TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 153, January 1971

σ-FINITE INVARIANT MEASURES ON INFINITE PRODUCT SPACES(1)

BY
DAVID G. B. HILL

Abstract. A necessary and sufficient condition in terms of Hellinger integrals is established for the existence of a σ-finite invariant measure on an infinite product space. Using this it is possible to construct a wide class of transformations on the unit interval which have no σ-finite invariant measure equivalent to Lebesgue measure. This class includes most of the previously known examples of such transformations.

1.1. **Introduction.** This paper deals with a special case of the following well-known problem from ergodic theory. Suppose that G is a group of measurable, invertible, nonsingular transformations of the σ-finite measure space $(X, \mathscr{B}, \mu)$. Under what conditions does there exist a σ-finite, G-invariant measure ν on $(X, \mathscr{B}, \mu)$ such that ν is equivalent to μ?

For our purposes we take $(X, \mathscr{B}, \mu)$ to be the infinite product space $(X, \mathscr{B}, \mu) = \prod_{k=1}^{\infty} (X_k, \mathscr{B}_k, \mu_k)$, and $G = \sum_{k=1}^{\infty} \oplus G_k$ where for each positive integer k, G_k is an ergodic group of measurable, invertible, nonsingular transformations on $(X_k, \mathscr{B}_k, \mu_k)$. In Theorem 3.20 we solve this problem in terms of the convergence of an infinite series of Hellinger integrals. In fact this result is derived from a more general theorem on the equivalence of certain types of σ-finite product measures with a finite product measure.

In §4 we apply this to the case when all the factor spaces are countable. In particular we are able to improve an existing theorem of C. C. Moore [9] by removing an unnecessary side condition from the hypotheses. Next, using our criterion, we construct a class of ergodic, measurable, invertible, nonsingular transformations on the unit interval which do not have any σ-finite invariant measures equivalent to Lebesgue measure. This class includes the examples of D. S. Ornstein [10], A. Brunel [2], R. V. Chacon [3], and L. K. Arnold [1]. Furthermore our method reveals the essential features these transformations have in common.

1.2. Definitions. Let $(X, \mathscr{B}, \mu)$ be a measure space. A measurable transformation of the space is a map $T: X \to X$ such that $T^{-1}\mathscr{B} \subset \mathscr{B}$. It is necessary only that

Received by the editors July 3, 1969 and, in revised form, February 10, 1970.

AMS 1969 *subject classifications*. Primary 2875, 2870; Secondary 2840.

Key words and phrases. Invariant measures, measures on infinite products, Hellinger integrals, Ornstein's transformation, transformations wihout invariant measures, σ-finite invariant measures.

(1) The results presented here are contained in the author's Ph.D. dissertation written at Yale University under the supervision of Professor S. Kakutani.

T be defined μ-almost everywhere. When the inverse function is also defined μ-a.e. and furthermore $T\mathscr{B}\subset\mathscr{B}$, T is said to be invertible. A measurable, invertible, transformation T on the space with the property that $\mu T^{-1}=\mu$ is called non-singular. Following the notation used by A. Ionescu Tulcea in [7], measurable, invertible, nonsingular transformations are called automorphisms of $(X, \mathscr{B}, \mu)$.

An automorphism T is called measure preserving iff $\mu T^{-1}=\mu$. In this case μ is said to be T-invariant. A T-invariant set $B\in\mathscr{B}$ has the property $T^{-1}B=B$ (μ-a.e.). An automorphism T is said to be ergodic iff for any $B\in\mathscr{B}$, B T-invariant implies that $\mu(B)=0$ or $\mu(X\backslash B)=0$. Similarly if G is a group of automorphisms of $(X, \mathscr{B}, \mu)$ we say that μ is G-invariant when for all $g\in G$, μ is g-invariant. And G is ergodic iff for all $B\in\mathscr{B}$, B G-invariant implies $\mu(B)=0$ or $\mu(X\backslash B)=0$.

2. In this section we state without proof a few well-known results needed as a preliminary to the main results in §3.

2.1. PROPOSITION. *Let G be a group of automorphisms of the finite measure space $(X, \mathscr{B}, \mu)$.*

(a) *G is ergodic iff the only G-invariant measurable functions are the constants.*

(b) *Suppose that G is ergodic. Then if there exists a σ-finite G-invariant measure ν which is equivalent to μ, ν is unique to within a scalar multiple. In particular we cannot have both a finite and an infinite solution to the invariant measure problem when G is ergodic.*

2.2. *Hellinger integrals.* Suppose that μ_1 and μ_2 are a pair of finite measures on the measurable space $(X, \mathscr{B})$ such that $\mu_1(X)=1$ and $\mu_2(X)=1$. By a countable measurable partition of X we shall mean a family of subsets, $\{B_n : n\in N\}\subset\mathscr{B}$, such that the B_n are mutually disjoint and $\bigcup_{n=1}^{\infty} B_n=X$. Let $\mathscr{D}$ be the family of countable, measurable partitions of X and direct $\mathscr{D}$ by refinement. Given $\Delta\in\mathscr{D}$, define

$$S(\Delta) = \sum_{B\in\Delta} [\mu_1(B)\mu_2(B)]^{1/2}.$$

Then $\{S(\Delta) : \Delta\in\mathscr{D}\}$ is a decreasing net and has a limit. This limit is called the Hellinger integral of μ_1 and μ_2 and we write

$$\int [d\mu_1\, d\mu_2]^{1/2} = \lim_{\Delta\in\mathscr{D}} S(\Delta).$$

The next theorem collects some of the useful properties of the Hellinger integral.

2.3. THEOREM. (a) *If μ is a σ-finite measure on $(X, \mathscr{B})$ such that $\mu_1\ll\mu$ and $\mu_2\ll\mu$ (e.g. $\mu=\mu_1+\mu_2$), then*

$$\int [d\mu_1\, d\mu_2]^{1/2} = \int [d\mu_1/d\mu\; d\mu_2/d\mu]^{1/2}\, d\mu,$$

where $d\mu_1/d\mu$ and $d\mu_2/d\mu$ are Radon-Nikodym derivatives.

(b) *In particular when* $\mu_1 \ll \mu_2$, *then*

$$\int [d\mu_1\, d\mu_2]^{1/2} = \int [d\mu_1/d\mu_2]^{1/2}\, d\mu_2.$$

(c) $0 \leqq \int [d\mu_1\, d\mu_2]^{1/2} \leqq 1$.
(d) $\int [d\mu_1\, d\mu_2]^{1/2} = 1$ *iff* $\mu_1 = \mu_2$.
(e) $\int [d\mu_1\, d\mu_2]^{1/2} = 0$ *iff* $\mu_1 \perp \mu_2$.

As long as μ_1 and μ_2 are finite measures we can still define the Hellinger integral as above and in place of (c) we will have $0 \leqq \int [d\mu_1\, d\mu_2]^{1/2} \leqq [\mu_1(X)\mu_2(X)]^{1/2}$. A similar equation will hold when one of the measures is infinite provided we replace X by some set of finite measure. Also, note that Theorem 2.3(e) gives a criterion for the singularity of a pair of finite measures. As long as we knew that the measures had to be either equivalent or singular, this would be also a criterion for equivalence. There are two important cases of this situation, one of which occurs in Kakutani's theorem (2.6 below) and the other in Proposition 2.4.

2.4. PROPOSITION. *Let* μ_1 *and* μ_2 *be σ-finite measures on* $(X, \mathscr{B})$ *and let G be an ergodic group of automorphisms with respect to both* μ_1 *and* μ_2. *Then either* $\mu_1 \perp \mu_2$ *or* $\mu_1 \sim \mu_2$.

2.5. *The Zero One Law.* This well-known result from probability theory is the principle upon which are based all the theorems on infinite product spaces in this paper. The version given below is taken from Halmos' book on Measure Theory [5, p. 201].

Let $(X, \mathscr{B}, \mu) = \prod_{k=1}^{\infty} (X_k, \mathscr{B}_k, \mu_k)$ where for each index k, $\mu_k(X_k) = 1$. Then if $A \in \mathscr{B}$ has the property that for any finite $F \subset N$ there is a measurable subset A_F^* of $\prod_{k \notin F} X_k$ such that $A = (\prod_{k \in F} X_k) \times A_F^*$, either $\mu(A) = 0$ or $\mu(A) = 1$.

The following theorem due to S. Kakutani [8, pp. 214–224] is an interesting consequence of the Zero One Law.

2.6. THEOREM. *For each* $k \in N$ *let* μ_k *and* μ_k' *be a pair of finite measures on the measurable space* $(X_k, \mathscr{B}_k)$ *with* $\mu_k(X_k) = 1 = \mu_k'(X_k)$. *Let* $(X, \mathscr{B}) = \prod_{k=1}^{\infty} (X_k, \mathscr{B}_k)$ *and* $\mu = \prod_{k=1}^{\infty} \mu_k$, $\mu' = \prod_{k=1}^{\infty} \mu_k'$. *Then the following statements are equivalent*:

(a) $\mu \sim \mu'$.
(b) *For each index* k, $\mu_k \sim \mu_k'$ *and* $\prod_{k=1}^{\infty} \int [d\mu_k\, d\mu_k']^{1/2} > 0$.
(c) *For each index* k, $\mu_k \sim \mu_k'$ *and*

$$\sum_{k=1}^{\infty} \left[1 - \int [d\mu_k\, d\mu_k']^{1/2}\right] < \infty.$$

(d) *For each index* k, $\mu_k \sim \mu_k'$ *and*

$$\sum_{k=1}^{\infty} \| 1 - \sqrt{f_k} \|_2^2 < \infty,$$

where f_k is the Radon-Nikodym $d\mu_k'/d\mu_k$ in $L_2(X_k, \mathscr{B}_k, \mu_k)$. On the other hand, if any one of the conditions in (b), (c), *or* (d) *fails to hold, then $\mu \perp \mu'$.*

Both 2.4 and 2.6 are proved by looking at the largest subset of X upon which the two measures are equivalent. In the first case this is a G-invariant set and in the second, a set of the sort to which the Zero One Law can be applied.

