ON SOME STARLIKE AND CONVEX FUNCTIONS

BY
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Abstract. In this paper we study functions of the form $\int_0^z (g(t)/\prod_{k=1}^n (1-tz_k)^{\alpha_k}) dt$ for $|z| < 1$ and show under what conditions such a function is convex, convex in one direction and hence univalent in $|z| < 1$. We also study the functions $g(z)$ where $g(0)=1$, $g(z) \neq 0$ and $\operatorname{Re} [zg'(z)/g(z)] \geq -\alpha$, $0 \leq \alpha < 1$, for $|z| < 1$.

1. Let S denote the set of all functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are holomorphic, univalent and starlike in the disk $D = \{z : |z| < 1\}$ in the complex z -plane. It is well known [3, p. 13] that a function f given by equation (1) is in S if and only if $\operatorname{Re} [zf'(z)/f(z)] > 0$ for $|z| < 1$, or equivalently, by Herglotz's representation [8, p. 570],

$$(2) \quad \frac{zf'(z)}{f(z)} = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\alpha(\theta),$$

where $\alpha(\theta)$ is a nondecreasing function of θ in $[0, 2\pi]$ and $\alpha(2\pi) - \alpha(0) = 1$. Equation (2) after simplification may be put in the form

$$(3) \quad f(z) = z \exp \left[-2 \int_0^{2\pi} \log(1 - e^{-i\theta} z) d\alpha(\theta) \right]$$

for $|z| < 1$. We choose here that branch of the logarithm which has the value zero at $z=0$.

In particular if we take $\alpha(\theta)$ to be a step function with jumps of $\alpha_k > 0$ at $\theta_k \in [0, 2\pi)$ ($k=1, 2, \dots, n$) such that $\sum_{k=1}^n \alpha_k = 1$ (to meet the requirement that $\alpha(2\pi) - \alpha(0) = 1$), in equation (3), it follows that the function

$$(4) \quad w = f(z) = z \prod_{j=1}^n (1 - \exp[-i\theta_j]z)^{2\alpha_j}$$

is in S . In fact $f(z)$ maps the disk D onto the w -plane with n radial slits [2, pp. 36–37].

A function h holomorphic in D is said to be convex and univalent in D if and only if there exists a function f in the class S such that $f(z) = zh'(z)$ for $z \in D$. We

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shall denote the class of such functions h by C . It follows from equation (3) now that $h \in C$ if and only if

$$(5) \quad h'(z) = \exp \left[-2 \int_0^{2\pi} \log(1 - e^{-i\theta} z) d\alpha(\theta) \right]$$

for $|z| < 1$. Once again that branch of the logarithm is taken in (5) which is zero at $z=0$.

We also have from equation (4) that functions h such that

$$(6) \quad h(z) = \int_0^z \left(1 / \prod_{j=1}^n (1 - \exp[i\theta_j]t)^{2\alpha_j} \right) dt,$$

with $\sum_{j=1}^n \alpha_j = 1$, $\alpha_j > 0$, $\theta_j \in [0, 2\pi)$ ($j=1, 2, \dots, n$) and $|z| < 1$, are members of C . In fact it follows by the Schwarz-Christoffel transformation [4, pp. 192-193] that h maps the disk $D = \{z : |z| < 1\}$ onto the interior of an n -sided convex polygon.

We remark here that functions f given by equation (4) are extremal in some sense in S and the functions h given by equation (6) are extremal in the class C [5].

In §2 we intend to study functions which reduce to functions of the type given by equations (4) and (6) as a special case. In §3 we study another class of functions $g(z)$ which are used in §2.

2. It is known that a function $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ holomorphic in D is in C if and only if $\operatorname{Re} [1 + zh''(z)/h'(z)] > 0$ for all z in D [3, p. 14]. A function $f(z)$ holomorphic in D is said to be convex in one direction in D if it maps $|z|=r$ for every r near 1 into a contour which is cut by every straight line parallel to this direction in not more than two points. If $h \in C$ then h is convex in every direction in D . We now prove the following result.

