr{H}'_b$ and is the unique element in $\mathscr{W}'_b \cap \mathscr{W}'^{0\perp}_b$. There is a similar statement if* $\inf_{x \in B} x = a > -\infty$.

Proof. Define G as the unique solution of the minimization problem

$$\inf_{h \in \mathscr{W}'_b} \sum_{p=1}^{q} \int_b^\infty (L_p h)^2 = \sum_{p=1}^{q} \int_b^\infty (L_p G)^2. \tag{8.19}$$

Notice that G is characterized by the orthogonality relation

$$\sum_{p=1}^{q} \int_b^\infty L_p G L_p g = 0$$

for all $g \in \mathscr{W}'^0_b$, i.e. $G \in \mathscr{W}'_b \cap \mathscr{W}'^{0\perp}_b$.

We show now that $G=F_b$. Indeed, consider the function

$$\begin{aligned} F_*(x) &= F(x), \quad x \leqq b, \\ &= G(x), \quad x > b, \end{aligned}$$

and observe that $F_*(x)=F(x)$ for $x \in B$ and $F_* \in \mathscr{H}'(\boldsymbol{R})$, hence now,

$$\begin{aligned} \sum_{p=1}^{q} \int_R (L_p F_*)^2 &= \sum_{p=1}^{q} \int_{-\infty}^{b} (L_p F)^2 + \sum_{p=1}^{q} \int_b^{\infty} (L_p G)^2 \\ &\leqq \sum_{p=1}^{q} \int_{-\infty}^{b} (L_p F)^2 + \sum_{p=1}^{q} \int_b^{\infty} (L_p F)^2 \end{aligned}$$

by (8.19). Hence

$$\sum_{p=1}^{q} \int_R (L_p F_*)^2 \leqq \sum_{p=1}^{q} \int_R (L_p F)^2$$

and since F uniquely solves

$$\inf_{h \in \mathscr{V}} \sum_{p=1}^{q} \int_R (L_p h)^2 = \sum_{p=1}^{q} \int_R (L_p F)^2$$

it follows that $F=F_*$. In particular, $F_b=G$. This completes the proof of the theorem.

The approximation theory of §7 carries over here in a similar manner. We need only assume that condition (8.4) is satisfied if $J=\boldsymbol{R}$.

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