3. Throughout this section, for each positive integer k, $(X_k, \mathscr{B}_k, \mu_k)$ will be a finite measure space with $\mu_k(X_k)=1$ and ν_k will be a σ-finite measure on $(X_k, \mathscr{B}_k)$. We denote the product measure space $\prod_{k=1}^{\infty}(X_k, \mathscr{B}_k, \mu_k)$ by $(X, \mathscr{B}, \mu)$. It follows that $\mu(X)=1$. By $\mathscr{F}$ we mean the collection of finite subsets of the positive integers directed by inclusion. F is used as a subscript in the following way:

$$X_F = \prod_{k\in F} X_k, \qquad \mu_F = \prod_{k\in F} \mu_k, \qquad \mathscr{B}_F = \prod_{k\in F} \mathscr{B}_k,$$

$$X_F^* = \prod_{k\notin F} X_k, \qquad \mu_F^* = \prod_{k\notin F} \mu_k, \qquad \mathscr{B}_F^* = \prod_{k\notin F} \mathscr{B}_k$$

and so on.

The usual definition of infinite product measure requires that at most a finite number of the factor measures be infinite. However, by adopting a slightly different procedure it is possible to construct a sort of product of σ-finite measures. This procedure is taken from C. C. Moore's paper [9].

3.1. *Restricted product measures.* For each index k, let Y_k be chosen in $\mathscr{B}_k$ such that $0<\nu_k(Y_k)<\infty$. Put $Y=\prod_{k=1}^{\infty} Y_k$, $Y_F=\prod_{k\in F} Y_k$ and $Y_F^*=\prod_{k\notin F} Y_k$. Also, for each $F\in\mathscr{F}$ define mixed product sets $S^{(Y,F)}=X_F\times Y_F^*$ and put $S^{(Y)}=\bigcup_{F\in\mathscr{F}} S^{(Y,F)}$. This is measurable since $\mathscr{F}$ is countable. Next, define normalized measures $\bar{\nu}_k=\nu_k/\nu_k(Y_k)$ and let λ_k be the restriction of $\bar{\nu}_k$ to Y_k (i.e. $\lambda_k(B_k)=\nu_k(B_k\cap Y_k)/\nu_k(Y_k)$ for any $B_k\in\mathscr{B}_k$). Note that $\bar{\nu}_k$ is a σ-finite measure on $(X_k, \mathscr{B}_k)$ and λ_k a finite measure with $\bar{\nu}_k(Y_k)=\lambda_k(X_k)$.

Now let $F\in\mathscr{F}$ and define the infinite product measure $\nu^{(Y,F)}=(\prod_{k\in F}\bar{\nu}_k)\times(\prod_{k\notin F}\lambda_k)=\bar{\nu}_F\times\lambda_F^*$. This is well defined since all but a finite number of the factors are unit measures. Notice that $\nu^{(Y,F)}$ is supported by $S^{(Y,F)}$ and $\nu^{(Y,F)}(Y)=1$. Finally, let $\mathscr{E}$ be the subfamily of $\mathscr{B}$ consisting of finite unions of sets of the form $R=\prod_{k=1}^{\infty} R_k$ where for each k, $R_k\in\mathscr{B}_k$.

3.2. THEOREM. (a) *For each $B\in\mathscr{B}$, $\{\nu^{(Y,F)}(B) : F\in\mathscr{F}\}$ defines an increasing net so $\lim_F \nu^{(Y,F)}(B)$ exists (though possibly infinite). This limit is called $\nu^{(Y)}(B)$.*

(b) *$\nu^{(Y)}$ is a countably additive, σ-finite measure on $(X, \mathscr{B})$ supported by $S^{(Y)}$ and $\nu^{(Y)}(Y)=1$. This measure is called the restricted product of $\{\nu_k : k\in N\}$ with respect to Y.*

(c) *For any $F\in\mathscr{F}$, $\nu^{(Y)}=\bar{\nu}_F\times\nu_F^*$ where $\bar{\nu}_F=\prod_{k\in F}\bar{\nu}_k$ and ν_F^* is the restricted product of $\{\nu_k : k\notin F\}$ with respect to Y_F^*.*

(d) *$\mathscr{E}$ is dense in $\mathscr{B}$ with respect to $\nu^{(Y)}$, i.e. if $\nu^{(Y)}(B)<\infty$ and $\varepsilon>0$, there is an $E\in\mathscr{E}$ such that $\nu^{(Y)}(E\Delta B)<\varepsilon$.*

Proof. (a) and (b) follow from standard calculations.

(c) Let $K \in \mathscr{F}$. Then,

$$\begin{aligned} \nu^{(Y)} &= \lim_F \nu^{(Y,F)} \\ &= \bar{\nu}_K \times \Big(\lim_{F \supset K} \bar{\nu}_{F\setminus K} \times \lambda_F^*\Big) \\ &= \bar{\nu}_K \times \nu_K^*, \end{aligned}$$

where ν_K^* is the restricted product of $\{\nu_k : k \notin K\}$ with respect to Y_K^*.

(d) Let $B \in \mathscr{B}$ such that $\nu^{(Y)}(B) < \infty$ and let $\varepsilon > 0$. From the definition of restricted product measure it follows that there exists $K \in \mathscr{F}$ such that if $F \supset K$ then $\nu^{(Y)}(B \setminus S^{(Y,F)}) < \frac{1}{2}\varepsilon$. Now $\nu^{(Y,K)} = \bar{\nu}_K \times \lambda_K^*$ and $\lambda_K^*(X_K^*) = 1$. Therefore there is an elementary set $E \in \mathscr{B}$ such that $\nu^{(Y,K)}(E \Delta B) < \frac{1}{2}\varepsilon$. But clearly, $E \cap S^{(Y,K)} \in \mathscr{E}$. And,

$$(E \cap S^{(Y,K)}) \Delta B \subset [(E\Delta B) \cap S^{(Y,K)}] \cup [B \setminus S^{(Y,K)}].$$

So

$$\nu^{(Y)}((E \cap S^{(Y,K)})\Delta B) \leqq \nu^{(Y,K)}(E\Delta B) + \nu^{(Y)}(B\setminus S^{(Y,K)}) \leqq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon.$$

3.3. Remark. In the particular case that each ν_k is actually a finite measure we may construct $\nu^{(X)}$. But clearly $\nu^{(X)} = \prod_{k\in N} \bar{\nu}_k$ where in this instance $\bar{\nu}_k = \nu_k/\nu_k(X_k)$ for each index k. So the restricted product concept is an extension of the normal product.

We now investigate equivalence of restricted products of the same measures $\{\nu_k : k \in N\}$ with respect to different subsets.

3.4. Theorem. *Let $Y = \prod_{k\in N} Y_k$ and $Z = \prod_{k\in N} Z_k$ where for each $k \in N$, Y_k, $Z_k \in \mathscr{B}_k$ and $0 < \nu_k(Y_k)$, $\nu_k(Z_k) < \infty$. Then the following conditions are equivalent*:

(a) $0 < \nu^{(Y)}(Z) < \infty$.

(b) $0 < \nu^{(Z)}(Y) < \infty$.

(c) $\sum_{k=1}^{\infty} \nu_k(Y_k \Delta Z_k)/\nu_k(Y_k) < \infty$.

(d) $\sum_{k=1}^{\infty} \nu_k(Y_k \Delta Z_k)/\nu_k(Z_k) < \infty$.

And furthermore, if one of these conditions holds, then not only do we have $\nu^{(Y)} \sim \nu^{(Z)}$ but also $\nu^{(Y)} = \nu^{(Y)}(Z)\nu^{(Z)}$ and

$$\nu^{(Y)}(Z) = \prod_{k\in N} \nu_k(Z_k)/\nu_k(Y_k) = 1/\nu^{(Z)}(Y).$$

On the other hand, if any one of the conditions fails, the measures are singular and either $\nu^{(Y)}(Z) = 0$ or $\nu^{(Z)}(Y) = 0$.

Proof. First we will show that (a) and (c) are equivalent. As

$$\nu^{(Y)}(Z) = \lim_F \Big[\prod_{k\in F} \nu_k(Z_k)/\nu_k(Y_k)\Big]\Big[\prod_{k\notin F} \nu_k(Y_k \cap Z_k)/\nu_k(Y_k)\Big],$$

the l.h.s. is positive and finite iff both

$$0 < \prod_{k\in N} \nu_k(Z_k)/\nu_k(Y_k) < \infty$$

and

$$0 < \prod_{k \notin K} \nu_k(Y_k \cap Z_k)/\nu_k(Y_k) < \infty,$$

for some $K \in \mathscr{F}$.

From the second relation,

$$\sum_{k=1}^{\infty} [1 - \nu_k(Y_k \cap Z_k)/\nu_k(Y_k)] < \infty,$$

and exchanging Y_k for Z_k in the denominators by means of the first product, $\sum_{k=1}^{\infty} [1 - \nu_k(Y_k \cap Z_k)/\nu_k(Z_k)] < \infty$ also, i.e.

$$\sum_{k=1}^{\infty} \nu_k(Y_k \backslash Z_k)/\nu_k(Y_k) < \infty$$

and

$$\sum_{k=1}^{\infty} \nu_k(Z_k \backslash Y_k)/\nu_k(Z_k) < \infty.$$

Finally, $\nu_k(Y_k)/\nu_k(Z_k) \to 1$, so we can replace the denominators in the latter series with the terms $\nu_k(Y_k)$. A similar reverse calculation yields the converse implication.

In the same way, (b) and (d) are equivalent. The remaining equivalence comes from the fact that both (a) and (b) imply that $\nu_k(Y_k)/\nu_k(Z_k) \to 1$.

Notice that if one of the four conditions holds then $\lim_F \prod_{k \notin F} \nu_k(Z_k \cap Y_k)/\nu_k(Y_k) = 1$ as this is the tail of a convergent infinite product. Thus,

$$\nu^{(Y)}(Z) = \prod_{k \in N} \nu_k(Z_k)/\nu_k(Y_k) = 1/\nu^{(Z)}(Y).$$

We want to show that in these circumstances $\nu^{(Y)} = \nu^{(Y)}(Z)\nu^{(Z)}$. From 3.2(d) it is sufficient to show that if $R = \prod_{k \in N} R_k$ where each $R_k \in \mathscr{B}_k$ and $0 < \nu^{(Y)}(R) < \infty$, then $\nu^{(Y)}(R) = \nu^{(Y)}(Z)\nu^{(Z)}(R)$. But this follows immediately from the infinite product formula for $\nu^{(Y)}(Z)$ above and the similar product for $\nu^{(Y)}(R)$ obtained by replacing Z with R.