THEOREM 1. *Let $g(z)$ be holomorphic in D with $g(0)=1$, $g(z) \neq 0$ and $\operatorname{Re} [zg'(z)/g(z)] \geq -\alpha$ for $|z| < 1$, where $0 \leq \alpha < 1$. Suppose also that*

$$(7) \quad h(z) = \int_0^z \left(g(t) / \prod_{k=1}^n (1 - z_k t)^{\alpha_k} \right) dt = z + \sum_{n=2}^{\infty} a_n z^n,$$

where $\alpha_k > 0$, $|z_k| \leq 1$ ($1 \leq k \leq n$). Then,

(a) $h(z)$ is convex in one direction and hence univalent in D if $\sum_{k=1}^n \alpha_k \leq 3 - 2\alpha$, and $|a_n| \leq n$ for all n .

(b) $h(z)$ is in C if $\sum_{k=1}^n \alpha_k \leq 2 - 2\alpha$ and $|a_n| \leq 1$ for all n .

Proof. We have $h'(z) = g(z) / \prod_{k=1}^n (1 - z z_k)^{\alpha_k}$ for $|z| < 1$, and hence

$$\frac{h''(z)}{h'(z)} = \frac{g'(z)}{g(z)} + \sum_{k=1}^n \frac{\alpha_k z_k}{1 - z z_k}.$$

Or,

$$\begin{aligned}\frac{zh''(z)}{h'(z)} &= \frac{zg'(z)}{g(z)} - \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{(1-zz_k)-(1+zz_k)}{1-zz_k} \\ &= \frac{zg'(z)}{g(z)} - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \frac{\alpha_k(1+zz_k)}{1-zz_k} \\ &= \frac{zg'(z)}{g(z)} - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{(1-|z|^2|z_k|^2)+2i \operatorname{Im}(zz_k)}{|1-zz_k|^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}(8) \quad \operatorname{Re} \left[1 + z \frac{h''(z)}{h'(z)} \right] &\geq 1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k \\ &\quad + \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{(1-|z|^2|z_k|^2)}{|1-zz_k|^2} \\ &> 1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k \quad \text{for } |z| < 1.\end{aligned}$$

It follows now by a result of Umezawa [10] that $h(z)$ is convex in one direction in D if $(1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k) \geq -\frac{1}{2}$ or if $\sum_{k=1}^n \alpha_k \leq 3 - 2\alpha$. That $h(z)$ is univalent in D and $|a_n| \leq n$ for all n follows from a result of Robertson [6].

Again from inequality (8) we have that $h(z)$ is convex in D if $1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k \geq 0$ or if $\sum_{k=1}^n \alpha_k \leq 2 - 2\alpha$. Thus $h(z) \in C$ and that $|a_n| \leq 1$ for all n is well known for functions in C [2, p. 12]. This completes the proof.

A function $f(z)$ holomorphic for $|z| < 1$ is said to be starlike in one direction if f maps $|z|=r$ for every r near 1 onto a contour C which is cut by a straight line passing through the origin in two, and not more than two points. From this definition, Theorem 1 and the relation between the members of S and C we have the following corollary.

COROLLARY 2. *If $g(z)$ and z_k ($1 \leq k \leq n$) are subject to the same conditions as in Theorem 1, $\alpha_k > 0$ ($1 \leq k \leq n$) and*

$$f(z) = zg(z) \left/ \prod_{k=1}^n (1-zz_k)^{\alpha_k} \right. = z + \sum_{n=2}^{\infty} b_n z^n,$$

then,

- (a) $f(z)$ is starlike in one direction if $\sum_{k=1}^n \alpha_k \leq 3 - 2\alpha$ and $|b_n| \leq n^2$ for all n ;
- (b) $f(z) \in S$ if $\sum_{k=1}^n \alpha_k \leq 2 - 2\alpha$ and $|b_n| \leq n$ for all n .

Proof. We have $f(z) = zh'(z)$, where h is given by equation (7). That $f(z)$ is starlike in one direction if $\sum_{k=1}^n \alpha_k \leq 3 - 2\alpha$ follows from part (a) of Theorem 1 and a result of Robertson [6]. Again from equation (7) we have $b_n = na_n$ for all n . Hence from (a) of Theorem 1 we have $|b_n| \leq n^2$ for all n . (b) follows from (b) of Theorem 1.