Finally, if any one of the four conditions does not hold, then either $\nu^{(Y)}(Z) = 0$ or $\nu^{(Z)}(Y) = 0$. If $\nu^{(Y)}(Z) = 0$, then $\nu_F^*(Z_F^*) = 0$, for any $F \in \mathscr{F}$, so

$$\nu^{(Y)}(S^{(Z)}) = \lim_F \bar{\nu}_F(X_F)\nu_F^*(Z_F^*) = 0.$$

As $\nu^{(Z)}$ is supported by $S^{(Z)}$, $\nu^{(Y)} \perp \nu^{(Z)}$.

3.5. Corollary. *With the same hypotheses as Theorem* 3.4 *the following statements are equivalent*:

(a) $\nu^{(Y)}$ *is a finite measure.*

(b) *For each index k, ν_k is finite and* $\nu^{(X)}(Y) > 0$.

(c) *For each index k, ν_k is finite and*

$$\sum_{k=1}^{\infty} [1 - \nu_k(Y_k)/\nu_k(X_k)] < \infty.$$

(d) *For each index k, ν_k is finite and* $\nu^{(Y)} \sim \nu^{(X)}$.

Proof. Note that if $\nu^{(Y)}$ is finite then $1=\nu^{(Y)}(Y)\leqq\nu^{(Y)}(X)<\infty$.

Now recalling Kakutani's criterion for the equivalence of a pair of unit product measures (Theorem 2.6) it seems likely that there should be a similar result for restricted product measures. Indeed there is as the next theorem demonstrates.

3.6. THEOREM. *If $\nu^{(Y)}$ is a restricted product measure and μ the unit product measure $\mu=\prod_{k\in N}\mu_k$, then the following conditions are equivalent*:

(a) $\nu^{(Y)}\sim\mu$.

(b) *For each index k, $\mu_k\sim\nu_k$ and the restrictions of μ and $\nu^{(Y)}$ to Y are equivalent.*

(c) *For each index k, $\mu_k\sim\nu_k$ and*

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\mu_k(Y_k)\nu_k(Y_k)]^{1/2}\right]<\infty.$$

(d) *For each index k, $\mu_k\sim\nu_k$ and*

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right]<\infty.$$

Conversely, if $\mu_k\sim\nu_k$ for each k, but one of the conditions above fails to hold, then $\mu\perp\nu^{(Y)}$.

Proof. First we shall show that (b), (c) and (d) are equivalent. Notice that the restrictions of $\nu^{(Y)}$ and $\mu/\mu(Y)$ to Y are both unit measures. Moreover, when $\mu(Y)>0$ we can express each of them in product form. $\nu^{(Y)}|\,Y=\prod_{k\in N}(\nu_k|\,Y_k)/\nu_k(Y_k)$ and $(\mu|\,Y)/\mu(Y)=\prod_{k\in N}(\mu_k|\,Y_k)/\mu_k(Y_k)$.

Now suppose that for each index k, $\mu_k\sim\nu_k$. Then their restrictions to Y_k are also equivalent and since $\nu_k(Y_k)>0$ each $\mu_k(Y_k)>0$ too and the product on the right above is well defined even if $\mu(Y)=0$. We now apply Theorem 2.6 to these products and conclude that (b) and (c) are equivalent statements.

To obtain (d) look at the equation

$$\begin{aligned}1-\int_{Y_k}&[d\mu_k\,d\nu_k]^{1/2}/[\mu_k(Y_k)\nu_k(Y_k)]^{1/2}\\&=\left[1-\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right]\Big/[\mu_k(Y_k)]^{1/2}+[1-1/[\mu_k(Y_k)]^{1/2}].\end{aligned}$$

Given (b) we know that $\mu(Y)>0$ since $\nu^{(Y)}(Y)=1$. Therefore, $\sum_{k=1}^{\infty}[1-\mu_k(Y_k)]<\infty$, and $\mu_k(Y_k)\to 1$. So, $\sum_{k=1}^{\infty}[1-1/[\mu_k(Y_k)]^{1/2}]<\infty$. It follows that

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right]<\infty.$$

Conversely, if this series does converge we consider the equation

$$1-\int_{Y_k} [d\mu_k\, d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}$$
$$= \left[1-\int_{Y_k} [d\mu_k\, d\nu_k]^{1/2}/[\mu_k(Y_k)\nu_k(Y_k)]^{1/2}\right][\mu_k(Y_k)]^{1/2}+[1-[\mu_k(Y_k)]^{1/2}].$$

It follows that $\sum_{k=1}^{\infty}[1-\mu_k(Y_k)]<\infty$, and

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\mu_k\, d\nu_k]^{1/2}/[\mu_k(Y_k)\nu_k(Y_k)]^{1/2}\right] < \infty.$$

Finally we show that (a) and (b) are equivalent. Suppose that $\mu\sim\nu^{(Y)}$. Then for each index k, $\mu=\mu_k\times\mu_k^*$ and $\nu^{(Y)}=\bar{\nu}_k\times\nu_k^*$ so $\mu_k\sim\bar{\nu}_k\sim\nu_k$. Also, $\nu^{(Y)}|\, Y\sim\mu|\, Y$ trivially.

Conversely, let $\nu^{(Y)}|\, Y\sim\mu|\, Y$ and $\nu_k\sim\mu_k$ for each index k. Take $F\in\mathscr{F}$. Then, $\mu=\mu_F\times\mu_F^*$ and $\nu^{(Y,F)}=\bar{\nu}_F\times\lambda_F^*$. But $\mu_F\sim\bar{\nu}_F$ and $\lambda_F^*\sim\mu_F^*|\, Y_F^*$ so it follows that $\nu^{(Y,F)}\sim\mu_F\times\mu_F^*|\, Y_F^*$. Now if $B\in\mathscr{B}$,

$$\begin{aligned}\nu^{(Y)}(B) &= 0 \text{ iff } \nu^{(Y,F)}(B) = 0 \text{ for all } F\in\mathscr{F};\\ &\text{iff } (\mu_F\times\mu_F^*|\, Y_F^*)(B) = 0 \text{ for all } F\in\mathscr{F};\\ &\text{iff } \mu(B) = 0.\end{aligned}$$

The very last statement of Theorem 3.6 is a consequence of the Zero One Law, which says that either $\mu(S^{(Y)})=0$ or $\mu(S^{(Y)})=1$. The latter can not hold if $\mu\nsim\nu^{(Y)}$.

So far we have constructed restricted product measures directly. However, there is another way to show that certain σ-finite measures on $(X,\mathscr{B})$ are actually restricted products.

3.7. Definition. Let ν_k be a σ-finite measure on $(X_k,\mathscr{B}_k)$ for each index k. Then a σ-finite measure ν on $(X,\mathscr{B})$ is said to be a product of $\{\nu_k : k\in N\}$ iff for each $F\in\mathscr{F}$, there exists a measure ν_F^*, σ-finite on $(X_F^*,\mathscr{B}_F^*)$ such that $\nu=\nu_F\times\nu_F^*$ where $\nu_F=\prod_{k\in F}\nu_k$ as usual.

Clearly the ordinary product of a finite number of σ-finite measures with an infinite number of unit measures fits in with this definition, as does the restricted product. Notice as well that if ν is a product in the sense of Definition 3.7 then the measures ν_F^* above are themselves products of $\{\nu_k : k\notin F\}$ on X_F^*.

3.8. Proposition. *Let ν_k be a σ-finite measure on $(X_k,\mathscr{B}_k)$ and $Y_k\in\mathscr{B}_k$ such that $0<\nu_k(Y_k)<\infty$, for each $k\in N$. Put $Y=\prod_{k\in N}Y_k$. Then, if ν is a product of $\{\nu_k : k\in N\}$, $\nu\sim\nu^{(Y)}$ iff there is an $a>0$ such that $a\nu=\nu^{(Y)}$.*

Proof. Obviously if $a\nu=\nu^{(Y)}$, then $\nu\sim\nu^{(Y)}$. To get the converse we first construct a finite measure $\alpha=\prod_{k\in N}\alpha_k$ such that $\alpha\sim\nu\sim\nu^{(Y)}$.

For each index k, choose a sequence of disjoint sets $\{Y_{k,j} : j\in N\}$ such that $0<\nu_k(Y_{k,j})<\infty$ for every j and $\bigcup_{j\in N}Y_{k,j}=X_k\setminus Y_k$. Define α_k by

$$\alpha_k(B_k) = [(2^k-1)/2^k][\nu_k(Y_k\cap B_k)/\nu_k(Y_k)]+\sum_{j=1}^{\infty}\nu_k(Y_{k,j}\cap B_k)/[\nu_k(Y_{k,j})2^{k+j}],$$

for each $B_k \in \mathscr{B}_k$. Clearly, $\alpha_k \sim \nu_k$ and $\alpha_k(X_k)=1$. Then,

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\alpha_k\, d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right] = \sum_{k=1}^{\infty}[1-[(2^k-1)/2^k]^{1/2}] < \infty.$$

So from Theorem 3.6, $\alpha \sim \nu^{(Y)} \sim \nu$.

To complete the proof let $h(x)$ be the Radon-Nikodym derivative $d\nu^{(Y)}/d\nu$, and $h_F=d\bar{\nu}_F/d\nu_F$, $h_F^*=d\lambda_F^*/d\nu_F^*$ where for any $F\in\mathscr{F}$ we can factorize the measures thus:

$$\nu = \nu_F \times \nu_F^* \quad \text{and} \quad \nu^{(Y)} = \bar{\nu}_F \times \lambda_F^*.$$

Then, given $x \in X$, $h(x)=h_F(x_F)h_F^*(x_F^*)$, where $x=(x_F; x_F^*)$. But, $h_F(x_F)=1/\nu_F(Y_F)$ and is independent of x_F. So for any $F\in\mathscr{F}$, $h(x)$ depends only on x_F^*. In this situation it follows from the Zero One Law that $h(x)$ is constant α-almost everywhere. But $\alpha\sim\nu\sim\nu^{(Y)}$ and so there exists an $a>0$ such that $\nu^{(Y)}=a\nu$.

However, it can also be demonstrated that any product measure equivalent to a finite product must be a multiple of a restricted product measure. This is the main result of this paper.

3.9. THEOREM. *Suppose that ν is a product of the σ-finite measures $\{\nu_k : k \in N\}$. Then there exists a finite product measure $\mu=\prod_{k\in N}\mu_k$ such that $\nu\sim\mu$ iff ν is a multiple of a restricted product $\nu^{(Y)}$ of $\{\nu_k : k\in N\}$ with respect to some $Y=\prod_{k\in N} Y_k$.*

Proof. Going one way is easy. If ν is a multiple of some $\nu^{(Y)}$ then using the construction in Proposition 3.8 we produce an equivalent finite product measure.

The converse result is much more difficult and for convenience the proof has been split up into four propositions. The method of proof is based on that used by C. C. Moore in [9] though this proof does differ from his in many respects other than just being in a more general setting. First we establish some notation.

The assumption is that ν is a product of $\{\nu_k : k\in N\}$ and there exists $\mu\sim\nu$ such that $\mu=\prod_{k\in N}\mu_k$ where each μ_k is a unit measure on $(X_k, \mathscr{B}_k)$. Define the Radon-Nikodym derivatives, $f=d\mu/d\nu$, $f_F^*=d\mu_F^*/d\nu_F^*$, $f_F=d\mu_F/d\nu_F$ and so on. For any $F\in\mathscr{F}$ we can factorize $f=f_F\times f_F^*$. Without loss of generality we can regard all functions as having domain X by making them constant in variables upon which they do not depend. However, the functions above may not be integrable with respect to μ and it is necessary to deal with the related functions:

$$\begin{aligned} g^t(x) &= \exp(-2\pi it\log f(x)), \\ g_F^t(x) &= \exp(-2\pi it\log f_F(x)), \\ g_F^{t*}(x) &= \exp(-2\pi it\log f_F^*(x)) \quad \text{where } t\in \boldsymbol{R}. \end{aligned}$$

These functions all have absolute value 1 and hence are all in $L_2(X, \mathscr{B}, \mu)$. Furthermore, the product formula is preserved, i.e. for each $F\in\mathscr{F}$ and for each $t\in\boldsymbol{R}$, $g^t(x)=g_F(x)g_F^*(x)$.

3.10. PROPOSITION A. (a) *Let* $\varphi(t)=\int_{g^t} d\mu$. *Then* φ *is continuous and* $\varphi(0)=1$.
(b) *There exists* $t_0>0$ *such that if* $|t|\leqq t_0$

$$\prod_{k\in N} |\langle g_k^t, 1\rangle| = |\varphi(t)| \geqq 1/e.$$

(c) *If* $|t|\leqq t_0$, *then* $\sum_{k=1}^{\infty} [1-|\langle g_k^t, 1\rangle|]\leqq 1$.

Proof. (a) follows immediately from the definition of φ.
(b) Using the continuity of φ at zero, there exists $t_0>0$ such that $|t|\leqq t_0$ implies $|\varphi(t)|\geqq 1/e$.

Now $\varphi(t)=\int g^t\, d\mu=\int g_F^t g_{F'}^t\, d\mu=\langle g_F^t, 1\rangle\langle g_{F'}^t, 1\rangle$ for any $F\in\mathcal{F}$. Let $P_F\colon L_2(X)\to L_2(X_F)$ be the orthogonal projection. By the Mean Fubini-Jessen Theorem (see [4, p. 207]) $P_F(g^t)$ converges to g^t in $L_2(X)$. Therefore $|P_F(g^t)| \xrightarrow[F]{} |g^t|=1$. But $P_F(g^t)=g_F^t\langle g_{F'}^t, 1\rangle$ and $|g_F^t|=1$, so $|\langle g_{F'}^t, 1\rangle| \xrightarrow[F]{} 1$. Hence $|\langle g_k^t, 1\rangle|>0$ for all but a finite number of values of k and $|\varphi(t)|=\prod_{k\in N} |\langle g_K^t, 1\rangle|$.

(c) From the inequality $1-u\leqq -\log u$ for $0<u\leqq 1$, we deduce that if $|t|\leqq t_0$, then

$$\begin{aligned}\sum_{k=1}^{\infty} [1-|\langle g_k^t, 1\rangle|] &\leqq \sum_{k=1}^{\infty} [-\log |\langle g_k^t, 1\rangle|]\\ &= -\log |\varphi(t)| \leqq 1.\end{aligned}$$

3.11. REMARK. The next step is to define the sets Y_k mentioned in the statement of Theorem 3.9. We take $c>0$ and let $Y_k=\{x_k\in X_k : |\log (f_k(x_k)/q_k)|\leqq c\}$, where the numbers $\{q_k : k\in N\}$ are to be chosen. In my original proof of the theorem the q_k were defined in terms of the means of the distributions of the functions $\log f_k$ on suitable subsets. However, the proof contained an error and was not true in the general case. I am indebted to Neil Rickert for noticing the mistake and suggesting an alternate approach which is also much neater. The idea is to use the median rather than the mean as a measure of the centre of the distribution of $\log f_k$. First we need the following lemma which estimates the spread of a probability measure on the line in terms of its Fourier transform.

3.12. LEMMA. *Let* γ *be a probability measure on the line. Then there exists* $s_0\in \boldsymbol{R}$ *such that for all* $c>0$,

$$1-\gamma([s_0-c, s_0+c]) \leqq \left[(1/t_0)\int_{-t_0}^{t_0} (1-|\hat\gamma(t)|)\, dt\right]\Big/(1-A_c),$$

where $t_0>0$ *and* $A_c=\sup\{(\sin u)/u : |u|\geqq 2\pi c t_0\}$.

Proof. $\hat\gamma(t)=\int \exp[-2\pi i t u]\gamma(du)$. Therefore,

$$|\hat\gamma(t)|^2 = \hat\gamma(t)\overline{\hat\gamma(t)} = \int \exp[-2\pi i t u]\gamma(du)\int \exp[2\pi i t s]\gamma(ds).$$

So,

$$\int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2\,dt = \int_{-t_0}^{t_0} \left[\iint \exp\left[-2\pi i t(u-s)\right]\gamma(du)\gamma(ds)\right] dt$$
$$= \iint \left[\int_{-t_0}^{t_0} \exp\left[-2\pi i t(u-s)\right] dt\right]\gamma(du)\gamma(ds)$$
$$= 2t_0 \iint \psi(u-s)\gamma(du)\gamma(ds),$$

where ψ is defined by $\psi(s)=\sin(2\pi t_0 s)/(2\pi t_0 s)$ if $s\neq 0$ and $\psi(0)=1$.

Now since γ is a probability measure, there exists $s_0 \in \boldsymbol{R}$ such that

$$(1/2t_0)\int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2\,dt \leqq \int \psi(u-s_0)\gamma(du)$$
$$\leqq \int_I \gamma(du) + \int_{\boldsymbol{R}\setminus I} A_c\gamma(du),$$

where $I=[s_0-c, s_0+c]$, and A_c is defined as above, i.e.

$$(1/2t_0)\int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2\,dt \leqq \gamma(I)+A_c[1-\gamma(I)].$$

Therefore,

$$1-\gamma(I) \leqq \left[1-(1/2t_0)\int_{-t_0}^{t_0} |\hat{\gamma}(t)|^2\,dt\right]\Big/(1-A_c)$$
$$= \left[(1/2t_0)\int_{-t_0}^{t_0} (1-|\hat{\gamma}(t)|^2)\,dt\right]\Big/(1-A_c).$$

But,

$$1-|\hat{\gamma}(t)|^2 = (1-|\hat{\gamma}(t)|)(1+|\hat{\gamma}(t)|)$$
$$\leqq 2(1-|\hat{\gamma}(t)|).$$

So,

$$1-\gamma(I) \leqq \left[(1/t_0)\int_{-t_0}^{t_0} (1-|\hat{\gamma}(t)|)\,dt\right]\Big/(1-A_c).$$

3.13. PROPOSITION B. *For each index k, there exists $q_k>0$ such that if $c>0$ and $Y_k=\{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leqq c\}$, then $\sum_{k=1}^{\infty} [1-u_k(Y_k)]<\infty$.*

Proof. Given $k \in \boldsymbol{N}$, define a probability measure γ_k on the real line by setting $\gamma_k(E)=\mu_k(\{x_k \in X_k : \log f_k(x_k) \in E\})$, for every Borel set E. Then,

$$\hat{\gamma}_k(t) = \int_{\boldsymbol{R}} \exp\left[-2\pi i t u\right]\gamma_k(du)$$
$$= \int_{X_k} \exp\left[-2\pi i t \log f_k(x_k)\right]\mu_k(dx_k) = \langle g_k^t, 1\rangle.$$

Applying Lemma 3.12 to γ_k, there exists $s_k \in \boldsymbol{R}$ such that for all $c>0$,

$$1-\gamma_k([s_k-c, s_k+c]) \leqq \left[(1/t_0)\int_{-t_0}^{t_0} (1-|\langle g_k^t, 1\rangle|)\,dt\right]\Big/(1-A_c).$$

But if $q_k = \exp(s_k)$,

$$\gamma_k([s_k - c, s_k + c]) = \mu_k(\{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leqq c\}) \\ = \mu_k(Y_k).$$

And $\sum_{k=1}^{\infty} [1 - |\langle g_k^t, 1\rangle|] \leqq 1$, for $|t| \leqq t_0$, from Proposition A.

This is a positive term series so we can integrate from $-t_0$ to t_0 and interchange the order of integration and summation to obtain

$$(1/t_0) \sum_{k=1}^{\infty} \int_{-t_0}^{t_0} (1 - |\langle g_k^t, 1\rangle|)\, dt \leqq 2.$$

Therefore, $\sum_{k=1}^{\infty} [1 - \mu_k(Y_k)] \leqq 2/(1 - A_c) < \infty$.

3.14. Remark. Choosing the sets Y_k as above, we can assume that $\mu_k(Y_k) > 0$, for all but a finite number of values of k. But, on Y_k, $e^{-c} q_k \leqq f_k(x_k) \leqq e^c q_k$. Integrating this with respect to ν_k, we get, $\nu_k(Y_k) e^{-c} q_k \leqq \mu_k(Y_k) \leqq \nu_k(Y_k) e^c q_k$, for any k. We change arbitrarily those Y_k for which $\nu_k(Y_k) = 0$ (i.e. $\mu_k(Y_k) = 0$) so that for all k, $0 < \nu_k(Y_k) < \infty$. There is no confusion in using the same notation for this modified set of Y_k. Later on in Proposition D we shall show that the q_k can actually be determined in terms of the medians of the functions $\log f_k$.

At this point we need a couple of elementary estimates to be used in the proof of Proposition C.

3.15. Lemma. (a) *$u^2 \leqq 12(1 - (\sin u)/u)$ if $|u| \leqq \pi/3$.*

(b) *If $|\log t| \leqq 2c$, then*

$$[(\sqrt{t} - 1)c/(e^c - 1)]^2 \leqq (\log t)^2.$$

Proof. Using Taylor's theorem, we get $\sin u = u - u^3/(3!) \cos\theta$ where $|\theta| \leqq |u| \leqq \pi/3$. So,

$$u^2 = 6(1 - (\sin u)/u)/\cos\theta \leqq 12(1 - (\sin u)/u).$$

(b) $|\log t| \leqq 2c$ iff $e^{-2c} \leqq t \leqq e^{2c}$.