If we take, in particular, $g(z) = 1$ for $|z| < 1$, $\alpha_k = m$ ($1 \leq k \leq n$) in Corollary 2 we have the following result of Rubinstein [7].

COROLLARY 3. The function $f(z) = z / \prod_{k=1}^n (1 - zz_k)^m$ is univalent (and starlike) in D if $m \leq 2/n$ and $|z_k| \leq 1$ ($1 \leq k \leq n$).

In the following result no restriction is put on the α_k except that these are positive. This in turn, however, shrinks the domain on which $f(z)$ is starlike or convex.

THEOREM 4. Let $g(z)$ be holomorphic in D with $g(0) = 1$ and $g(z) \neq 0$ and

$$\operatorname{Re} [zg'(z)/g(z)] \geq -\alpha, \quad 0 \leq \alpha < 1.$$

Suppose that

$$(9) \quad h(z) = \int_0^z \left(g(t) / \prod_{k=1}^n (1 - tz_k)^{\alpha_k} \right) dt,$$

with $\alpha_k > 0$, $|z_k| \leq 1$ ($1 \leq k \leq n$), and $R = \max_{1 \leq k \leq n} |z_k|$. Then

(a) $h(z)$ is convex in one direction and hence univalent for

$$|z| \leq (3 - 2\alpha) / R \left(2 \sum_{k=1}^n \alpha_k + 2\alpha - 3 \right);$$

(b) $h(z)$ is convex (and univalent) for

$$|z| \leq (1 - \alpha) / R \left(\sum_{k=1}^n \alpha_k + \alpha - 1 \right).$$

Proof. As in the proof of Theorem 1, we have

$$(10) \quad \operatorname{Re} \left[1 + \frac{zh''(z)}{h'(z)} \right] \geq 1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{1 - |z|^2 |z_k|^2}{|1 - zz_k|^2}.$$

It is clear from the proof of Theorem 1 that $f(z)$ is convex in one direction and univalent for $|z| \leq R_{1/2}$ if the right side of the inequality (10) is $\geq -\frac{1}{2}$ for $|z| \leq R_{1/2}$ and that $f(z)$ is convex for $|z| < R_0$ if the right side of the inequality (10) ≥ 0 for $|z| < R_0$.

We intend to find R_β such that

$$(11) \quad 1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{1 - |z|^2 |z_k|^2}{|1 - zz_k|^2} \geq -\beta \quad \text{for } |z| \leq R_\beta,$$

where $\beta = \frac{1}{2}$ or 0. Let $z = re^{i\theta}$, $z_k = r_k \exp[i\theta_k]$, $1 \leq k \leq n$. The inequality (11) is true if

$$1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \alpha_k \frac{1 - r^2 r_k^2}{1 - 2rr_k \cos(\theta + \theta_k) + r^2 r_k^2} \geq -\beta$$

for $0 \leq r \leq R_\beta$, $0 \leq \theta \leq 2\pi$. The last inequality holds if

$$1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \frac{\alpha_k (1 - r^2 r_k^2)}{1 + r^2 r_k^2 + 2rr_k} \geq -\beta \quad \text{for } 0 \leq r \leq R_\beta,$$

or, if

$$1 - \alpha - \frac{1}{2} \sum_{k=1}^n \alpha_k + \frac{1}{2} \sum_{k=1}^n \frac{\alpha_k(1 - rr_k)}{1 + rr_k} \geq -\beta \quad \text{for } 0 \leq r \leq R_\beta.$$

If we let $F_k(r) = (1 - rr_k)/(1 + rr_k)$ ($1 \leq k \leq n$) then the last inequality becomes

$$(12) \quad 2(1 - \alpha + \beta) - \sum_{k=1}^n \alpha_k + \sum_{k=1}^n \alpha_k F_k(r) \geq 0 \quad \text{for } 0 \leq r \leq R_\beta.$$