Now if $e^{-2c} \leqq t \leqq 1$, $|\sqrt{t} - 1| = 1 - \sqrt{t} \leqq -\log t$. So since $c/(e^c - 1) < 1$, $(\log t)^2 \geqq [(\sqrt{t} - 1)c/(e^c - 1)]^2$.

On the other hand, if $1 \leqq t \leqq e^{2c}$, then putting

$$W(t) = \log t - (\sqrt{t} - 1)c/(e^c - 1),$$
$$W'(t) = 1/t - [\tfrac{1}{2}c/(e^c - 1)][1/\sqrt{t}].$$

Now $W'(1) > 0$, $W(1) = 0$ and $W'(t)$ has only one zero and is eventually negative. But $W(e^{2c}) = c$, so clearly $W(t) > 0$, for $1 \leqq t \leqq e^{2c}$.

The next proposition essentially completes the proof of Theorem 3.9.

3.16. Proposition C. *With the modified set of Y_k chosen as in Remark 3.15, and $c \leqq 1/(6t_0)$,*

$$\sum_{k=1}^{\infty} \left[1 - \int_{Y_k} [d\mu_k\, d\nu_k]^{1/2} / [\nu_k(Y_k)]^{1/2}\right] < \infty.$$

Hence, $\mu \sim \nu^{(Y)}$ and ν is a multiple of $\nu^{(Y)}$.

Proof. If $\sum_{k=1}^{\infty}[1-\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}]<\infty$, then $\nu^{(Y)}\sim\mu$ by Theorem 3.6 and ν is a multiple of $\nu^{(Y)}$ by Proposition 3.8. So we have to show this series converges.

Choose $c>0$ and form the sets Y_k as in 3.13. (Choose t_0 as in Proposition A, but suppose furthermore that $0<t_0\leqq 1/(6c)$, so that $2\pi ct_0\leqq\pi/3$.) Then if $\psi(s)=(\sin(2\pi t_0 s))/(2\pi t_0 s)$, $s\neq 0$ and $\psi(0)=1$, we can apply the calculation in Lemma 3.12 to the probability measures γ_k on the real line and obtain

$$\int_R [1-\psi(u-\log q_k)]\gamma_k(du) \leqq (1/t_0)\int_{-t_0}^{t_0}(1-|\hat{\gamma}(t)|)\,dt.$$

Here, γ_k is defined by $\gamma_k(E)=\mu_k(\{x_k\in X_k : \log f_k(x_k)\in E\})$. Now, repeating the calculation in Proposition B,

$$\sum_{k=1}^{\infty}\int_{X_k}[1-\psi(\log(f_k/q_k))]\,d\mu_k \leqq \sum_{k=1}^{\infty}(1/t_0)\int_{-t_0}^{t_0}(1-|\langle g_k^t, 1\rangle|)\,dt \leqq 2.$$

Therefore, $\sum_{k=1}^{\infty}\int_{Y_k}[1-\psi(\log(f_k/q_k))]\,d\mu_k\leqq 2$.

But if $x_k\in Y_k$, $|\log(f_k(x_k)/q_k)|\leqq c$, so by Lemma 3.15(a),

$$[2\pi t_0\log(f_k(x_k)/q_k)]^2 \leqq 12[1-\psi(\log(f_k(x_k)/q_k))]$$

for all $k\geqq p$, where $\mu_k(Y_k)>0$ if $k\geqq p$. Hence, $\sum_{k=1}^{\infty}\int_{Y_k}(\log(f_k/q_k))^2\,d\mu_k<\infty$.

Now applying Lemma 3.15(b) and the fact that $f_k=d\mu_k/d\nu_k$,

$$\sum_{k=1}^{\infty}\int_{Y_k}(\sqrt{q_k}-\sqrt{f_k})^2\,d\nu_k = \sum_{k=1}^{\infty}\int_{Y_k}(\sqrt{q_k}/\sqrt{f_k}-1)^2\,d\mu_k < \infty.$$

But

$$\int_{Y_k}\left[\sqrt{f_k}-\int_{Y_k}\sqrt{f_k}\,d\nu_k/\nu_k(Y_k)\right]^2 d\nu_k \leqq \int_{Y_k}(\sqrt{f_k}-\sqrt{q_k})^2\,d\nu_k,$$

for any index k. Thus,

$$\sum_{k=1}^{\infty}\int_{Y_k}\left[\sqrt{f_k}-\int_{Y_k}\sqrt{f_k}\,d\nu_k/\nu_k(Y_k)\right]^2 d\nu_k < \infty,$$

i.e.

$$\sum_{k=1}^{\infty}\left[\mu_k(Y_k)-\left[\int_{Y_k}\sqrt{f_k}\,d\nu_k/[\nu_k(Y_k)]^{1/2}\right]^2\right] < \infty.$$

And since from Proposition B, $\sum_{k=1}^{\infty}[1-\mu_k(Y_k)]<\infty$, then

$$\sum_{k=1}^{\infty}\left[1-\left[\int_{Y_k}\sqrt{f_k}\,d\nu_k/[\nu_k(Y_k)]^{1/2}\right]^2\right] < \infty,$$

i.e.

$$\sum_{k=1}^{\infty}\left[1-\left[\int_{Y_k}[d\mu_k\,d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right]^2\right] < \infty.$$

So,

$$\sum_{k=1}^{\infty}\left[1-\int_{Y_k}[d\mu_k\, d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right] < \infty,$$

as required.

The next proposition shows how we can take the numbers $\log q_k$ to be the medians of the functions $\log f_k$ and hence gives a method by which the sets Y_k could actually be computed, for any $c>0$.

3.17. PROPOSITION D. *For each index k, let m_k be the median of $\log f_k$ with respect to μ_k. Then Proposition C and hence Theorem 3.9 remain true if we take $\log q_k = m_k$, and any $c>0$.*

Proof. m_k is a number such that

$$\mu_k(\{x_k \in X_k : \log(f_k(x_k)) \leqq m_k\}) \geqq \tfrac{1}{2}$$

and

$$\mu_k(\{x_k \in X_k : \log(f_k(x_k)) < m_k\}) \leqq \tfrac{1}{2}.$$

Choose the q_k as in Proposition B. If $c>0$, let

$$Y_k(c) = \{x_k \in X_k : |\log(f_k(x_k)/q_k)| \leqq c\}$$

and

$$Z_k(c) = \{x_k \in X_k : |\log(f_k(x_k))-m_k| \leqq c\}.$$

Then for all but a finite number of k, $Y_k(\frac{1}{2}c)\subset Z_k(c)\subset Y_k(2c)$. For we know that $\sum_{k=1}^{\infty}[1-\mu_k(Y_k(\frac{1}{2}c))]<\infty$ and hence that $\mu_k(Y_k(\frac{1}{2}c))>\frac{1}{2}$, for every $k \geqq$ some k_0. So clearly $|m_k - \log q_k| \leqq \frac{1}{2}c$ if $k \geqq k_0$. Now if $x_k \in Y_k(\frac{1}{2}c)$,

$$\begin{aligned}|\log(f_k(x_k))-m_k| &\leqq |\log(f_k(x_k)/q_k)|+|\log q_k - m_k|\\ &\leqq \tfrac{1}{2}c+\tfrac{1}{2}c,\end{aligned}$$

i.e. $Y_k(\frac{1}{2}c)\subset Z_k(c)$. Also if $x_k \in Z_k(c)$,

$$\begin{aligned}|\log(f_k(x_k)/q_k)| &\leqq |\log f_k(x_k)-m_k|+|\log q_k - m_k|\\ &\leqq c+\tfrac{1}{2}c < 2c.\end{aligned}$$

Now if $k \geqq k_0$, let $Y_k = Y_k(\frac{1}{2}c)$, $Z_k = Z_k(c)$ and $Y'_k = Y_k(2c)$. Otherwise, choose $Y_k \subset Z_k \subset Y'_k$ arbitrarily so that they all have positive finite ν_k measure. Put $Y=\prod_{k\in N} Y_k$, $Z=\prod_{k\in N} Z_k$, $Y'=\prod_{k\in N} Y'_k$.

By Theorem 3.9, if $2c \leqq 1/(6t_0)$, $\nu^{(Y)}\sim\nu\sim\nu^{(Y')}$. And $\nu^{(Y)}(Y')<\infty$, by Theorem 3.4. But $Y\subset Z\subset Y'$, so $\nu^{(Y)}(Y)\leqq\nu^{(Y)}(Z)\leqq\nu^{(Y)}(Y')$. Using 3.4 again, this means $\nu^{(Y)}\sim\nu^{(Z)}$. Finally, since the numbers m_k are independent of t_0, and since we may choose t_0 arbitrarily small, any $c>0$ will do.

3.18. This work was initiated by a study of invariant measures on infinite product spaces. We are now in a position to show that certain invariant measures have the product property and to use Theorem 3.9 to obtain a criterion for invariance.

Suppose that for each k, G_k is an ergodic group of automorphisms on $(X_k, \mathscr{B}_k, \mu_k)$. Form the direct sum, $G=\sum_{k=1}^{\infty} \oplus G_k$. Then G acts on $(X, \mathscr{B})$ in the obvious way and the groups G_k may be regarded as subgroups of G. Furthermore, G is ergodic with respect to μ and acts freely in the sense that if $g \in G$ and g is not the identity, then $\mu(\{x \in X : g \cdot x = x\})=0$. This follows directly from the Zero One Law and the ergodicity of the factor groups.

3.19. Proposition. *Let ν be a σ-finite measure on $(X, \mathscr{B}, \mu)$ such that $\nu \sim \mu$. Then ν is G-invariant iff ν is a product of invariant measures. More precisely, ν is G-invariant iff for each index k there exists $\nu_k \sim \mu_k$, σ-finite and G_k-invariant, and for each $F \in \mathscr{F}$ there exists $\nu_F^* \sim \mu_F^*$, σ-finite and G_F^*-invariant, such that $\nu = \nu_F \times \nu_F^*$. Here $G_F^* = \sum_{k \notin F} \oplus G_k$ and $\nu_F = \prod_{k \in F} \nu_k$ as before.*

Proof. If ν can be factorized as above then for any k, $\nu = \nu_k \times \nu_k^*$ which is G_k-invariant. The converse is more tedious and depends on the ergodicity of the factor groups. Suppose that $\nu \sim \mu$ and ν is G-invariant. As usual let $f = d\mu/d\nu$.

Let $B_0 = \{x \in X : f(x) = 0 \text{ or } \infty\}$. Then $0 = \mu(B_0) = (\mu_1 \times \mu_1^*)(B_0) = \int \mu_1[(B_0)x_1^*]\, d\mu_1^*$, where $(B_0)x_1^* = \{x_1 \in X_1 : (x_1; x_1^*) \in B_0\}$. Therefore $\mu_1[(B_0)x_1^*]=0$ for μ_1^*-almost all $x_1^* \in X_1^*$, i.e. for μ_1^*-almost all $x_1^* \in X_1^*$ $0 < f(x_1; x_1^*) < \infty$, for μ_1-almost all $x_1 \in X_1$.

But this means that we can define a measure on $(X_1, \mathscr{B}_1, \mu_1)$ which is equivalent to μ_1 by

$$\nu_1(x_1^*; B_1) = \int_{B_1} f(x_1; x_1^*)^{-1}\, d\mu_1 \quad \text{where } B_1 \in \mathscr{B}_1.$$

This measure is σ-finite, G_1-invariant and defined for μ_1^*-almost all $x_1^* \in X_1^*$. However G_1 is an ergodic group of automorphisms so as in Proposition 2.1 any G_1-invariant measure is unique to within a positive multiple. Therefore there exists $\nu_1 \sim \mu_1$, a σ-finite G_1-invariant measure and a function f_1^* defined μ_1^*-almost everywhere on X_1^*, such that $\nu_1(x_1^*;\ ) = \nu_1/f_1^*(x_1^*)$. Of course f_1^* is positive and finite where defined. Now taking Radon-Nikodym derivatives with respect to μ_1 we see that $f(x_1; x_1^*)^{-1} = d\nu_1(x_1^*;\ )/d\mu_1 = f_1^*(x_1^*)^{-1}\, d\nu_1/d\mu_1$ at $x_1 \in X_1$.

Hence if $f_1 = d\mu_1/d\nu_1$, $f(x_1; x_1^*) = f_1(x_1) f_1^*(x_1^*)$. This equation also implies that f_1^* is measurable. Now we can define ν_1^* on $(X_1^*, \mathscr{B}_1^*)$ by $d\mu_1^*/d\nu_1^* = f_1^*$, and $\nu = \nu_1 \times \nu_1^*$. Clearly ν_1^* is σ-finite and G_1^*-invariant. Applying the same argument to ν_1^* we obtain invariant measures ν_2 and ν_2^* on the appropriate spaces. An obvious induction completes the proof.

The main theorem of this paper is an immediate consequence.

3.20. Theorem. *Suppose that for each positive integer k, we have a finite measure space $(X_k, \mathscr{B}_k, \mu_k)$ with $\mu_k(X_k)=1$ and an ergodic group of automorphisms G_k on this space. Let $G = \sum_{k \in N} \oplus G_k$ and $(X, \mathscr{B}, \mu) = \prod_{k \in N} (X_k, \mathscr{B}_k, \mu_k)$. Then the following statements are equivalent:*

(a) *There exists a σ-finite, G-invariant measure ν on $(X, \mathscr{B}, \mu)$ such that $\nu \sim \mu$.*

(b) *For each index k, there exists a σ-finite, G_k-invariant measure $\nu_k \sim \mu_k$ and there* *exists* $Y_k \in \mathscr{B}_k$, $0 < \nu_k(Y_k) < \infty$, *such that if* $Y = \prod_{k \in N} Y_k$, *then* $\nu^{(Y)}$ *is G-invariant and* $\nu^{(Y)} \sim \mu$.

(c) *For each index k, there exists a σ-finite, G_k-invariant measure* $\nu_k \sim \mu_k$ *and there exists* $Y_k \in \mathscr{B}_k$, $0 < \nu_k(Y_k) < \infty$, *such that*

$$\sum_{k=1}^{\infty} \left[1 - \int_{Y_k} [d\mu_k \, d\nu_k]^{1/2}/[\nu_k(Y_k)]^{1/2}\right] < \infty.$$

There will exist a finite G-invariant measure ν equivalent to μ iff for each index k, there exists a finite G_k-invariant measure $\nu_k \sim \mu_k$ *such that*

$$\sum_{k=1}^{\infty} \left[1 - \int_{X_k} [d\mu_k \, d\nu_k]^{1/2}/[\nu_k(X_k)]^{1/2}\right] < \infty.$$

Proof. All the work has been done already. The statement about a finite measure comes from Corollary 3.5 and the rest from Theorem 3.9 and Proposition 3.19.

4. The discrete factor case, Moore's theorem and the examples of Ornstein, Brunel, Arnold and Chacon.

4.1. *The space and the group.* Now each factor space is assumed to be countable. Without loss of generality, for each index k, either $X_k = \{1, 2, 3, \ldots, n_k\}$ where $n_k \in N$ or $X_k = N$ and then we say $n_k = \infty$. X_k is given the usual measure structure with every subset measurable and a unit measure μ_k is chosen for X_k so that each point has positive measure. Thus for each k, there exists a sequence $\{p_{k,i} : 1 \leqq i \leqq n_k\}$ such that $\mu_k(\{i\}) = p_{k,i} > 0$ and $\sum_{i=1}^{n_k} p_{k,i} = 1$.

For the groups G_k we take $G_k = \boldsymbol{Z}(n_k)$, the cyclic group of order n_k, if n_k is finite. Otherwise we take G_k as the integers. Once again $G = \sum_{k \in N} \oplus G_k$, the direct sum.

To describe the action of G_k on X_k let g_k be the generator. Then, if $n_k < \infty$, $g_k(i) = i + 1 \bmod (n_k)$. When $n_k = \infty$, the situation is a little complicated since to avoid double summations we have chosen to take X_k as $\boldsymbol{N}$ not $\boldsymbol{Z}$. The action of g_k is defined in this instance by $\cdots 7 \to 5 \to 3 \to 1 \to 2 \to 4 \to 6 \cdots$ under g_k. Now G acts on X in the obvious way, i.e. if $g \in G$, then $g = \prod_{k \in F} g_k^{r_k}$ for some $F \in \mathscr{F}$, so given $x = (x_1, x_2, \ldots) \in X$,

$g \cdot x$ has kth coordinate x_k if $k \notin F$ and

$g \cdot x$ has kth coordinate $g_k^{r_k} \cdot x_k$ if $k \in F$.

This messy notation is not used in the sequel.

The following theorem was proved by C. C. Moore in [9].

4.2. Theorem. *Take* $(X, \mathscr{B}, \mu)$ *and G as above.*

(a) *There exists a finite G-invariant measure equivalent to μ iff for each index k,* $n_k < \infty$, *and*

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} [\sqrt{p_{k,i}} - 1/\sqrt{n_k}]^2 < \infty.$$

(b) *Assume that for each k, $p_{k,i}$ is decreasing with respect to i and there exists $a>0$ such that all $p_{k,1} \geqq a$. Then there is a σ-finite, G-invariant measure equivalent to μ iff for some $c>0$,*

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} p_{k,i}|p_{k,1}/p_{k,i}-1|_c^2 < \infty,$$

where $|t|_c$ means $\min\{|t|, c\}$ *for any $t \in \mathbf{R}$.*

REMARK. Notice that if we define $s_k \leqq n_k$ by, for each index k, $1 \leqq i \leqq s_k$ iff $p_{k,1}/p_{k,i}-1 \leqq c$, then the convergence of the series in (b) is equivalent to the convergence of the two series,

$$\sum_{k=1}^{\infty} \sum_{i=1}^{s_k} p_{k,i}(p_{k,1}/p_{k,i}-1)^2$$

and

$$\sum_{k=1}^{\infty} \sum_{i=s_k+1}^{n_k} (p_{k,i}).$$

Also, in this theorem that the condition $\{p_{k,1} : k \in N\}$ has a uniform, positive lower bound is immediately suspect since it does not apply in the finite case. In fact, if for each k, $n_k=k$, and $p_{k,i}=1/k$, $1 \leqq i \leqq k$, then we have an example in which a finite invariant measure exists by 4.2(a) and yet $p_{k,1} \to 0$.

The original purpose of this paper was to free Moore's theorem from this condition. However, once this had been achieved it became clear that the improved result also applied to the more general situation described in Theorem 3.20. So as a corollary to 3.20 we have the following:

4.3. THEOREM. *If $(X, \mathscr{B}, \mu)$ and G are described as in* 4.1 *then the following conditions are equivalent:*

(a) *There exists a σ-finite, G-invariant measure ν equivalent to μ.*

(b) *For each index k, there exists $Y_k \subset X_k$ such that $|Y_k| = r_k < \infty$ and*

$$\sum_{k=1}^{\infty} \left[1-\left(\sum_{i\in Y_k} \sqrt{p_{k,i}}\right)\Big/\sqrt{r_k}\right] < \infty.$$

(c) *For each index k, there exists $Y_k \subset X_k$ such that $|Y_k| = r_k < \infty$ and both*

$$\sum_{k=1}^{\infty} \sum_{i\notin Y_k} (p_{k,i}) < \infty$$

and

$$\sum_{k=1}^{\infty} \sum_{i\in Y_k} [\sqrt{p_{k,i}} - 1/\sqrt{r_k}]^2 < \infty.$$

There will exist a finite invariant measure equivalent to μ iff for each k, $n_k < \infty$ and

$$\sum_{k=1}^{\infty} \left[1-\left(\sum_{i=1}^{n_k} \sqrt{p_{k,i}}\right)\Big/\sqrt{n_k}\right] < \infty.$$

Proof. This is just Theorem 3.20 restated in the discrete factor case. To note the equivalence of the conditions in (b) and (c) look at the equality

$$2\Big[1-\Big(\sum_{i\in Y_k}\sqrt{p_{k,i}}\Big)\Big/\sqrt{r_k}\Big] = \Big[1-\sum_{i\in Y_k}p_{k,i}\Big]+\sum_{i\in Y_k}[\sqrt{p_{k,i}}-1/\sqrt{r_k}]^2.$$

4.4. Remark. A long and tedious calculation shows that Moore's theorem can be derived directly from Theorem 4.3. Furthermore, the single condition "there exists $c>0$ such that $\sum_{k=1}^{\infty}\sum_{i=1}^{n_k}p_{k,i}|p_{k,1}/p_{k,i}-1|_c^2<\infty$," while a sufficient condition for the existence of a σ-finite, G-invariant measure, ν equivalent to μ, is not a necessary condition except under the additional hypothesis "there exists $a>0$ such that $p_{k,1}\geqq a$ for all $k\in \boldsymbol{N}$." These results are not used in what follows and for further details the reader is referred to the author's thesis. We turn now to an interesting application of Theorem 4.3.