For a fixed r_k , $0 \leq r_k \leq R$, $F_k(0) = 1$ and $F_k(r)$ is a decreasing function of r , and hence $\min_{0 \leq r \leq R_\beta} F_k(r) = (1 - R_\beta r_k)/(1 + R_\beta r_k) = F_k(R_\beta)$. Again $F_k(R_\beta)$, when treated as a function of r_k , is a decreasing function of r_k and hence

$$\min_{0 \leq r_k \leq R} F_k(R_\beta) = (1 - R_\beta R)/(1 + R_\beta R).$$

Thus the inequality (12) holds if

$$2(1 - \alpha + \beta) - \sum_{k=1}^n \alpha_k + \left(\frac{1 - R_\beta R}{1 + R_\beta R} \right) \left(\sum_{k=1}^n \alpha_k \right) \geq 0,$$

or, if

$$2(1 + RR_\beta)(1 - \alpha + \beta) - 2 \sum_{k=1}^n \alpha_k (RR_\beta) \geq 0,$$

or if

$$R_\beta \leq (1 - \alpha + \beta) / R \left(\sum_{k=1}^n \alpha_k - 1 + \alpha - \beta \right).$$

Hence by taking $\beta = \frac{1}{2}$ and 0 we have that $h(z)$ is convex in one direction and univalent for

$$|z| \leq (3 - 2\alpha) / R \left(2 \sum_{k=1}^n \alpha_k + 2\alpha - 3 \right)$$

and $h(z)$ is convex for

$$|z| \leq (1 - \alpha) / R \left(\sum_{k=1}^n \alpha_k + \alpha - 1 \right).$$

We deduce immediately from Theorem 4 the following.

COROLLARY 5. *If $g(z)$ and z_k ($1 \leq k \leq n$) and R are the same as in Theorem 4 and $f(z) = zg(z)/\prod_{k=1}^n (1 - zz_k)^{\alpha_k}$ with $\alpha_k > 0$ ($1 \leq k \leq n$). Then*

(a) *$f(z)$ is starlike in one direction for*

$$|z| \leq (3 - 2\alpha) / R \left(2 \sum_{k=1}^n \alpha_k + 2\alpha - 3 \right),$$

(b) *$f(z)$ is starlike (and univalent) for*

$$|z| \leq (1 - \alpha) / R \left(\sum_{k=1}^n \alpha_k + \alpha - 1 \right).$$

By taking $g(z)=1$ for $|z|<1$, $\alpha_k=1$ ($1\leq k\leq n$) in Corollary 5 we obtain the following result of Rubinstein [7].

COROLLARY 6. *The function $z/\prod_{k=1}^n(1-zz_k)$ is univalent (and starlike) for $|z|<1/R(n-1)$ where $R=\max_{1\leq k\leq n}|z_k|\leq 1$.*

We use Theorem 1 to prove the next result.

THEOREM 7. *If $g(z)$ is holomorphic in D , $g(0)=1$ and $g(z)\neq 0$, $\operatorname{Re}[zg'(z)/g(z)]\geq -\alpha$ where $0\leq\alpha\leq 1$, for $|z|<1$, then the function*

$$F(z) = zg(z) / \left(\prod_{k=1}^n (1-zz_k)^{\alpha} \int_0^z \left(g(t) / \prod_{k=1}^n (1-z_k t)^{\alpha_k} \right) dt \right),$$

where $\alpha_k>0$, $|z_k|\leq 1$, $1\leq k\leq n$, and $\sum_{k=1}^n \alpha_k\leq 2-2\alpha$, is holomorphic in $|z|<1$ and if $F(z)=1+\sum_{n=1}^{\infty} b_n z^n$ then $|b_n|\leq 1$ for all n .