4.5. *A class of transformations which lack an invariant measure*. Theorem 4.3 can be used in a rather nice way to construct a class of ergodic automorphisms on the unit interval which have no invariant σ-finite measure equivalent to Lebesgue measure. As special cases, this class includes the transformations constructed by D. S. Ornstein [10], A. Brunel [2], L. K. Arnold [1] and R. V. Chacon [3]. Moreover, Theorem 4.3 shows directly why these transformations have the required property and reveals their common features.

We commence by setting up a correspondence between the unit interval and a product space of finite factors, which identifies the product structure with an easily visualized family of subintervals. Let I be the unit interval $[0, 1)$, $\mathscr{B}_0$ the Borel sets, and m Lebesgue measure. Let $\{n_k : k\in \boldsymbol{N}\}\subset \boldsymbol{N}$. Successively partition the unit interval as follows:

Stage 1. We divide I into n_1 equal subintervals.

Put $P_{1,j}=[(j-1)/n_1, j/n_1)$, $1\leqq j\leqq n_1$, and put $\mathscr{P}_1=\{P_{1,j} : 1\leqq j\leqq n_1\}$.

Stage 2. We divide each $P_{1,j}$ into n_2 equal subintervals.

Put $P_{2,j}=[(j-1)/m_2, j/m_2)$, $1\leqq j\leqq m_2=n_1n_2$, and put $\mathscr{P}_2=\{P_{2,j} : 1\leqq j\leqq m_2\}$.

By an induction argument we have an increasing (with respect to refinement) sequence of partitions $\{\mathscr{P}_k : k\in \boldsymbol{N}\}$ such that $P_{k,j}=[(j-1)/m_k, j/m_k)$, $1\leqq j\leqq m_k$, and $\mathscr{P}_k=\{P_{k,j} : 1\leqq j\leqq m_k\}$, where $m_0=1$ and $m_k=n_1n_2\cdots n_k$. $\mathscr{P}_k$ is obtained by dividing each of the subintervals of $\mathscr{P}_{k-1}$ into n_k equal parts.

Clearly, for each $t\in I$, we have a unique sequence $\{P_{k,j}(t) : k\in \boldsymbol{N}\}$ such that $\{t\}=\bigcap_{k\in \boldsymbol{N}}P_{k,j}(t)$ and the smallest σ-algebra containing all the $P_{k,j}$ is just $\mathscr{B}_0$.

Action of G on $(I,\mathscr{B}_0)$. As in 4.1, $G=\sum_{k=1}^{\infty}\oplus G_k$, where G_k is the cyclic group of order n_k and g_k is the generator.

Now, for all $t\in I$, $g_1(t)=t+1/n_1 \bmod (1)$, and $g_k(t)=t+1/m_k \bmod (1/m_{k-1})$, for $k>1$, i.e. g_k translates mod $(1/m_{k-1})$ the elements of each $\mathscr{P}_k\cap P_{k-1,j}$.

Definition of the map T. A map $T: I\to I$ is defined in terms of the generators of G. T will have the property that elements of G can be expressed in terms of powers of T. Firstly, we construct the countable partition $\mathscr{R}=\{R_k : k \in N\}$, of I where

$$R_k = [(m_{k-1}-1)/m_{k-1}, (m_k-1)/m_k) \quad \text{for all } k \in N.$$

Then $T(t)=g_1(t)=t+1/n_1$ if $t \in R_1$. And $T(t)=g_{k-1}g_k(t)=t+1/m_{k-1}+1/m_k$ mod (1) if $t \in R_k$ and $k>1$. What T does is to reverse the order of the subintervals R_k, while leaving their length and direction unchanged. A map T defined as above is said to be of type $\{n_k : k \in N\}$.

Compatible measures for T. Take a set of positive numbers

$$\{p_{k,i} : 1 \leqq i \leqq n_k, k \in N\},$$

with the property that for all k, $\sum_{i=1}^{n_k} p_{k,i}=1$. Using these we define a measure μ' on $(I, \mathscr{B}_0)$ by giving the values of μ' on the sets $P_{k,j}$.

Stage 1. $\mu'(P_{1,j})=p_{1,j}$, $1 \leqq j \leqq n_1$, i.e. we divide I into n_1 equal subintervals and give them measures in the ratio $p_{1,1}:p_{1,2}:\cdots p_{1,n_1}$.

Stage k. If $k>1$, divide each of the $P_{k,j}$ into n_k equal subintervals and assign them measures in the ratio $p_{k,1}:p_{k,2}:\cdots p_{k,n_k}$. A measure μ' defined as above is said to be compatible with respect to the map T.

The correspondence between $(X, \mathscr{B}, \mu)$ and $(I, \mathscr{B}_0, \mu')$. Let T and μ' be defined as above. Form the product space $(X, \mathscr{B}, \mu)$ in the usual way and let $U: X \to I$ be defined by $U(x)=\sum_{k=1}^{\infty} (x_k-1)/m_k$ for all $x \in X$.

4.6. Proposition. (a) *The map U is measurable, invertible and preserves the action of G. Furthermore, $\mu=\mu' U$ and G acts on $(I, \mathscr{B}_0)$ as an ergodic group of automorphisms.*

(b) *T is an ergodic automorphism of $(I, \mathscr{B}_0, \mu')$.*

(c) *If ν is a σ-finite measure on $(I, \mathscr{B}_0)$, then ν is G-invariant iff ν is T-invariant.*

Proof. (a) This is an easy generalization of the case when all the n_k are equal which is well known (e.g. decimals). The statement about G is an immediate consequence of the properties of U.

(b) From the way T was defined it is clearly both measurable and invertible. To show that T is nonsingular let $B \in \mathscr{B}_0$ and suppose that $\mu'(B)=0$. Then $B=\bigcup_{k\in N} B \cap R_k$ and these sets are disjoint. Let g_0 be the identity on I. Then

$$\mu'(TB) = \mu'\Big(T \bigcup_{k\in N} B \cap R_k\Big) = \mu'\Big(\bigcup_{k\in N} g_{k-1}g_k(B \cap R_k)\Big)$$
$$= \sum_{k\in N} \mu'(g_{k-1}g_k(B \cap R_k)) = 0,$$

since each g_k is an automorphism.

To show that T is ergodic, suppose that $B \in \mathscr{B}_0$ is a T-invariant set. Then, $U^{-1}B$ is $U^{-1}TU$-invariant in $(X, \mathscr{B}, \mu)$. But invariant sets of the map $U^{-1}TU$ are of the type to which we can apply the Zero One Law, hence have measure zero or one.

(c) This is the most interesting part of the previous proposition. Suppose that ν is a G-invariant measure on $(I, \mathscr{B}_0, \mu')$. Let $B \in \mathscr{B}_0$. Then, $B=\bigcup_{k\in N} B \cap R_k$ and these sets are disjoint. So,

$$\nu(TB) = \sum_{k\in N} \nu(g_{k-1}g_k(B \cap R_k)) = \sum_{k\in N} \nu(B \cap R_k) = \nu(B).$$

To get the converse we must express the generators of G in terms of powers of T. Look at g_1.

$$\begin{aligned} g_1(t) &= T(t) && \text{if } t \in R_1, \\ &= T^{-n_1}(t) && \text{if } t \notin R_1. \end{aligned}$$

In the general case if we let, for each index k,

$$Q_k = \bigcup \{P_{k,jn_k} : 1 \leqq j \leqq m_{k-1}\},$$

then

$$\begin{aligned} g_k(t) &= T^{m_{k-1}}(t) && \text{if } t \notin Q_k, \\ &= T^{-m_k}(t) && \text{if } t \in Q_k. \end{aligned}$$

Now suppose that ν is T-invariant. If $B \in \mathscr{B}_0$ and $k \in N$,

$$\begin{aligned} \nu(g_k B) &= \nu(T^{m_{k-1}}(B \cap Q_k)) + \nu(T^{-m_k}(B \backslash Q_k)) \\ &= \nu(B \cap Q_k) + \nu(B \backslash Q_k) = \nu(B). \end{aligned}$$

It follows that ν is G-invariant also.

The last step in this construction is to map I into itself in such a way that the measure μ is exchanged with Lebesgue measure. Suppose that T is a transformation of type $\{n_k : k \in N\}$ and μ is a compatible measure. Corresponding to $\{\mathscr{P}_k : k \in N\}$, we define another sequence of increasing partitions $\{\mathscr{P}_k(\mu) : k \in N\}$ by the following process.

Stage 1. Divide I into n_1 subintervals $\mathscr{P}_1(\mu)=\{P_{1,j}(\mu) : 1 \leqq j \leqq n_1\}$. These subintervals are half open on the right and have lengths in the ratio $p_{1,1}:p_{1,2}:\cdots p_{1,n_1}$, i.e. $P_{1,1}(\mu)=[0, p_{1,1})$, $P_{1,j}(\mu)=[\sum_{i=1}^{j-1} p_{1,i}, \sum_{i=1}^{j} p_{1,i})$ if $1<j\leqq n_1$.

Stage k. We divide each of the $P_{k-1,j}(\mu)$ into n_k subintervals half open on the right and with lengths in the ratio $p_{k,1}:p_{k,2}:\cdots p_{k,n_k}$. This partition is called $\mathscr{P}_k(\mu)=\{P_{k,j}(\mu) : 1 \leqq j \leqq m_k\}$.

The map V_μ. We define a map $V_\mu : \mathscr{B}_0 \to \mathscr{B}_0$ by $V_\mu(P_{k,j})=P_{k,j}(\mu)$, $1 \leqq j \leqq m_k$ and $k \in N$. This clearly extends to a unique map on $\mathscr{B}_0$ and a point to point transformation on I if μ is nonatomic.

4.7. Proposition. *Suppose that μ is nonatomic.*

(a) *V_μ is a measurable, invertible transformation on $(I, \mathscr{B}_0, m)$ and the measures are related by $mV_\mu=\mu$.*

(b) *If $T_\mu=V_\mu T V_\mu^{-1}$, then T_μ is an ergodic automorphism of $(I, \mathscr{B}_0, m)$.*

Proof. (a) Let $t \in I$. Then for each index k, there exists a unique $P_{k,j(t)}(\mu)$ containing t. If μ is nonatomic, then

$$\{t\} = \bigcap \{P_{k,j(t)}(\mu) : k \in N\}.$$

That V_μ is a measurable, invertible, point to point transformation follows immediately. Also, for any $P_{k,j}$, $mV_\mu(P_{k,j}) = m(P_{k,j}(\mu)) = \mu(P_{k,j})$, so $mV_\mu = \mu$.