Proof. By Theorem 1, $h(z)=\int_0^z (g(t)/\prod_{k=1}^n (1-z_k t)^{\alpha_k}) dt$ is holomorphic, univalent and convex for $|z|<1$. It follows by a result of Strohcker [9] that

$$\operatorname{Re}[zh'(z)/h(z)]\geq \frac{1}{2},$$

or equivalently, $\operatorname{Re} F(z)\geq \frac{1}{2}$ for $|z|<1$. Since $h(0)=0$, $h(z)$ is univalent for $|z|<1$, it follows that $h(z)\neq 0$ for $0<|z|<1$ and hence $F(z)$ is holomorphic for $|z|<1$. Let $G(z)=F(z)-1$. Then $G(0)=0$, $G(z)=\sum_{n=1}^{\infty} b_n z^n$ and $\operatorname{Re}(G(z))\geq -\frac{1}{2}$ for $|z|<1$. The function $w=z/(1-z)=\sum_{n=1}^{\infty} z^n$ is holomorphic, univalent and convex for $|z|<1$ and maps $|z|<1$ onto the half-plane $\operatorname{Re}(w)>-\frac{1}{2}$. It follows now by a known result [2, p. 12] that $|b_n|\leq 1$ for all n .

3. In this section we study the functions $g(z)$ which we have used in the last section.

THEOREM 8. *If $g(z)$ is holomorphic in D , $g(0)=1$, and $g(z)\neq 0$, $\operatorname{Re}[zg'(z)/g(z)]\geq -\alpha$, with $0\leq\alpha<1$, for $|z|<1$, then*

$$g(z) = \exp \left[2(\alpha + \rho) \int_0^{2\pi} \log \left(\frac{e^{i\theta}}{e^{i\theta} - z} \right) d\beta(\theta) \right],$$

for some $\rho>0$, where $\beta(\theta)$ is a nondecreasing function of θ in $[0, 2\pi]$ with $\beta(2\pi)-\beta(0)=1$. (Here that branch of the logarithm is chosen which vanishes at $z=0$.) Also

$$\frac{1}{(1+r)^{2\alpha}} \leq |g(re^{i\theta})| \leq \frac{1}{(1-r)^{2\alpha}} \quad \text{for } 0 \leq r < 1, 0 \leq \theta \leq 2\pi.$$

Proof. For $0\leq\alpha<1$, we have that $\operatorname{Re}[zg'(z)/g(z)]\geq -\alpha>-(\alpha+\rho)$ for any $\rho>0$ and hence $\operatorname{Re}[zg'(z)/(\alpha+\rho)g(z)+1]>0$. Let

$$(13) \quad \frac{zg'(z)}{(\alpha+\rho)g(z)} + 1 = F(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{for } |z| < 1.$$

It is easy to see that $F(0)=1$ and by Herglotz's representation [8, p. 570] for such functions F we have

$$\frac{zg'(z)}{(\alpha+\rho)g(z)} + 1 = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\beta(\theta) \quad \text{for } |z| < 1,$$

where $\beta(\theta)$ is nondecreasing in $[0, 2\pi]$ and $\beta(2\pi) - \beta(0) = 1$. Thus

$$\frac{zg'(z)}{g(z)} = (\alpha+\rho) \int_0^{2\pi} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} - 1 \right) d\beta(\theta)$$

or,

$$\frac{g'(z)}{g(z)} = 2(\alpha+\rho) \int_0^{2\pi} \frac{1}{e^{i\theta} - z} d\beta(\theta).$$

Integrating the last equation from 0 to z , $|z| < 1$, with respect to z , we obtain

$$\begin{aligned} \log g(z) - \log g(0) &= 2(\alpha+\rho) \int_0^z \int_0^{2\pi} \frac{1}{e^{i\theta} - z} d\beta(\theta) dz \\ &= 2(\alpha+\rho) \int_0^{2\pi} \int_0^z \frac{1}{e^{i\theta} - z} dz d\beta(\theta) \\ &= 2(\alpha+\rho) \int_0^{2\pi} \log \left(\frac{e^{i\theta}}{e^{i\theta} - z} \right) d\beta(\theta). \end{aligned}$$

Hence,

$$(14) \quad g(z) = \exp \left[2(\alpha+\rho) \int_0^{2\pi} \log \left(\frac{e^{i\theta}}{e^{i\theta} - z} \right) d\beta(\theta) \right],$$

for $|z| < 1$. In equation (14) we take that branch of the logarithm in the integral which vanishes at $z=0$.