(b) is obvious.

4.8. Remark. It is the transformations T_μ in which we are mainly interested. T_μ acts linearly with respect to the $P_{k,j}(\mu)$ in the same way that T translated the $P_{k,j}$.

It would of course have been possible to construct T directly on the product space (i.e. $U^{-1}TU$) and then proceeded straight to $(I, \mathscr{B}_0, m)$. However, the map is more easily pictured on the interval and the notation less messy.

The best that can be done in the atomic case is to take an atomic subalgebra $\mathscr{B}_\mu$ of $\mathscr{B}_0$ such that $\mathscr{B}_\mu(m) \overset{V_\mu}{\approx} \mathscr{B}_0(\mu)$. Then V_μ and T_μ will not be point to point transformations, but they will still be invertible transformations of the algebras. However, we might just as well use $(I, \mathscr{B}_0, \mu, T)$ as the model for the system.

If μ is atomic there will always be a σ-finite, T-invariant measure equivalent to μ. This measure is finite only when all but a finite number of the $n_k = 1$ and the infinite product collapses down to a finite product. This situation is still covered by the criterion in the next theorem.

4.9. Theorem. *Let T be a transformation of type $\{n_k : k \in N\}$, and let μ be the compatible measure for T determined by $\{p_{k,i} : 1 \leqq i \leqq n_k, k \in N\}$. Form T_μ as above.*

(a) *There exists $\nu \sim m$, σ-finite and T_μ-invariant iff for each $k \in N$, there exists $Y_k \subset \{1, 2, \ldots, n_k\}$, $|Y_k| = r_k$, such that*

$$\sum_{k=1}^{\infty} \left[1 - \Big(\sum_{i \in Y_k} \sqrt{p_{k,i}}\Big)\Big/\sqrt{r_k}\right] < \infty.$$

(b) *Assume that for each index k, $p_{k,i}$ is decreasing in i and $p_{k,1} \geqq a > 0$, for some $a > 0$. Then there exists $\nu \sim m$, σ-finite and T_μ-invariant iff*

$$\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} p_{k,i} |p_{k,1}/p_{k,i} - 1|_c^2 < \infty \quad \textit{for some } c > 0.$$

(c) *There exists $\nu \sim m$, finite and T_μ-invariant, iff*

$$\sum_{k=1}^{\infty} \left[1 - \Big(\sum_{i=1}^{n_k} \sqrt{p_{k,i}}\Big)\Big/\sqrt{n_k}\right] < \infty.$$

Proof. If ν is a σ-finite measure on $(I, \mathscr{B}_0)$, so too is νV_μ and $\nu \sim m$ iff $\nu V_\mu \sim m V_\mu$. Now we apply 4.2 and 4.3.

4.10. *The transformations of Ornstein, Brunel, Arnold and Chacon.* The first example of an automorphism with no σ-finite invariant measure was constructed by D. S. Ornstein in 1960 [10]. Later examples have been found by A. Brunel [2], R. V. Chacon [3] and L. K. Arnold [1]. In addition, Arnold demonstrated that both Ornstein's and Brunel's transformations could be approached more simply using his method. However, we do not want to go into the details of the original constructions of these transformations here. We wish only to indicate how they can be regarded as special cases of the transformations T_μ.

D. Ornstein's Transformation. For each index k, let $n_k \geqq 3$, $p_{k,1}=\frac{1}{2}$, and $p_{k,i}=1/(2n_k-2)$, $1<i\leqq n_k$. We apply the criterion in Theorem 4.9(a) using the fact that convergence of

$$\sum_{k=1}^{\infty}\left[1-\Big(\sum_{i\in Y_k}\sqrt{p_{k,i}}\Big)\Big/\sqrt{r_k}\right]$$

is equivalent to the joint convergence of the series

$$\sum_{k=1}^{\infty}\sum_{i\notin Y_k} p_{k,i} \quad\text{and}\quad \sum_{k=1}^{\infty}\sum_{i\in Y_k}(\sqrt{p_{k,i}}-1/\sqrt{r_k})^2.$$

Suppose that we can find sets $Y_k\subset\{1,2,\ldots,n_k\}$ such that $\sum_{k=1}^{\infty}\sum_{i\notin Y_k}p_{k,i}<\infty$. Then clearly, there is a $k_0\in N$ such that if $k\geqq k_0$, $1\in Y_k$ and

$$\sum_{k=k_0}^{\infty}(n_k-r_k)/(2n_k-2)=\sum_{k=k_0}^{\infty}\sum_{i\notin Y_k}p_{k,i}<\infty.$$

In particular $r_k/n_k\to 1$ and $r_k\geqq 3$ for $k\geqq$ some $k_1\in N$. Hence

$$\sum_{k=k_1}^{\infty}\sum_{i\in Y_k}(\sqrt{p_{k,i}}-1/\sqrt{r_k})^2 \geqq \sum_{k=k_1}^{\infty}(1/\sqrt{2}-1/\sqrt{3})^2=\infty.$$

So there does not exist any σ-finite measure on I which is T_μ-invariant and equivalent to m.

A. Brunel's Transformation. For each index k, let $n_k=3$, and $p_{k,1}=\frac{1}{4}=p_{k,3}$, while $p_{k,2}=\frac{1}{2}$.

Interchanging $p_{k,1}$ and $p_{k,2}$, and using exactly the same argument as above, we arrive at the same conclusion.

L. K. Arnold's Transformation. For each index k, let $n_k=2$, $p_{k,1}=1/(\alpha+1)>\frac{1}{2}$, and $p_{k,2}=\alpha/(\alpha+1)$, where $\alpha\in(0,1)$. Clearly, $\sum_{k=1}^{\infty}\sum_{i\notin Y_k}p_{k,i}<\infty$ iff $Y_k=\{1,2\}$, for all $k\geqq$ some k_0. Then

$$\sum_{k=k_0}^{\infty}\sum_{i\in Y_k}(\sqrt{p_{k,i}}-1/\sqrt{r_k})^2 \geqq \sum_{k=k_0}^{\infty}(1/\sqrt{(\alpha+1)}-1/\sqrt{2})^2=\infty.$$

Once again, there is no σ-finite T_μ-invariant measure on I equivalent to m.

It is obvious from the form of the criterion in Theorem 4.9 that if the factor spaces are taken to be identical as in the last two transformations above, there can not exist any σ-finite invariant measure unless $p_{k,i}=1/n_k$, $1\leqq i\leqq n_k$, and then there is a finite invariant measure.

R. V. Chacon's Transformation. Chacon proves a result (Lemma 2 in his paper) repeated applications of which can be used to construct an automorphism with the required properties. He does not go into details of the construction and there would appear to be more than one way to achieve this. However, by what seems to be the most simple interpretation we get the following transformation of the sort T_μ. Details of his construction not essential to our purpose have been omitted.

Let $\varepsilon \in (0, 1)$ and let $\{b_k : k \in N\}$ be a positive sequence such that $b_k \to \infty$. Take $n_1 > 2$ and an arbitrary set of positive numbers $\{p_{1,j} : 1 \leqq j \leqq n_1\}$ summing to 1. For $k > 1$, take $p_{k,1} = \varepsilon$ and $0 < p_{k,j} \leqq 1/b_k$ if $1 < j \leqq n_k$. Here all the $n_k > 2$.

Now suppose that we can find sets $Y_k \subset \{1, 2, \ldots, n_k\}$ with $|Y_k| = r_k$, such that $\sum_{k=1}^{\infty} \sum_{i \notin Y_k} p_{k,i} < \infty$. Then, except for a finite number of values of k, $1 \in Y_k$. Also, if

$$\sum_{k=1}^{\infty} \sum_{i \in Y_k} (\sqrt{p_{k,i}} - 1/\sqrt{r_k})^2 < \infty,$$

then $p_{k,i} \to \varepsilon$ uniformly ($i \in Y_k$). But from our hypotheses, $p_{k,i} \to 0$ uniformly ($i > 1$).

It follows that $Y_k = \{1\}$ for all but a finite number of values of k, and

$$\sum_{k=1}^{\infty} \sum_{i \notin Y_k} p_{k,i} = \infty.$$

From Chacon's basic lemma, we can also construct a more complicated automorphism which, though not itself of the T_μ type, induces a T_μ type transformation on a subinterval of I. This induced transformation is somewhat similar to the above and can be reached by our criterion.

In general, this criterion can always be applied to a system $(X, \mathscr{B}, \mu, T)$ with the following properties.

(a) T is an ergodic automorphism of the space.

(b) There exists a countable family of independent, finite, measurable partitions of $(X, \mathscr{B}, \mu)$, $\{\mathscr{I}_k : k \in N\}$ which generate $\mathscr{B}$.

(c) For each $n \in N$, $\bigvee_{k=1}^{n} \mathscr{I}_k$ is a T-invariant partition.

Under these conditions, T will be isomorphic to some T_μ on $(I, \mathscr{B}_0, m)$.

Bibliography

1. L. K. Arnold, *On σ-finite invariant measures*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 85–97.

2. A. Brunel, *Sur les mesures invariantes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **6** (1966), 300–303.

3. R. V. Chacon, *A class of linear transformations*, Proc. Amer. Math. Soc. **15** (1964), 560–564.

4. N. Dunford and J. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

5. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR **11**, 504.

6. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc., Japan, no. 3, Math. Soc. Japan, 1956; reprint, Chelsea, New York, 1960. MR **20** #3958; MR **22** #2677.

7. A. Ionescu Tulcea, *On the category of certain classes of transformations in ergodic theory*, Trans. Amer. Math. Soc. **114** (1965), 261–279. MR **31** #3575.

8. S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. (2) **49** (1948), 214–224. MR **9**, 340.

9. C. C. Moore, *Invariant measures on product spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965), vol. II, part 2, Univ. of California Press, Berkeley, Calif., 1967, pp. 447–459. MR **37** #2951.

10. D. S. Ornstein, *On invariant measures*, Bull. Amer. Math. Soc. **66** (1960), 297–300. MR **26** #3872.

Yale University,
New Haven, Connecticut 06520
University of Otago,
Dunedin, New Zealand