To prove the other assertion we have from equation (13) that $F(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $\operatorname{Re}(F(z)) > 0$ for $|z| < 1$ and hence by a known result [1, p. 44]

$$\frac{1-r}{1+r} \leq \operatorname{Re}(F(re^{i\theta})) \leq \frac{1+r}{1-r} \quad \text{for } 0 \leq r < 1, 0 \leq \theta \leq 2\pi.$$

In terms of g these inequalities become

$$\frac{-2(\alpha+\rho)r}{1+r} \leq \operatorname{Re} \left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})} \right) \leq \frac{2(\alpha+\rho)r}{1-r}, \quad 0 \leq r < 1, 0 \leq \theta \leq 2\pi.$$

Since

$$\operatorname{Re} \left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})} \right) = r \frac{\partial}{\partial r} \operatorname{Re}(\log g(re^{i\theta})),$$

we have

$$\frac{-2(\alpha+\rho)}{1+r} \leq \frac{\partial}{\partial r} \log |g(re^{i\theta})| \leq \frac{2(\alpha+\rho)}{1-r}, \quad 0 < r < 1, 0 \leq \theta \leq 2\pi.$$

Integrating with respect to r from 0 to r , we get

$$-2(\alpha + \rho) \log(1+r) \leq \log |g(re^{i\theta})| \leq -2(\alpha + \rho) \log(1-r),$$

$$0 < r < 1, 0 \leq \theta \leq 2\pi.$$

Or,

$$(15) \quad \frac{1}{(1+r)^{2(\alpha+\rho)}} \leq |g(re^{i\theta})| \leq \frac{1}{(1-r)^{2(\alpha+\rho)}}, \quad 0 \leq r < 1, 0 < \theta < 2\pi,$$

since the relations hold trivially for $r=0$. Since inequalities (15) hold for any $\rho > 0$, we let $\rho \rightarrow 0$ and we finally get

$$\frac{1}{(1+r)^{2\alpha}} \leq |g(re^{i\theta})| \leq \frac{1}{(1-r)^{2\alpha}}, \quad 0 \leq r < 1, 0 \leq \theta \leq 2\pi.$$

THEOREM 9. *If $g(z)$ is subject to the same conditions as in Theorem 8 and $g(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \leq n+1$ for all positive integral values of n .*

Proof. Choose a positive number ρ such that $\alpha + \rho \leq 1$, and let

$$(16) \quad \frac{zg'(z)}{(\alpha + \rho)g(z)} + 1 = F(z) = \sum_{n=0}^{\infty} b_n z^n.$$

As in the proof of Theorem 8, we have $\operatorname{Re}(F(z)) > 0$ for $|z| < 1$, $F(0) = b_0 = 1$ and hence by a known result of Carathéodory $|b_n| \leq 2$ for $n = 1, 2, 3, \dots$. Substituting in equation (16) the power series expansion for $g(z)$ we obtain

$$\sum_{n=1}^{\infty} n a_n z^n = (\alpha + \rho) \sum_{n=1}^{\infty} \left(\sum_{k=1}^n b_k a_{n-k} \right) z^n \quad \text{with } a_0 = 1.$$

Equating the coefficients of z^n we have

$$(17) \quad n a_n = (\alpha + \rho) \sum_{k=1}^n b_k a_{n-k} \quad (n = 1, 2, 3, \dots).$$

For $n=1$, equation (17) gives, $|a_1| = |\alpha + \rho| |b_1 a_0| \leq |b_1 a_0| \leq 2$. For $n=2$, we have $2|a_2| = |\alpha + \rho| |b_1 a_1 + b_2 a_0| \leq 2|a_1| + 2|a_0| \leq 4 + 2 = 6$ and hence $|a_2| \leq 3$.

Suppose now that $|a_n| \leq n+1$ for $1 \leq n \leq j$, then again from equation (17) we have

$$\begin{aligned} (j+1)|a_{j+1}| &\leq |\alpha + \rho| \sum_{k=1}^{j+1} |b_k| |a_{j+1-k}| \\ &\leq 2 \sum_{k=1}^{j+1} |a_{j+1-k}| \leq 2 \sum_{k=1}^{j+1} (j+2-k) \\ &= 2[(j+1) + j + \dots + 1] = (j+1)(j+2). \end{aligned}$$

Thus $|a_{j+1}| \leq j+2$ and the result follows by induction.

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